

General Formulas for Explicit Evaluations of Ramanujan's Cubic Continued Fraction

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ABSTRACT. On page 366 of his lost notebook [15], Ramanujan recorded a cubic continued fraction and several theorems analogous to Rogers-Ramanujan's continued fractions. In this paper, we derive several general formulas for explicit evaluations of Ramanujan's cubic continued fraction, several reciprocity theorems, two formulas connecting $V(q)$ and $V(q^3)$ and also establish some explicit evaluations using the values of remarkable product of theta-function.

1. Introduction

In Chapter 16, of his second notebook, Ramanujan defines his theta-function as

$$\begin{aligned} f(a, b) &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1, \\ &= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \end{aligned}$$

where

$$(a; q)_{\infty} = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

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Following Ramanujan for $|q| < 1$, we define

$$(1.1) \quad \varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty},$$

$$(1.2) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

$$(1.3) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q)_{\infty}$$

and

$$(1.4) \quad \chi(q) := (-q; q^2)_{\infty}.$$

On page 366 of his lost notebook [15], Ramanujan recorded the cubic continued fraction

$$(1.5) \quad V := V(q) := \frac{q^{\frac{1}{3}}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \cdots, \quad |q| < 1,$$

and claimed that there are many results of $V(q)$ which are analogous to the Rogers-Ramanujan continued fraction. H. H. Chan [7] established the relation between $V(q)$ and the other three continued fractions $V(-q)$, $V(q^2)$ and $V(q^3)$. N. D. Baruah [3] established the relation between $V(q)$ and the two continued fractions $V(q^5)$ and $V(q^7)$.

In [4], Baruah and Nipen Saikia have also established some general theorems to find explicit evaluations of cubic continued fraction. In [1], C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudan derived a modular equation relating $V(q)$ with $V(q^3)$, a transformation formula and several explicit evaluations of $V(q)$. For more discussions of (1.5), see [2] and [9].

On page 338 in his first notebook [14], Ramanujan defined

$$(1.6) \quad a_{M, N} = \frac{N e^{-\frac{(N-1)\pi}{4}} \sqrt{\frac{M}{N}} \psi^2(e^{-\pi\sqrt{MN}}) \varphi^2(-e^{-2\pi\sqrt{MN}})}{\psi^2(e^{-\pi\sqrt{\frac{M}{N}}}) \varphi^2(-e^{-2\pi\sqrt{\frac{M}{N}}})}.$$

He then, on pages 338 and 339, offered a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [6]. In [10] and [11], Mahadeva Naika et al. derived general theorems to establish explicit new evaluations of Ramanujan's remarkable product of theta-function. In [12], Mahadeva Naika and M. C. Maheshkumar have also obtained other explicit evaluations of Ramanujan's remarkable product of theta-function.

In [13], Mahadeva Naika, Maheshkumar and K. Sushan Bairy defined

$$(1.7) \quad b_{M,N} = \frac{Ne^{-\frac{(N-1)\pi}{4}\sqrt{\frac{M}{N}}}\psi^2(-e^{-\pi\sqrt{MN}})\varphi^2(-e^{-2\pi\sqrt{MN}})}{\psi^2(-e^{-\pi\sqrt{\frac{M}{N}}})\varphi^2(-e^{-2\pi\sqrt{\frac{M}{N}}})}.$$

and established several general formulas for evaluating $b_{M,N}$.

The ordinary hypergeometric series ${}_2F_1(a, b; c; x)$ is defined for $|x| < 1$,

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!},$$

where

$$(a)_0 = 1 \text{ and } (a)_n = a(a+1)(a+2)\cdots(a+n-1), \text{ for } n \geq 1.$$

Let

$$(1.8) \quad z(r) := z(r; x) := {}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)$$

and

$$q_r := q_r(x) := \exp\left(-\pi \operatorname{csc}\left(\frac{\pi}{r}\right) \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)}\right),$$

where $r = 2, 3, 4, 6$ and $0 < x < 1$.

Let N denote a fixed natural number, and assume that

$$(1.9) \quad N \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)},$$

where $r = 2, 3, 4$ or 6 . Then a modular equation of degree N in the theory of elliptic functions of signature r is a relation between α and β induced by (1.9). We often say that β is of degree N over α and $m(r) := \frac{z(r; \alpha)}{z(r; \beta)}$ is called the multiplier. We also use the notations $z_1 := z_1(r) = z(r; \alpha)$ and $z_N := z_N(r) = z(r; \beta)$ to indicate that β has degree N over α . When the context is clear, we omit the argument r in $q_r, z(r)$ and $m(r)$.

In Section 2, we collect some preliminary results which are useful in proving our main results. In Section 3, we establish our main results. In Section 4, we prove several reciprocity theorems for $V(q)$. In Section 5, we establish two formulas connecting $V(q)$ and $V(q^3)$. In Section 6, we also establish several explicit evaluations of $V(q)$.

2. Preliminary results

In this section, we collect some of the identities which are very useful to prove

our main results.

Theorem 2.1. *If z is defined as in (1.8), then we have*

$$(2.1) \quad \varphi(e^{-y}) = \sqrt{z}$$

$$(2.2) \quad \varphi(-e^{-2y}) = \sqrt{z}(1-x)^{\frac{1}{8}}$$

$$(2.3) \quad \psi(e^{-y}) = \sqrt{\frac{1}{2}z(xe^y)^{\frac{1}{8}}}$$

$$(2.4) \quad \psi(-e^{-y}) = \sqrt{\frac{1}{2}z\{x(1-x)e^y\}^{\frac{1}{8}}}$$

$$(2.5) \quad 1 + \frac{1}{V} = \frac{\psi(q^{\frac{1}{3}})}{q^{\frac{1}{3}}\psi(q^3)} \quad \text{and} \quad 1 + \frac{1}{V^3} = \frac{\psi^4(q)}{q\psi^4(q^3)}$$

$$(2.6) \quad 2V = 1 - \frac{\varphi(-q^{\frac{1}{3}})}{\varphi(-q^3)}$$

$$(2.7) \quad 1 - 8V^3 = \frac{\varphi^4(-q)}{\varphi^4(-q^3)}, \quad \text{where } V \text{ is defined as in (1.5)}$$

Proof. For a proof of (2.1) see [5, Entry 10(i), ch.17, p.122]; for a proof of (2.2) see [5, Entry 10(iii), ch.17, p.123]; for a proof of (2.3) see [5, Entry 11(i), ch.17, pp.123-124]; for a proof of (2.4) see [5, Entry 11(ii), ch.17, pp.123-124]; for a proof of (2.5) see [5, Entry 1(i), ch.20, pp.345-346]; for a proof of (2.6) see [5, Entry 1(ii), ch.20, pp.345-347]; for a proof of (2.7) see proof of Entry 1(iii) in [5, ch.20, p.347]. \square

3. General formulas for explicit evaluations of $V(q)$

Lemma 3.1. *If β is of degree N over α , then*

$$(3.1) \quad \frac{N}{a_{M,N}} = m^2 \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{\frac{1}{4}}.$$

Proof. The identity (1.6) can be written as

$$(3.2) \quad a_{M,N} = \frac{Nq^{\frac{(N-1)}{4}}\psi^2(q^N)\varphi^2(-q^{2N})}{\psi^2(q)\varphi^2(-q^2)}, \quad q = e^{-\pi\sqrt{\frac{M}{N}}}.$$

Using (2.2) and (2.3), we obtain (3.1). \square

Theorem 3.1. *We have*

$$(3.3) \quad 8V^3(-e^{-\pi\sqrt{\frac{M}{3}}}) - \frac{1}{V^3(-e^{-\pi\sqrt{\frac{M}{3}}})} + 7 = \frac{9}{a_{M,3}^2}.$$

Proof. Using (2.1) and (2.4) in (2.5) and (2.7), we find that

$$(3.4) \quad -1 - \frac{1}{V^3(-q)} = m^2 \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{\frac{1}{2}}$$

and

$$(3.5) \quad 1 - 8V^3(-q) = \frac{z_1^2}{z_3^2} = m^2.$$

Using (3.4) and (3.5) in (3.1) with $N = 3$, we obtain (3.3). □

It is clear from the above Theorem 3.1 that to evaluate $V(-e^{-\pi\sqrt{\frac{M}{3}}})$, one has to compute $a_{M,3}$.

Corollary 3.1. *We have*

$$(3.6) \quad V^3(-e^{-\frac{\pi}{\sqrt{3}}}) = -\frac{1}{4}.$$

Proof. Putting $M = 1$ and $N = 3$ in (3.2), we find that

$$(3.7) \quad a_{1,3} = 1.$$

Using (3.7) in (3.3), we obtain (3.6). □

This was also evaluated by Adiga et al. [2] and Mahadeva Naika [9].

Corollary 3.2. *We have*

$$(3.8) \quad V^3(-e^{-\pi}) = \frac{(5 - 3\sqrt{3})}{4}.$$

Proof. Putting $M = 3$ and $N = 3$ in (3.2), we find that

$$(3.9) \quad a_{3,3} = \frac{1}{\sqrt{3}}.$$

Using (3.9) in (3.3), we obtain (3.8). □

This was also evaluated by Baruah et al. [4] and Mahadeva Naika [9].

Theorem 3.2. *We have*

$$(3.10) \quad 2V(-e^{-\pi\sqrt{M}}) - \frac{1}{V(-e^{-\pi\sqrt{M}})} + 1 = \frac{3}{\sqrt{a_{M,9}}}.$$

Proof. Using (2.1) and (2.4) in (2.5) and (2.6), we find that

$$(3.11) \quad 1 - 2V(-q^3) = \sqrt{\frac{z_1}{z_9}} = \sqrt{m}.$$

$$(3.12) \quad -1 - \frac{1}{V(-q^3)} = \sqrt{m} \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{\frac{1}{8}}.$$

and putting $N = 9$ in (3.1), we find that

$$(3.13) \quad \frac{9}{a_{M,9}} = m^2 \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)} \right)^{\frac{1}{4}},$$

where β is of degree 9 over α . Using (3.11) and (3.12) in (3.13), we obtain (3.10). \square

It is clear from the above Theorem 3.2 that to evaluate $V(-e^{-\pi\sqrt{M}})$, one has to compute $a_{M,9}$.

Corollary 3.3. *We have*

$$(3.14) \quad V(-e^{-\pi}) = \frac{1 - \sqrt{3}}{2}.$$

Proof. Putting $M = 1$ and $N = 9$ in (2.2), we find that

$$(3.15) \quad a_{1,9} = 1.$$

Using (3.15) in (3.10), we obtain (3.14). \square

This was also evaluated by Chan [7], Baruah et al. [4] and Mahadeva Naika [9].

Remark. In [9], Mahadeva Naika has also derived a different general formula to evaluate $V(-e^{-\pi\sqrt{n}})$.

Theorem 3.3. *We have*

$$(3.16) \quad \frac{1}{V^3(e^{-\pi\sqrt{\frac{M}{3}}})} - 8V^3(e^{-\pi\sqrt{\frac{M}{3}}}) - 7 = \frac{9}{b_{M,3}^2}.$$

Proof. Using (2.1) and (2.4) in (2.5) and (2.7), we find that

$$(3.17) \quad 1 + \frac{1}{V^3(q)} = m^2 \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}}$$

and

$$(3.18) \quad 1 - 8V^3(q) = m^2 \left(\frac{1-\alpha}{1-\beta} \right),$$

where β is of degree 3 over α . Putting $N = 3$ in (1.7) with $q = e^{-\pi\sqrt{\frac{M}{3}}}$ and then using (2.1) and (2.4), we find that

$$(3.19) \quad \frac{9}{b_{M,3}^2} = m^4 \left(\frac{\alpha(1-\alpha)^2}{\beta(1-\beta)^2} \right)^{\frac{1}{2}}.$$

Using (3.17) and (3.18) in (3.19), we obtain (3.16). □

It is clear from the above Theorem 3.3 that to evaluate $V(e^{-\pi\sqrt{\frac{M}{3}}})$, one has to compute $b_{M,3}$.

Corollary 3.4. *We have*

$$(3.20) \quad V^3(e^{-\sqrt{\frac{2}{3}}\pi}) = \frac{3 - 2\sqrt{2}}{2\sqrt{2}}.$$

Proof. Putting $M = 2$ and $N = 3$ in (1.7), we find that

$$(3.21) \quad b_{2,3} = 1.$$

Using (3.21) in (3.16), we obtain (3.20). □

Theorem 3.4. *We have*

$$(3.22) \quad \frac{1}{V(e^{-\pi\sqrt{M}})} - 2V(e^{-\pi\sqrt{M}}) - 1 = \frac{3}{\sqrt{b_{M,9}}}.$$

Proof. Using (2.1) and (2.4) in (2.5) and (2.6), we find that

$$(3.23) \quad 1 - 2V(q^3) = \sqrt{m} \left(\frac{1 - \alpha}{1 - \beta} \right)^{\frac{1}{4}}$$

and

$$(3.24) \quad 1 + \frac{1}{V(q^3)} = \sqrt{m} \left(\frac{\alpha}{\beta} \right)^{\frac{1}{8}}.$$

and putting $N = 9$ in (1.7) with $q = e^{-\pi\sqrt{\frac{M}{9}}}$ and then using (2.1) and (2.4), we find that

$$(3.25) \quad \frac{9}{\sqrt{b_{M,9}}} = m \left(\frac{\alpha(1 - \alpha)^2}{\beta(1 - \beta)^2} \right)^{\frac{1}{8}},$$

where β is of degree 9 over α . Using (3.23), (3.24) in (3.25), we obtain (3.22). □

It is clear from the above Theorem 3.4 that to evaluate $V(e^{-\pi\sqrt{M}})$, one has to compute $b_{M,9}$.

Corollary 3.5. *We have*

$$(3.26) \quad V(e^{-\sqrt{2}\pi}) = \frac{\sqrt{3} - \sqrt{2}}{\sqrt{2}}.$$

Proof. Putting $M = 2$ and $N = 9$ in (1.7), we find that

$$(3.27) \quad b_{2,9} = 1.$$

Using (3.27) in (3.22), we obtain (3.26). \square

This was also evaluated by Chan [7].

4. Some reciprocity theorems for $V(q)$

In this section, we establish several reciprocity theorems for $V(q)$.

Theorem 4.1. *We have*

$$(4.1) \quad V^3(-e^{-\pi\sqrt{\frac{1}{3M}}}) = \frac{1 + V^3(-e^{-\pi\sqrt{\frac{M}{3}}})}{-1 + 8V^3(-e^{-\pi\sqrt{\frac{M}{3}}})}.$$

Proof. Replacing M by $\frac{1}{M}$ in (3.3), we find that

$$(4.2) \quad 8V^3(-e^{-\pi\sqrt{\frac{1}{3M}}}) - \frac{1}{V^3(-e^{-\pi\sqrt{\frac{1}{3M}}})} + 7 = \frac{9}{a_{\frac{1}{M},3}^2}.$$

Using (3.3) and (4.2), we deduce that

$$(4.3) \quad (8x - \frac{1}{x} + 7)(8y - \frac{1}{y} + 7) = 81, \quad x = V^3(-e^{-\pi\sqrt{\frac{M}{3}}}), y = V^3(-e^{-\pi\sqrt{\frac{1}{3M}}}).$$

On simplification of the above identity (4.3), we find that

$$(4.4) \quad (8x + 8xy + 8y - 1)(-y + 8xy - x - 1) = 0.$$

The first factor does not vanish in the neighborhood of $q = e^{-\frac{\pi}{\sqrt{3}}}$. But the second factor vanishes in the neighborhood of $q = e^{-\frac{\pi}{\sqrt{3}}}$. So by the identity theorem the second factor vanishes identically. Hence, we complete the proof. \square

Corollary 4.1. *We have*

$$(4.5) \quad V^3(-e^{-\frac{\pi}{3}}) = \frac{-(\sqrt{3} + 1)}{4}.$$

Proof. Putting $M = 3$ in (4.1), we find that

$$(4.6) \quad V^3(-e^{-\frac{\pi}{3}}) = \frac{1 + V^3(-e^{-\pi})}{-1 + 8V^3(-e^{-\pi})}.$$

Using (3.14) in (4.6), we obtain (4.5). \square

Theorem 4.2. *We have*

$$(4.7) \quad V(-e^{-\pi\sqrt{\frac{1}{M}}}) = \frac{1 + V(-e^{-\pi\sqrt{M}})}{-1 + 2V(-e^{-\pi\sqrt{M}})}.$$

Proof. Replacing M by $\frac{1}{M}$ in (3.10), we find that

$$(4.8) \quad 2V(-e^{-\pi\sqrt{\frac{1}{M}}}) - \frac{1}{V(-e^{-\pi\sqrt{\frac{1}{M}}})} + 1 = \frac{3}{\sqrt{a_{\frac{1}{M},9}}}.$$

Using (3.10) and (4.8), we deduce that

$$(4.9) \quad (2x - \frac{1}{x} + 1)(2y - \frac{1}{y} + 1) = 9, \quad x = V(-e^{-\pi\sqrt{M}}), y = V(-e^{-\pi\sqrt{\frac{1}{M}}}).$$

On simplification of the above identity (4.9), we find that

$$(4.10) \quad (2x + 2xy + 2y - 1)(-y + 2xy - x - 1) = 0.$$

The first factor does not vanish in the neighborhood of $q = e^{-\pi}$. But the second factor vanishes in the neighborhood of $q = e^{-\pi}$. So by the identity theorem the second factor vanishes identically. Hence, we complete the proof. \square

Corollary 4.2. *We have*

$$(4.11) \quad V(-e^{-3\pi}) = \frac{-2 + \sqrt[3]{2(\sqrt{3} + 1)}}{2\left(1 + \sqrt[3]{2(\sqrt{3} + 1)}\right)}.$$

Proof. Using (4.5) in (4.7) with $M = 9$ and solving the resultant equation, we obtain (4.11). \square

Theorem 4.3. *We have*

$$(4.12) \quad \frac{1}{8V^3(e^{-\pi\sqrt{\frac{2}{3M}}})} = \frac{1 + V^3(e^{-\pi\sqrt{\frac{2M}{3}}})}{1 - 8V^3(e^{-\pi\sqrt{\frac{2M}{3}}})}.$$

Proof. Replacing M by $2M$ in (3.16), we find that

$$(4.13) \quad \frac{1}{V^3(e^{-\pi\sqrt{\frac{2M}{3}}})} - 8V^3(e^{-\pi\sqrt{\frac{2M}{3}}}) - 7 = \frac{9}{b_{2M,3}^2}.$$

Replacing M by $\frac{1}{M}$ in (4.13), we find that

$$(4.14) \quad \frac{1}{V^3(e^{-\pi\sqrt{\frac{2}{3M}}})} - 8V^3(e^{-\pi\sqrt{\frac{2}{3M}}}) - 7 = \frac{9}{b_{\frac{2}{M},3}^2}.$$

Using (4.13) and (4.14), we deduce that

$$(4.15) \quad \left(\frac{1}{x} - 8x - 7\right)\left(\frac{1}{y} - 8y - 7\right) = 81, \quad x = V^3(e^{-\pi\sqrt{\frac{2M}{3}}}), y = V^3(e^{-\pi\sqrt{\frac{2}{3M}}}).$$

On simplification of the above identity (4.15), we find that

$$(4.16) \quad (8x + 8xy + 8y - 1)(-y + 8xy - x - 1) = 0.$$

The second factor does not vanish in the neighborhood of $q = e^{-\pi\sqrt{\frac{2}{3}}}$. But the first factor vanishes in the neighborhood of $q = e^{-\pi\sqrt{\frac{2}{3}}}$. So by the identity theorem the first factor vanishes identically. Hence, we complete the proof. \square

Corollary 4.3. *We have*

$$(4.17) \quad V^3(e^{-\frac{\pi\sqrt{2}}{3}}) = \frac{\sqrt{3} - \sqrt{2}}{2\sqrt{2}}.$$

Proof. Using (3.26) in (4.12) with $M = 3$ and solving the resulting equation, we obtain the required result. \square

Theorem 4.4. *We have*

$$(4.18) \quad \frac{1}{2V(e^{-\pi\sqrt{\frac{2}{M}}})} = \frac{1 + V(e^{-\pi\sqrt{2M}})}{1 - 2V(e^{-\pi\sqrt{2M}})}.$$

Proof. Replacing M by $2M$ in (3.22), we find that

$$(4.19) \quad \frac{1}{V(e^{-\pi\sqrt{2M}})} - 2V(e^{-\pi\sqrt{2M}}) - 1 = \frac{3}{\sqrt{b_{2M,9}}}.$$

Replacing M by $\frac{1}{M}$ in (4.19), we find that

$$(4.20) \quad \frac{1}{V(e^{-\pi\sqrt{\frac{2}{M}}})} - 2V(e^{-\pi\sqrt{\frac{2}{M}}}) - 1 = \frac{3}{\sqrt{b_{\frac{2}{M},9}}}.$$

Using (4.19) and (4.20), we deduce that

$$(4.21) \quad \left(\frac{1}{x} - 2x - 1\right)\left(\frac{1}{y} - 2y - 1\right) = 9, \quad x = V(e^{-\pi\sqrt{2M}}), y = V(e^{-\pi\sqrt{\frac{2}{M}}}).$$

On simplification of the above identity (4.21), we find that

$$(4.22) \quad (2x + 2xy + 2y - 1)(-y + 2xy - x - 1) = 0.$$

The second factor does not vanish in the neighborhood of $q = e^{-\pi\sqrt{2}}$. But the first factor vanishes in the neighborhood of $q = e^{-\pi\sqrt{2}}$. So by the identity theorem the first factor vanishes identically. Hence, we complete the proof. \square

Corollary 4.4. *We have*

$$(4.23) \quad V(e^{-\pi\sqrt{6}}) = \frac{1 - \sqrt{2}\sqrt[3]{3 - 2\sqrt{2}}}{\sqrt{2}(\sqrt{2} + \sqrt[3]{3 - 2\sqrt{2}})}.$$

Proof. Using (3.20) in (4.18) with $M = 3$ and solving the resulting equation, we obtain the result (4.23). \square

5. Formulas relating $V(q)$ and $V(q^3)$

In this section, we prove two formulas relating $V(q)$ and $V(q^3)$.

Theorem 5.1. *We have*

$$(5.1) \quad V(q^3) = \frac{-1 + t}{1 + 2t},$$

where

$$t = \sqrt[3]{\frac{1 + V^3(q)}{1 - 8V^3(q)}}.$$

Proof. Using (4.18) with replacing M by $3M$ in (4.12), we find that

$$(5.2) \quad \frac{1 + V(e^{-\pi\sqrt{6M}})}{1 - 2V(e^{-\pi\sqrt{6M}})} = \sqrt[3]{\frac{1 + V^3(e^{-\pi\sqrt{\frac{2M}{3}}})}{1 - 8V^3(e^{-\pi\sqrt{\frac{2M}{3}}})}}$$

On simplification of the above identity (5.2) with $q = e^{-\pi\sqrt{\frac{2M}{3}}}$, we obtain the required result (5.1). \square

Replacing q by $-q$ in Theorem 5.1 and observe that t is equal to $-y$, then we obtain the following theorem.

Theorem 5.2. *We have*

$$(5.3) \quad V(-q^3) = \frac{1 + y}{-1 + 2y},$$

where

$$y = \sqrt[3]{\frac{1 + V^3(-q)}{-1 + 8V^3(-q)}}.$$

6. Some explicit evaluations of $V(q)$

In this section, we establish following explicit evaluations of Ramanujan's cubic continued fractions.

Theorem 6.1. *We have*

$$(6.1) \quad V^3(-e^{-\pi\sqrt{\frac{14}{3}}}) = \frac{-7x^2 + 9 - 3\sqrt{9x^4 - 14x^2 + 9}}{16x^2},$$

where

$$x = \sqrt{\frac{4125\sqrt{7} - 10887 - 2376\sqrt{21} + 6300\sqrt{3}}{2}} - \sqrt{\frac{4125\sqrt{7} - 10889 - 2376\sqrt{21} + 6300\sqrt{3}}{2}}.$$

$$(6.2) \quad V^3(-e^{-\pi\sqrt{\frac{3}{5}}}) = \frac{9\sqrt{6} - 15\sqrt{2} + 12\sqrt{3} - 22}{4},$$

$$(6.3) \quad V(-e^{-\pi\sqrt{2}}) = \frac{1}{4} \left[\sqrt{97 - 51\sqrt{3} - 63\sqrt{2} + 39\sqrt{6}} - \sqrt{105 - 51\sqrt{3} - 63\sqrt{2} + 39\sqrt{6}} \right],$$

and

$$(6.4) \quad V(-e^{-\pi\sqrt{3}}) = \frac{1 - \sqrt[3]{4}}{2 + \sqrt[3]{4}}.$$

Proof of (6.1). From Ramanujan's first notebook [14, p.316], we have

$$(6.5) \quad g_{42} = (2\sqrt{2} + \sqrt{7})^{\frac{1}{6}} \left(\frac{\sqrt{3} + \sqrt{7}}{2} \right)^{\frac{1}{2}}.$$

From the Theorem 4.1(i) in [8], we have

$$(6.6) \quad 2\sqrt{2} \left[g_n^3 g_{9n}^3 + \frac{1}{g_n^3 g_{9n}^3} \right] = \frac{g_{9n}^6}{g_n^6} - \frac{g_n^6}{g_{9n}^6}.$$

Using (6.5) in (6.6) with $n = \frac{14}{3}$, we find that

$$(6.7) \quad g_{\frac{14}{3}} = (2\sqrt{2} - \sqrt{7})^{\frac{1}{6}} \left(\frac{\sqrt{3} + \sqrt{7}}{2} \right)^{\frac{1}{2}}.$$

From the Theorem 4.3(i) in [8] with slight modification, we have

$$(6.8) \quad G_n G_{9n} = (2g_n g_{9n})^{-\frac{1}{2}} \sqrt{g_n^3 g_{9n}^3 + \sqrt{g_n^6 g_{9n}^6 + 2}}.$$

Using (6.5) and (6.7) in (6.8) with $n = \frac{14}{3}$, we deduce that

$$(6.9) \quad G_{\frac{14}{3}} G_{42} = (2 + \sqrt{3})^{\frac{1}{2}} \left(\frac{3 + \sqrt{7}}{2} \right)^{\frac{1}{2}} \left(\frac{\sqrt{7} - \sqrt{3}}{2} \right)^{\frac{1}{2}}.$$

From the Theorem 4.2(i) in [8], we have

$$(6.10) \quad 2\sqrt{2} \left[G_n^3 G_{9n}^3 - \frac{1}{G_n^3 G_{9n}^3} \right] = \frac{G_{9n}^6}{G_n^6} + \frac{G_n^6}{G_{9n}^6}.$$

Using (6.9) in (6.10) with $n = \frac{14}{3}$, we find that

$$(6.11) \quad \frac{G_{\frac{14}{3}}^6}{G_{42}^6} = \sqrt{\frac{37125\sqrt{7} + 56700\sqrt{3} - 97999 - 21384\sqrt{21}}{2}} - 3\sqrt{\frac{4125\sqrt{7} + 6300\sqrt{3} - 10889 - 2376\sqrt{21}}{2}}.$$

From Theorem 2.1 in [6], we have

$$(6.12) \quad a_{M,3} - \frac{1}{a_{M,3}} = \frac{1}{3} \left[\frac{G_{\frac{M}{3}}^6}{G_{3M}^6} - \frac{G_{3M}^6}{G_{\frac{M}{3}}^6} \right].$$

Using (6.11) in (6.12) with $M = 14$, we have

$$(6.13) \quad a_{14,3} = \sqrt{\frac{4125\sqrt{7} - 10887 - 2376\sqrt{21} + 6300\sqrt{3}}{2}} - \sqrt{\frac{4125\sqrt{7} - 10889 - 2376\sqrt{21} + 6300\sqrt{3}}{2}}.$$

Using (6.13) in (3.3), we obtain the required result.

As the proofs of the identities (6.2)-(6.4) being similar to the proof of the identity (6.1). So we omit the details. □

Theorem 6.2. *We have*

$$(6.14) \quad V^3(e^{-\pi\sqrt{\frac{5}{3}}}) = \frac{-209 - 93\sqrt{5} + 54\sqrt{15} + 120\sqrt{3}}{2},$$

$$(6.15) \quad V^3(e^{-\pi\sqrt{8}}) = \frac{1}{8}(19 - 9\sqrt{3} + 9\sqrt{6} - 18\sqrt{2}),$$

$$(6.16) \quad V^3(e^{-\pi\sqrt{10}}) = \frac{1}{4\sqrt[4]{5}}(158\sqrt{2} + 128\sqrt{3})^{\frac{1}{4}} \\ \times \left[\sqrt{6\sqrt{10\sqrt{3} + 20\sqrt{2} + 30\sqrt[4]{3}} - \sqrt{2\sqrt{2108\sqrt{2} - 1494\sqrt{3} + 30\sqrt[4]{3}}}} \right],$$

and

$$(6.17) \quad V^3(e^{-\pi\sqrt{\frac{7}{3}}}) = \frac{-(7x^2 + 9) + 3\sqrt{9x^4 + 14x^2 + 9}}{16x^2},$$

where

$$x = \sqrt{\frac{207 + 78\sqrt{7} - 3\sqrt{9149 + 3458\sqrt{7}}}{2}} - \sqrt{\frac{205 + 78\sqrt{7} - 3\sqrt{9149 + 3458\sqrt{7}}}{2}}.$$

Proof of (6.14). From Ramanujan's first notebook [14, p.289], we have

$$(6.18) \quad G_{15} = 2^{\frac{1}{4}} \left(\frac{1 + \sqrt{5}}{2} \right)^{\frac{1}{3}}.$$

Using (6.18) in (6.10) with $n = \frac{5}{3}$, we find that

$$(6.19) \quad G_{\frac{5}{3}} = 2^{\frac{1}{4}} \left(\frac{-1 + \sqrt{5}}{2} \right)^{\frac{1}{3}}.$$

From Theorem 3.4(i) in [8], with replacing q by $-q$ and using the fact that $g_{4n} = 2^{1/4}g_nG_n$, we deduce that

$$(6.20) \quad g_n^2g_{9n}^2 = \frac{1}{2} \left[G_n^2G_{9n}^2 + \sqrt{G_n^4G_{9n}^4 - \frac{2}{G_n^2G_{9n}^2}} \right].$$

Using (6.18) and (6.19) with $n = \frac{5}{3}$, we deduce that

$$(6.21) \quad g_{15}g_{\frac{5}{3}} = \frac{\sqrt{3} + 1}{2}.$$

Using (6.21) in (6.6) with $n = \frac{5}{3}$, we find that

$$(6.22) \quad \frac{g_{15}^6}{g_{\frac{5}{3}}^6} = \frac{1}{8}(\sqrt{5} - 1)^2(9\sqrt{6} + 7\sqrt{10}).$$

From Theorem 2.5 in [13], we have

$$(6.23) \quad \frac{1}{b_{M,3}} + b_{M,3} = \frac{1}{3} \left(\frac{g_{3M}^6}{g_{\frac{M}{3}}^6} + \frac{g_{\frac{M}{3}}^6}{g_{3M}^6} \right).$$

Using (6.22) in (6.23) with $M = 5$, we deduce that

$$(6.24) \quad b_{5,3} = \frac{1}{4}(\sqrt{5} + \sqrt{3})(7\sqrt{2} - 3\sqrt{10}).$$

Using (6.24) in (3.16), we obtain the required result.

As the proofs of the identities (6.15)-(6.17) being similar to the proof of the identity (6.14). So we omit the details. \square

Remark. One can find several interesting other explicit values of $V(q)$ using the values of $a_{M,3}$, $a_{M,9}$, $b_{M,3}$ and $b_{M,9}$ in [10], [11], [12] and [13].

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References

- [1] C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, *On Ramanujan's cubic continued fraction and explicit evaluations of theta-function*, Indian J. Pure and Appl. Math., **35(9)**(2004), 1047-1062.
- [2] C. Adiga, K. R. Vasuki, and M. S. Mahadeva Naika, *Some new explicit evaluations of Ramanujan's cubic continued fraction*, The New Zealand J. Math., **31**(2002), 1-6.
- [3] N. D. Baruah, *Modular equations for Ramanujan's cubic continued fraction*, J. Math. Anal. and Appl., **268**(2002), 244-255.
- [4] N. D. Baruah and Nipen Saikia, *Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction*, J. Compu. and Appl. Math., **160**(2003), 37-51.
- [5] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.
- [6] B. C. Berndt, H. H. Chan and L. -C. Zhang, *Ramanujan's remarkable product of the theta-function*, Proc. Edinburgh Math. Soc., **40**(1997), 583-612.
- [7] H. H. Chan, *On Ramanujan's cubic continued fraction*, Acta Arith., **73**(1995), 343-355.
- [8] M. S. Mahadeva Naika, *P-Q eta-function identities and computation of Ramanujan-Webber class invariants*, J. Indian Math. Soc., **70(1-4)**, (2003), 121-134.
- [9] M. S. Mahadeva Naika, *Some theorems on Ramanujan's cubic continued fraction and related identities*, Tamsui Oxford J. Math. Sci., **24(3)**(2008), 243-256.

- [10] M. S. Mahadeva Naika and B. N. Dharmendra, *On some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-function*, The Ramanujan J., **15(3)**(2008), 349-366.
- [11] M. S. Mahadeva Naika, B. N. Dharmendra and K. Shivashankara, *On some new explicit evaluations of Ramanujan's remarkable product of theta-function*, South East Asian J. Math. and Math. Sci., **5(1)**(2006), 107-119.
- [12] M. S. Mahadeva Naika and M. C. Maheshkumar, *Explicit evaluations of Ramanujan's remarkable product of theta-function*, Adv. Stud. Contemp. Math., **13(2)**(2006), 235-254.
- [13] M. S. Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy, *On some remarkable product of theta-function*, Aust. J. Math. Anal. Appl., **5(1)**(2008), Art. 13, 1-15.
- [14] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
- [15] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.