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# GENERAL FORMULAE FOR THE LOWER BOUND OF THE FIRST TWO DIRICHLET EIGENVALUES

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**Abstract.** This note presents general formulae for the lower bound of the first two Dirichlet eigenvalues on a regular domain. As applications, the positivity of top spectrum and the gap between these two Dirichlet eigenvalues are studied.

## 1. Introduction

Let $M$ be a complete Riemannian manifold of dimension $d$, and let $\Omega \subset M$ be a regular domain. Next, let $L = \Delta + \nabla V$ for some $V \in C^2(M)$. Consider the Dirichlet eigenvalue problem on $\Omega$:

$$Lu = -\lambda u, \quad u|_{\partial\Omega} = 0.$$

Denote by $\lambda_1$ and $\lambda_2$ the first two Dirichlet eigenvalues. We have $0 < \lambda_1 < \lambda_2$.

The purpose of this note is to present general formulae for the lower bound of $\lambda_i (i = 1, 2)$. The motivation of the study comes from [4] and [5] in which a general lower bound formula was presented for the spectral gap of an elliptic operator. We first recall the formula for the first Neumann eigenvalue due to [4], which will be used later on to study the lower bound of $\lambda_2$.

Let $K(V) = \inf\{r : (\mathrm{Hess}_V - \mathrm{Ric})(X, X) \le r|X|,\ X \in T\Omega\}$, and simply denote $K(0) = K$. Define

$$a(r) = \sup\{\langle \nabla\rho(x, \cdot)(y), \nabla V(y)\rangle + \langle \nabla\rho(\cdot, y)(x), \nabla V(x)\rangle : x, y \in \Omega, \rho(x, y) = r, y \notin \mathrm{cut}(x)\}$$

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for $r \in (0, D]$, where $\rho$ is the Riemannian distance and $D$ is the diameter of $\Omega$, and set $a(0) = 0$.

Next, choose $\gamma \in C[0, D]$ such that

$$\gamma(r) \geq \min\Big\{K(V)r,\ 2\sqrt{K^+(d-1)}\tanh\Big[\tfrac{r}{2}\sqrt{K^+/(d-1)}\Big] \\ -2\sqrt{K^-(d-1)}\tan\Big[\tfrac{r}{2}\sqrt{K^-/(d-1)}\Big] + a(r)\Big\}.$$

For simplicity, one may take $\gamma(r) = K(V)r$.

Finally, let $C(r) = \exp[\frac{1}{4}\int_0^r \gamma(s)\mathrm{d}s],\ r \in [0, D]$.

**Theorem 1.1 (Chen-Wang[4]).** *Suppose that $\Omega$ is convex. Let $n_1$ denote the first Neumann eigenvalue of $L$ on $\Omega$. We have*

$$n_1 \geq 4 \inf_{r\in(0,D)} f(r)\Big\{\int_0^r C(s)^{-1}\mathrm{d}s \int_s^D C(u)f(u)\mathrm{d}u\Big\}^{-1} \tag{1.1}$$

*for any $f \in C[0, D]$ with $f > 0$ in $(0, D)$.*

Theorem 1.1 provides a formula for the lower bound of $n_1$ in the sense that for each test function, one has a nontrival lower bound estimate. We will try to present formulae for the lower bound of $\lambda_1$ and $\lambda_2$ in the same spirit.

For the study of the lower bound of $\lambda_1$, a very famous tool is Barta's inequality, which says that (see also [9])

$$\lambda_1 \geq \inf_\Omega(-f^{-1}Lf) \tag{1.2}$$

for any $f \in C^2(\bar\Omega)$ with $f > 0$ in $\Omega$ and $f|_{\partial\Omega} = 0$. But the lower bound may be negative for some test function. In Section 2, we will establish the exact form of the formula such that the lower bound is nontrivial for each test function. Furthermore, by comparing $\lambda_2$ with the first Neumann eigenvalue, we obtain the lower bound formula for $\lambda_2$ in Section 3. Consequently, the positivity of the top spectrum and the lower bound estimates of $\lambda_2 - \lambda_1$ are also considered.

## 2. The Formula for the Lower Bound of $\lambda_1$

In this section, we assume that $\Omega = B(p, R)$, the open geodesic ball with centre $p$ and radius $R$. Let $\rho(x) = \rho(p, x)$, and let $i(p)$ be the injectivity radius of $p$.

**Theorem 2.1.** *Suppose that $i(p) > R$. Let $\gamma \in C[0, R]$ be such that $L\rho(x) \geq \gamma(\rho(x)),\ 0 < \rho(x) < R$. For any positive $f \in C[0, R]$, we have*

$$\lambda_1 \geq \inf_{r\in(0,R)} f(r)\Big\{\int_r^R \exp\Big[-\int_0^s \gamma(u)\,du\Big]\,ds \int_0^s \exp\Big[\int_0^t \gamma(u)\,du\Big]f(t)\,dt\Big\}^{-1},$$

*where $\lambda_1$ denotes the first Dirichlet eigenvalue of $L$ on $B(p,R)$.*

*Proof.* Let $\delta$ denote the lower bound given by Theorem 2.1, and take

$$h(x)=\int_{\rho(x)}^R \exp\left[-\int_0^s \gamma(u)\mathrm{d}u\right]\mathrm{d}s\int_0^s \exp\left[\int_0^t \gamma(u)\mathrm{d}u\right]f(t)\mathrm{d}t,\ x\in B(p,R).$$

Then $h>0$ in $B(p,R)$, $h|_{\partial B(p,R)}=0$, and

$$Lh(x)\le -f(\rho(x))\le -\delta h(x),\quad x\in B(p,R).$$

By (1.2) we prove the theorem. ∎

Suppose that the sectional curvatures of $\Omega$ are not larger than $k$. We have (see, e.g., [2, pp.69–96])

$$\Delta\rho\ge (d-1)K'(\rho)/K(\rho),\quad 0<r\le R,$$

where

$$K(r)=\begin{cases}\sin\sqrt{k}r, & \text{if } k>0,\\ r, & \text{if } k=0,\\ \sinh\sqrt{-k}r, & \text{if } k<0.\end{cases}$$

Then, by taking $\gamma(r)=\min\{n,(d-1)K'(r)/K(r)\}$ and $f=1$ in Theorem 2.1, and letting $n\to\infty$, we obtain

$$\lambda_1\ge\left\{\int_0^R K(r)^{1-d}\mathrm{d}r\int_0^r K(s)^{d-1}\mathrm{d}s\right\}^{-1},$$

which is exactly the first estimate of [9, Theorem 1.2].

Now, let $\sigma(V)$ be the top spectrum of $-L$ on $M$. We have $\lambda_1\downarrow\sigma(V)$ as $R\uparrow\infty$. Then the following is a direct consequence of Theorem 2.1.

**Corollary 2.2.** *Suppose that p is a pole, i.e., $i(p)=\infty$. If $\underline{\lim}_{\rho\to\infty}L\rho>0$, then $\sigma(V)>0$.*

*Proof.* If $\underline{\lim}_{\rho(x)\to\infty}L\rho(x)>0$, then there exist $r_0,c_0>0$ such that $\gamma(r)\ge c_0$ for $r\ge r_0$. Next, we know that $L\rho\to\infty$ as $\rho\to 0$ for $d>1$, and $L\rho$ is locally bounded for $d=1$. Hence $L\rho$ is bounded from below, i.e., $L\rho\ge -N_0$ for some $N_0\ge 0$. Choose nondecreasing function $\gamma\in C([0,\infty);[-N_0,c_0])$ such that $L\rho\ge\gamma(\rho)$ and $\gamma|_{[0,r_0]}\equiv -N_0, \gamma|_{[r_0+1,\infty)}\equiv c_0$. Taking $f(r)=\mathrm{e}^{-c_0r/2}$, we have

$$\begin{aligned}&\int_r^\infty \exp\left[-\int_0^s\gamma(u)\right]\mathrm{d}s\int_0^s\exp\left[\int_0^t\gamma(u)\right]f(t)\mathrm{d}t\\ &\le\int_r^\infty \exp\left[(r_0+1)(N_0+c_0)-c_0s\right]\mathrm{d}s\int_0^s \mathrm{e}^{c_0t/2}\mathrm{d}t\\ &\le\frac{4}{c_0^2}\mathrm{e}^{(r_0+1)(N_0+c_0)}f(r),\quad r\ge 0.\end{aligned}$$

By Theorem 1.1, we obtain $\sigma(V) \geq \frac{c_0^2}{4} \mathrm{e}^{-(r_0+1)(N_0+c_0)} > 0$. ∎

A very interesting problem is to seek for good geometric conditions on $M$ such that $\sigma(0) > 0$. The well-known result of McKean [8] implies that $\sigma(0) > 0$ provided $M$ is a CH-manifold with sectional curvatures uniformly negative. More recently, Kifer [6] proved that if $M$ is simply connected with hyperbolic metric (see his paper for the definition) and without focal points, then $\sigma(0) > 0$. Here, by using Corollary 2.2, we provide some sufficient conditions on curvatures. Let

$$K_p(x) = -\mathrm{Ric}(\nabla\rho(x), \nabla\rho(x)) \quad \text{and}$$
$$k_p(r) = \inf_{\rho(x)=r} \big\{ -\langle R(Y, \nabla\rho(x))Y, \nabla\rho(x)\rangle : Y \in T_xM, |Y| = 1, \langle \nabla\rho(x), Y\rangle = 0\big\}.$$

**Corollary 2.3.** *We have $\sigma(0) > 0$ provided one of the following holds:*

1) *$M$ has no focal points and $\underline{\lim}_{\rho(x)\to\infty} K_p(x) > 0$.*
2) *There exists $r_0 > 0$ such that $k_p(r) \geq 0$ for $r \geq r_0$ and $\frac{\pi^2}{4r_0^2} \geq -\inf_{r\geq 0} k_p(r)$, and $\underline{\lim}_{r\to\infty} k_p(r) > 0$.*

*Proof.* a) Suppose that 1) holds. For any $\xi \in T_pM$ with $|\xi| = 1$, let $X(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{t\xi}$, $t \geq 0$. Let $U(t)$ be the operator of the second fundamental form of $\partial B(p, t)$ at point $\mathrm{e}^{t\xi}$. We have (see [2, p. 72]) $\Delta\rho(\mathrm{e}^{t\xi}) = \mathrm{tr}U(t)$ and

$$U'(t) + U(t)^2 + \langle R(\cdot, X)\cdot, X\rangle(t) = 0, \quad t > 0.$$

Since $M$ has no focal points, $(\mathrm{tr}U(t))^2 \geq \mathrm{tr}U(t)^2$ by the proof of [6, Lemma 2.11]. Let $\phi(t) = \Delta\rho(\mathrm{e}^{t\xi})$. We obtain

$$\phi'(t) \geq K_p(\mathrm{e}^{t\xi}) - \phi^2(t), \quad t > 0. \tag{2.1}$$

By 1), there exist $t_0, K_0 > 0$ such that $K_p(x) \geq K_0$ for $\rho(x) \geq t_0$. Once again, since there are no focal points, $c_0 := \inf_{|\xi|=1} \phi(t_0) > 0$ (see [6, (2.5)]). We conclude from (2.1) that $\phi(t) \geq c_1 := \min\{c_0, \sqrt{K_0}\}$ for $t \geq t_0$. Actually, if there exists $t_1 > t_0$ such that $\phi(t_1) < c_1$, there exists $t_2 \in (t_0, t_1]$ such that $\phi(t_2) = \min_{[t_0,t_1]} \phi$. By (2.1) we have $\phi'(t_2) > 0$. Then there exists $t_3 \in (t_0, t_2)$ such that $\phi(t_3) < \phi(t_2)$. This is a contradiction. Therefore $\Delta\rho(x) \geq c_1$ for $\rho(x) \geq t_0$, and hence by Corollary 2.2 we obtain $\sigma(0) > 0$.

b) Suppose that 2) holds. For $x \in M$, let $l : [0, \rho(x)] \to M$ be the regular geodesic from $p$ to $x$ with unit tangent vector field $X$. Choose parallel vector fields $e_i (2 \leq i \leq d)$ along $l$ such that $\{X, e_2, \cdots, e_d\}$ is an orthonormal basis. Let $J_i$ be the Jacobi field along $l$ with $J_i(0) = 0, J_i(\rho(x)) = e_i$, $2 \leq i \leq d$. We have (see [3])

$$\Delta\rho(x) = \sum_{i=2}^{d} \int_0^{\rho(x)} (|\nabla_X J_i|^2 - \langle R(J_i, X)J_i, X\rangle). \tag{2.2}$$

Let $f_i(s) = |J_i(s)|$, $s \in [0, \rho(x)]$, $2 \le i \le d$. Since $\mathrm{cut}(p) = \emptyset, f_i > 0$ in $(0, \rho(x)]$. Note that $f_i' = f_i^{-1}\langle \nabla_T J_i, J_i\rangle$. We have $|\nabla_T J_i| \ge |f_i'|$. Hence

$$\Delta\rho(x) \ge \sum_{i=2}^{d} \int_0^{\rho(x)} [f_i'^2(s) + k_p(s) f_i^2(s)]\mathrm{d}s. \tag{2.3}$$

Consider the mixed eigenvalue problem of $d^2/dr^2$ on $[0, r_0]$, with Dirichlet condition at 0 and Neumann condition at $r_0$. We see that the first eigenvalue is $\pi^2/4r_0^2$ with eigenfunction $\sin[\pi s/2r_0]$. Hence, for $\rho(x) \ge r_0$,

$$\int_0^{r_0} f_i'^2(s)\mathrm{d}s \ge \frac{\pi^2}{4r_0^2}\int_0^{r_0} f_i^2(s)\mathrm{d}s.$$

Noting that $\frac{\pi^2}{4r_0^2} \ge -\inf k_p$, we obtain

$$\int_0^{r_0} [f_i'^2(s) + k_p(s) f_i^2(s)]\mathrm{d}s \ge \int_0^{r_0} \Big(\frac{\pi^2}{4r_0^2} + \inf k_p\Big) f^2(s)\mathrm{d}s \ge 0. \tag{2.4}$$

Next, since $\varliminf_{r\to\infty} k_p(r) > 0$, there exist $r_1 > r_0$ and $c_2 > 0$ such that $k_p(r) \ge c_2$ for $r \ge r_1$. By (2.3) and (2.4), for $\rho(x) \ge r_1 + 1$, we have

$$\Delta\rho(x) \ge \sum_{i=2}^{d} \int_{r_1}^{\rho(x)} [f_i'^2(s) + c_2 f_i^2(s)]\mathrm{d}s. \tag{2.5}$$

If $\int_{r_1}^{\rho(x)} f_i'^2(s)\mathrm{d}s \le 1/2$, then, for $s \in [\rho(x) - 1, \rho(x)]$, we have

$$\begin{aligned} f(s) &= 1 - \int_s^{\rho(x)} f_i'(s)\mathrm{d}s \ge 1 - \sqrt{\rho(x) - s}\Big(\int_s^{\rho(x)} f_i'^2(s)\mathrm{d}s\Big)^{1/2} \\ &\ge 1 - \frac{\sqrt{2}}{2}. \end{aligned}$$

Therefore,

$$\Delta\rho(x) \ge (d-1)\min\Big\{\frac{1}{2}, \frac{(\sqrt{2}-1)^2}{2}c_2\Big\}, \quad \rho(x) \ge r_1 + 1.$$

Hence we have $\sigma(0) > 0$. ■

## 3. The Formula for the Lower Bound of $\lambda_2$

We first extend Barta's inequality (1.2) to the second eigenvalue $\lambda_2$ of $L$ on $\Omega$. For any $\phi \in C^2(\bar{\Omega})$ with $\phi > 0$ in $\Omega$, let $n_1(\phi)$ be the first Neumann eigenvalue of $L_\phi := L + 2\nabla \log\phi$ on $\Omega$.

**Theorem 3.1.** *For any $\phi \in C^2(\bar{\Omega})$ with $\phi > 0$ in $\Omega$, we have*

$$\lambda_2 \geq \inf(-\phi^{-1}L\phi) + n_1(\phi). \tag{3.1}$$

*Proof.* Without loss of generality, we assume that $\inf_\Omega \phi > 0$. If $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$, then $L_\phi$ is symmetric on $L^2(\Omega, \phi^2\mathrm{d}\mu)$ with Neumann boundary condition. By the variational formula, we have

$$n_1(\phi) \leq \frac{\int_\Omega |\nabla f|^2\phi^2\mathrm{d}\mu}{\int_\Omega f^2\phi^2\mathrm{d}\mu} \tag{3.2}$$

for any $f \in C^1(\bar{\Omega})$ with $\int_\Omega f\phi^2\mathrm{d}\mu = 0$.

Next, let $u_i$ denote the $i$th Dirichlet eigenfunction of $L$ on $\Omega$ with $\mu(u_i^2) = 1(i = 1, 2)$ such that $u_1 > 0$ in $\Omega$ and $\int_\Omega u_2\phi\mathrm{d}\mu \leq 0$. Let $c \geq 0$ be such that $\int_\Omega(u_2 + cu_1)\phi\mathrm{d}x = 0$. Take $f = (u_2 + cu_1)/\phi$. We have $\int_\Omega f\phi^2\mathrm{d}\mu = 0$ and

$$-fL_\phi f = (\lambda_2 + \phi^{-1}L\phi)f^2 + (\lambda_2 - \lambda_1)cu_1f\phi^{-1}.$$

Let $\delta = \inf(-\phi^{-1}L\phi)$. We have

$$\begin{aligned} -\int_\Omega(fL_\phi f)\phi^2\mathrm{d}\mu &\leq \int_\Omega(\lambda_2 - \delta)f^2\phi^2\mathrm{d}\mu + (\lambda_2 - \lambda_1)c\int_\Omega(u_1u_2 + cu_1^2)\mathrm{d}\mu \\ &\leq \int_\Omega(\lambda_2 - \delta)f^2\phi^2\mathrm{d}\mu \end{aligned} \tag{3.3}$$

since $\int_\Omega u_1u_2\mathrm{d}\mu = 0$. Let $N$ be the outward unit normal vector field of $\partial\Omega$. We have $\phi^2\mathrm{e}^V fNf|_{\partial\Omega} = 0$. By Green's formula, we obtain

$$\begin{aligned} -\int_\Omega(fL_\phi f)\phi^2\mathrm{d}\mu &= \int_\Omega\{\langle\nabla f, \nabla(f\phi^2\mathrm{e}^V)\rangle - f\phi^2\mathrm{e}^V\langle\nabla V + 2\nabla\log\phi, \nabla f\rangle\}\mathrm{d}x \\ &= \int_\Omega|\nabla f|^2\phi^2\mathrm{d}\mu. \end{aligned}$$

By combining this with (3.2) and (3.3), we complete the proof. ■

Now, by simply taking $\gamma(r) = K(V + \log\phi)r$ in Theorem 1.1 for the lower bound of $n_1(\phi)$, and then combining with Theorem 3.1, we obtain the following result.

**Corollary 3.2.** *Suppose that $\Omega$ is convex. We have*

$$\begin{aligned} \lambda_2 \geq &\inf(-\phi^{-1}L\phi) \\ &+4\inf_{r\in(0,D)} f(r)\left\{\int_0^r \mathrm{e}^{-K(V+\log\phi)s^2/8}\mathrm{d}s\int_s^D \mathrm{e}^{K(V+\log\phi)u^2/8}f(u)\mathrm{d}u\right\}^{-1} \end{aligned}$$

*for any $\phi \in C^2(\bar{\Omega})$ with $\phi > 0$ in $\Omega$ and positive $f \in C[0, D]$.*

**Remarks** 1) By taking $\phi = 1$ in Theorem 3.1, we obtain $\lambda_2 \geq n_1(0)$, which is well-known by the domain monotonicity of eigenvalues (see [2, p.18]).

2) By taking $\phi = u_1$ in Theorem 3.1, we obtain $\lambda_2 - \lambda_1 \geq n_1(u_1)$, which gives a general formula for the lower bound esetimate of $\lambda_2 - \lambda_1$ by Corollary 3.2 when $\Omega$ is convex. Especially, when $V = 0$ and $M = \mathbb{R}^d$ or $\mathbb{S}^d$, we know that $u_1$ is log-concave (see [1] and [7]). Then $K(\log u_1) \leq 0$ for $M = \mathbb{R}^d$ and $\leq 1 - d$ for $M = \mathbb{S}^d$. By taking $f(r) = \sin[\pi r/(2D)]$, we have (see [4])

$$4f(r)\bigg\{\int_0^r \mathrm{e}^{-Ks^2/8}\mathrm{d}s\int_s^D \mathrm{e}^{Ku^2/8}f(u)\mathrm{d}u\bigg\}^{-1} \geq \frac{\pi^2}{D^2} - \frac{\pi-2}{\pi}K$$

for $K \leq 0$. Therefore, by Corollary 3.2,

$$\lambda_2 - \lambda_1 \geq \begin{cases} \frac{\pi^2}{D^2}, & \text{if } M = \mathbb{R}^d, \\ \frac{\pi^2}{D^2} + \frac{\pi-2}{\pi}(d-1), & \text{if } M = \mathbb{S}^d. \end{cases}$$

This recovers Yu-Zhong's estimate [11] ($M = \mathbb{R}^d$) and improves Lee-Wang's estimate [7] ($M = \mathbb{S}^d$). Refer to [10] for further research in this direction.

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