symmetry

MDPI

Article

General k-Dimensional Solvable Systems of Difference Equations

Stevo Stević

Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia; sstevic@ptt.rs

Received: 6 December 2017; Accepted: 21 December 2017; Published: 28 December 2017

Abstract: The solvability of a k-dimensional system of difference equations of interest, which extends several recently studied ones, is investigated. A general sufficient condition for the solvability of the system is given, considerably extending some recent results in the literature.

Keywords: system of difference equations; system solvable in closed form; characteristic polynomial

MSC: 39A14; 39A45

1. Introduction

One of the oldest topics related to difference equations and systems of difference equations is their solvability. Many classical methods for solving some classes of equations and systems, including linear ones, can be found in the following known books: [1–7].

There has been some renewed recent interest in the topic, especially in the solvability of various classes of nonlinear difference equations and systems. One of the reasons for this is that our method for solving the following second-order nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + b x_{n-1} x_n}, \quad n \in \mathbb{N}_0,$$

from 2004, has attracted some attention. Namely, for the case when $x_n \neq 0$ for every $n \geq -1$, the equation can be transformed to a linear one by using a suitable change of variables. Some generalizations of the equation, which are studied by developing the method, can be found in [8–10] (see also [11] where a slight extension of the equation was studied in another way). A related solvable system of difference equations was treated in [12]. Since that time various modifications of the method have been often used (see [13,14] and the references therein for some related difference equations, as well as [15–17] and the references therein for some related systems of difference equations). It should be pointed out that the systems are usually symmetric or close-to-symmetric, whose study was popularized by Papaschinopoulos and Schinas ([18–24]). In some of their papers, such as [19–21,23], they study the solvability and the long-term behaviour of solutions to the equations and systems by finding their invariants. For some applications of solvability and related matters see [6,25–29]. Some product-type difference equations and systems have been essentially solved by using the linear ones, but in a more complex way ([30–32]). Several methods, including the method of transformation and methods connected to product-type equations and systems, can be found in the representative paper [33].

Recall that a sequence $(x_n^{(1)}, \ldots, x_n^{(k)})_{n \geq l}$, $l \in \mathbb{Z}$, is a solution to a k-dimensional system of difference equations if it satisfies the system for every $n \geq l$. If every solution to a system can be obtained from a finite family of formulas, then such a system is *solvable in closed form*.

It is not difficult to see that the following nonlinear system of difference equations

$$x_{n+1} = \frac{a_1}{x_n} + \frac{a_2}{y_n}, \quad y_{n+1} = \frac{b_1}{x_n} + \frac{b_2}{y_n}, \quad n \in \mathbb{N}_0, \tag{1}$$

where the parameters a_1, a_2, b_1, b_2, and the initial values x_0, y_0 are complex numbers, is solvable in closed form (see, for example, [34]).

It is easily seen that system (1) is related to the equation

$$z_{n+1} = \frac{a_2 z_n + a_1}{b_2 z_n + b_1}, \tag{2}$$

(the bilinear one) with the initial condition

$$z_0 = \frac{x_0}{y_0},$$

(note that for every well-defined solution to system (1), z_0 is defined, since in this case it must be $x_0 \neq 0 \neq y_0$). It is well-known that there are several methods for solving Equation (2). For example, the equation can be solved by transforming it into a two-dimensional linear system (see [4]), which can be solved by several methods (for solving more general linear systems, see, for example, [2,5]). The original Russian version of the book [4] from 1937, which, at the moment, can be freely found on the internet, gives the solution. In [6,7,35], a solution is presented to the equation by transforming it, by using a suitable changes of variables, to a homogeneous linear second-order difference equation with constant coefficients. The idea was later used in our paper [36] where, among other things, a representation of the general solution to the equation was given in terms of the, so called, generalized Fibonacci sequence. Equation (2) is also connected to finding the nth power of the following matrix

$$\widehat{A} = \begin{bmatrix} a_2 & a_1 \\ b_2 & b_1 \end{bmatrix}, \tag{3}$$

associated to the equation, which can also be used to get the general solution to Equation (2).

All these facts, show the importance of the system of difference Equation (1) and suggest that there are many interesting things behind the system which could be studied in detail. Having noticed these facts a natural question arises of finding some related three-dimensional systems of difference equations which are solvable in closed form.

Motivated by the problem, we have investigated some natural extensions of system (1) and, recently in [37], have shown the solvability of the following three-dimensional system

$$\begin{aligned} x_{n+1} &= \frac{a_1}{x_n y_n} + \frac{b_1}{y_n z_n} + \frac{c_1}{z_n x_n} \\ y_{n+1} &= \frac{a_2}{x_n y_n} + \frac{b_2}{y_n z_n} + \frac{c_2}{z_n x_n} \\ z_{n+1} &= \frac{a_3}{x_n y_n} + \frac{b_3}{y_n z_n} + \frac{c_3}{z_n x_n}, \end{aligned} \tag{4}$$

$n \in \mathbb{N}_0$, where the parameters a_i, b_i, c_i, $i = \overline{1,3}$, and initial values x_0, y_0, z_0, are complex numbers.

We want to point out that in [37], we showed that system (4) is *practically* solvable, in the sense that the set of closed-form formulas for its solutions can be explicitly given for all possible values of parameters a_i, b_i, c_i, $i = \overline{1,3}$, and initial values x_0, y_0, z_0. To clarify the notion, say that, for example, a homogeneous linear difference equation with constant coefficients of order greater or equal to five is an example of a *theoretically* solvable difference equation, which is not always practically solvable one, because the characteristic polynomial associated to the equation need not be solvable by radicals in this case ([38]).

Pushing further this line of investigations, quite recently, in [39], we have shown that the following four-dimensional system

$$\begin{aligned} x_{n+1} &= \frac{a_1}{x_n y_n z_n} + \frac{b_1}{y_n z_n u_n} + \frac{c_1}{z_n u_n x_n} + \frac{d_1}{u_n x_n y_n} \\ y_{n+1} &= \frac{a_2}{x_n y_n z_n} + \frac{b_2}{y_n z_n u_n} + \frac{c_2}{z_n u_n x_n} + \frac{d_2}{u_n x_n y_n} \\ z_{n+1} &= \frac{a_3}{x_n y_n z_n} + \frac{b_3}{y_n z_n u_n} + \frac{c_3}{z_n u_n x_n} + \frac{d_3}{u_n x_n y_n} \\ u_{n+1} &= \frac{a_4}{x_n y_n z_n} + \frac{b_4}{y_n z_n u_n} + \frac{c_4}{z_n u_n x_n} + \frac{d_4}{u_n x_n y_n}, \end{aligned} \tag{5}$$

$n \in \mathbb{N}_0$, where the parameters a_i, b_i, c_i, d_i, $i = \overline{1,4}$, and initial values x_0, y_0, z_0, u_0, are complex numbers, is also practically solvable, by giving a detailed description of how closed-form formulas, in all possible cases, can be found.

The line of investigations in [34,37,39] has motivated us to try to find what it is that decides the solvability of the systems studied therein, that is, of systems of difference Equations (1), (4) and (5). The fact that in the study of the systems in [37,39] appeared matrices consisting of the parameters which appeared in the systems, as well as the fact that, as we have already mentioned, the general solution to system (1) can be solved by using the matrix (3), have strikingly suggested that matrices have some important role in the solvability of these systems. This also suggested to us that systems (4) and (5), should be written in the following, somewhat nicer, forms

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \frac{1}{x_n y_n z_n} \begin{bmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ u_{n+1} \end{bmatrix} = \frac{1}{x_n y_n z_n u_n} \begin{bmatrix} b_1 & c_1 & d_1 & a_1 \\ b_2 & c_2 & d_2 & a_2 \\ b_3 & c_3 & d_3 & a_3 \\ b_4 & c_4 & d_4 & a_4 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ z_n \\ u_n \end{bmatrix}, \tag{7}$$

respectively.

Our aim is to unify and extend the results in [34,37,39], by explaining what is behind the solvability of systems (1), (4) and (5). To do this, motivated by the forms of systems (4) and (5) given in (6) and (7), here we consider the following general system of difference equations

$$\begin{bmatrix} y_{n+1}^{(1)} \\ y_{n+1}^{(2)} \\ \vdots \\ y_{n+1}^{(k)} \end{bmatrix} = \frac{1}{f(y_n^{(1)}, \dots, y_n^{(k)})} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{bmatrix} \cdot \begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(k)} \end{bmatrix}, \tag{8}$$

for $n \in \mathbb{N}_0$, where f is a complex-valued function on $\mathbb{C}^k$, such that

$$f(0, 0, \dots, 0) = 0 \tag{9}$$

and

$$f(w_1, w_2, \ldots, w_k) \neq 0, \tag{10}$$

when $w_j \neq 0$, $j = \overline{1,k}$.

By C_k^n, $0 \leq k \leq n$, we denote the *Binomial coefficients*. They can be defined algebraically as the coefficients of the polynomial $P_n(x) = (1+x)^n$, $n \in \mathbb{N}_0$, that is, we have

$$P_n(x) = C_0^n + C_1^n x + \cdots + C_n^n x^n.$$

Recall that

$$C_k^n = \frac{n!}{k!(n-k)!}, \tag{11}$$

where

$$m! = \prod_{j=1}^{m} j \quad \text{and} \quad 0! := 1.$$

By comparing the coefficients in the following identity $(1+x)^{n-1}(1+x) = (1+x)^n$, the following recurrence relation

$$C_k^n = C_k^{n-1} + C_{k-1}^{n-1}, \tag{12}$$

is obtained for $k, n \in \mathbb{N}$, such that $1 \leq k < n$.

For more information regarding the coefficients consult the following classics: [4,7,40–43]. It is interesting that the recurrence relation (12) is also a solvable difference equation, but with two independent variables which are usually called partial difference equations (for some results on solvability of the equations see [3,5,44]), and that there is a closed form formula for the general solution to the equation on its natural domain, the, so called, *combinatorial domain* (see [45]). Namely, in [45] was devised a method, which is called the *method of half-lines* for which it turned out that can be used for solving several other important difference equations with two independent variables (see, [46] and the related references therein).

Throughout the paper we will use the standard convention $\sum_{j=k}^{l} c_i = 0$, when $k < l$, and $\prod_{j=k}^{k-1} c_i = 1$.

2. Analysis of Solvability of System (8) and the Main Result

For every complex square matrix A of order k there is a nonsingular matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \ldots & t_{1k} \\ t_{21} & t_{22} & \ldots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \ldots & t_{kk} \end{bmatrix},$$

a transition matrix, such that

$$T^{-1}AT = J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_l \end{bmatrix}, \tag{13}$$

where J_i, $1 \le i \le l$, are matrices of the following form

$$J_i = \begin{bmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{bmatrix}, \tag{14}$$

the, so called, Jordan blocks. If the submatrices J_i, $i = \overline{1,l}$, are of orders ρ_i, $i = \overline{1,l}$, respectively, then, of course, it must be $\sum_{i=1}^{l} \rho_i = k$. The matrix J in (13) is called the Jordan normal form of matrix A.

Remark 1. *Recall that for a given matrix A its normal form is not unique. Namely, the Jordan blocks of a Jordan matrix corresponding to matrix A, can be permutated and the obtained block diagonal matrix is also a Jordan matrix of A (see, for example, [47,48]).*

Using the change of variables

$$\begin{bmatrix} y_n^{(1)} \\ y_n^{(2)} \\ \vdots \\ y_n^{(k)} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1k} \\ t_{21} & t_{22} & \dots & t_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ t_{k1} & t_{k2} & \dots & t_{kk} \end{bmatrix} \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} \tag{15}$$

in (8), and if in the obtained system we employ equality (13), it follows that

$$\begin{bmatrix} x_{n+1}^{(1)} \\ x_{n+1}^{(2)} \\ \vdots \\ x_{n+1}^{(k)} \end{bmatrix} = \frac{1}{f\left(\sum_{j=1}^{k} t_{1j} x_n^{(j)}, \dots, \sum_{j=1}^{k} t_{kj} x_n^{(j)}\right)} \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_l \end{bmatrix} \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}, \tag{16}$$

for $n \in \mathbb{N}_0$.

Let

$$k_i := \sum_{j=1}^{i} \rho_j, \quad i = \overline{1,l}, \tag{17}$$

and $k_0 = 0$.

The system (16) can be written as a set of l systems each of which corresponds to a Jordan block of matrix J, that is, for the Jordan block J_i, where $i \in \{1, \dots, l\}$ is fixed, we have

$$\begin{bmatrix} x_{n+1}^{(k_{i-1}+1)} \\ \vdots \\ x_{n+1}^{(k_i-1)} \\ x_{n+1}^{(k_i)} \end{bmatrix} = \frac{1}{f\left(\sum_{j=1}^{k} t_{1j} x_n^{(j)}, \dots, \sum_{j=1}^{k} t_{kj} x_n^{(j)}\right)} \begin{bmatrix} \lambda_i & 1 & \dots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \lambda_i \end{bmatrix} \begin{bmatrix} x_n^{(k_{i-1}+1)} \\ \vdots \\ x_n^{(k_i-1)} \\ x_n^{(k_i)} \end{bmatrix}, \tag{18}$$

for every $n \in \mathbb{N}_0$.

First, we assume that

$$\lambda_j \ne 0, \quad j = \overline{1,l}, \tag{19}$$

that is, that none of the characteristic values of the matrix A is equal to zero.

Then for every solution to system (16) which has no zero component, we have

$$\begin{aligned}
&\frac{x_{n+1}^{(1)}}{\lambda_1 x_n^{(1)}+x_n^{(2)}}=\frac{x_{n+1}^{(2)}}{\lambda_1 x_n^{(2)}+x_n^{(3)}}=\cdots=\frac{x_{n+1}^{(k_1-1)}}{\lambda_1 x_n^{(k_1-1)}+x_n^{(k_1)}}=\frac{x_{n+1}^{(k_1)}}{\lambda_1 x_n^{(k_1)}}\\
&=\frac{x_{n+1}^{(k_1+1)}}{\lambda_2 x_n^{(k_1+1)}+x_n^{(k_1+2)}}=\frac{x_{n+1}^{(k_1+2)}}{\lambda_2 x_n^{(k_1+2)}+x_n^{(k_1+3)}}=\cdots=\frac{x_{n+1}^{(k_2-1)}}{\lambda_2 x_n^{(k_2-1)}+x_n^{(k_2)}}=\frac{x_{n+1}^{(k_2)}}{\lambda_2 x_n^{(k_2)}}\\
&\qquad\vdots\\
&=\frac{x_{n+1}^{(k_{i-1}+1)}}{\lambda_i x_n^{(k_{i-1}+1)}+x_n^{(k_{i-1}+2)}}=\frac{x_{n+1}^{(k_{i-1}+2)}}{\lambda_i x_n^{(k_{i-1}+2)}+x_n^{(k_{i-1}+3)}}=\cdots=\frac{x_{n+1}^{(k_i-1)}}{\lambda_i x_n^{(k_i-1)}+x_n^{(k_i)}}=\frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}}\\
&\qquad\vdots\\
&=\frac{x_{n+1}^{(k_{l-2}+1)}}{\lambda_{l-1} x_n^{(k_{l-2}+1)}+x_n^{(k_{l-2}+2)}}=\frac{x_{n+1}^{(k_{l-2}+2)}}{\lambda_{l-1} x_n^{(k_{l-2}+2)}+x_n^{(k_{l-2}+3)}}=\cdots=\frac{x_{n+1}^{(k_{l-1}-1)}}{\lambda_{l-1} x_n^{(k_{l-1}-1)}+x_n^{(k_{l-1})}}=\frac{x_{n+1}^{(k_{l-1})}}{\lambda_{l-1} x_n^{(k_{l-1})}}\\
&=\frac{x_{n+1}^{(k_{l-1}+1)}}{\lambda_l x_n^{(k_{l-1}+1)}+x_n^{(k_{l-1}+2)}}=\frac{x_{n+1}^{(k_{l-1}+2)}}{\lambda_l x_n^{(k_{l-1}+2)}+x_n^{(k_{l-1}+3)}}=\cdots=\frac{x_{n+1}^{(k_l-1)}}{\lambda_l x_n^{(k_l-1)}+x_n^{(k_l)}}=\frac{x_{n+1}^{(k_l)}}{\lambda_l x_n^{(k_l)}}\\
&=\frac{1}{f\left(\sum_{j=1}^{k} t_{1j}x_n^{(j)},\ldots,\sum_{j=1}^{k} t_{kj}x_n^{(j)}\right)},
\end{aligned}\tag{20}$$

for $n \in \mathbb{N}_0$.

Let

$$a_{s,t} := \frac{x_0^{(s)}}{x_0^{(t)}},\tag{21}$$

for $s,t \in \{1,2,\ldots,k\}$.

Note that

$$a_{s,s}=1, \quad s=\overline{1,k}.$$

Since

$$\frac{x_{n+1}^{(k_i-1)}}{\lambda_i x_n^{(k_i-1)}+x_n^{(k_i)}}=\frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}},$$

$i=\overline{1,l}$, we have

$$\frac{x_{n+1}^{(k_i-1)}}{x_{n+1}^{(k_i)}}=\frac{x_n^{(k_i-1)}}{x_n^{(k_i)}}+\frac{1}{\lambda_i},\tag{22}$$

for $i=\overline{1,l}$.

From (22), it follows that

$$\frac{x_n^{(k_i-1)}}{x_n^{(k_i)}}=a_{k_i-1,k_i}+\frac{n}{\lambda_i},$$

that is,

$$x_n^{(k_i-1)}=\left(a_{k_i-1,k_i}+\frac{n}{\lambda_i}\right)x_n^{(k_i)},\tag{23}$$

for $i=\overline{1,l}$.

Further, we have

$$\frac{x_{n+1}^{(k_i-2)}}{\lambda_i x_n^{(k_i-2)}+x_n^{(k_i-1)}}=\frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}},\tag{24}$$

for $i = \overline{1,l}$.

Employing (23) in (24), we obtain

$$\frac{x_{n+1}^{(k_i-2)}}{\lambda_i x_n^{(k_i-2)} + (a_{k_i-1,k_i} + \frac{n}{\lambda_i}) x_n^{(k_i)}} = \frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}},$$

$i = \overline{1,l}$, that is,

$$\frac{x_{n+1}^{(k_i-2)}}{x_{n+1}^{(k_i)}} = \frac{x_n^{(k_i-2)}}{x_n^{(k_i)}} + \frac{a_{k_i-1,k_i}}{\lambda_i} + \frac{n}{\lambda_i^2}, \tag{25}$$

$i = \overline{1,l}$.

From (25), we obtain

$$\frac{x_n^{(k_i-2)}}{x_n^{(k_i)}} = a_{k_i-2,k_i} + a_{k_i-1,k_i} \frac{n}{\lambda_i} + \frac{(n-1)n}{2\lambda_i^2}, \tag{26}$$

for $i = \overline{1,l}$.

Motivated by (23) and (26), we assume that

$$\frac{x_n^{(k_i-m)}}{x_n^{(k_i)}} = \sum_{j=0}^{m} a_{k_i-m+j,k_i} \frac{C_j^n}{\lambda_i^j}, \tag{27}$$

for every $m \in \{1, \ldots, k_i - k_{i-1} - 2\}$ and each $i \in \{1, \ldots, l\}$.

By using (27) in the following equality

$$\frac{x_{n+1}^{(k_i-m-1)}}{\lambda_i x_n^{(k_i-m-1)} + x_n^{(k_i-m)}} = \frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}},$$

$i = \overline{1,l}$, we get

$$\frac{x_{n+1}^{(k_i-m-1)}}{\lambda_i x_n^{(k_i-m-1)} + x_n^{(k_i)} \sum_{j=0}^{m} a_{k_i-m+j,k_i} \frac{C_j^n}{\lambda_i^j}} = \frac{x_{n+1}^{(k_i)}}{\lambda_i x_n^{(k_i)}},$$

and consequently

$$\frac{x_{n+1}^{(k_i-m-1)}}{x_{n+1}^{(k_i)}} = \frac{x_n^{(k_i-m-1)}}{x_n^{(k_i)}} + \sum_{j=0}^{m} a_{k_i-m+j,k_i} C_j^n \frac{1}{\lambda_i^{j+1}}, \tag{28}$$

for $i = \overline{1,l}$.

From (28), we obtain

$$\begin{aligned} \frac{x_n^{(k_i-m-1)}}{x_n^{(k_i)}} &= a_{k_i-m-1,k_i} + \sum_{s=0}^{n-1} \sum_{j=0}^{m} a_{k_i-m+j,k_i} C_j^s \frac{1}{\lambda_i^{j+1}} \\ &= a_{k_i-m-1,k_i} + \sum_{j=0}^{m} a_{k_i-m+j,k_i} \frac{1}{\lambda_i^{j+1}} \sum_{s=0}^{n-1} C_j^s. \end{aligned} \tag{29}$$

Now by using (12), we have

$$\sum_{s=0}^{n-1} C_j^s = \sum_{s=0}^{n-1} \left(C_{j+1}^{s+1} - C_{j+1}^s\right) = C_{j+1}^n. \tag{30}$$

Employing (30) in (29), we obtain

$$\begin{aligned}\frac{x_n^{(k_i-m-1)}}{x_n^{(k_i)}} &= a_{k_i-m-1,k_i} + \sum_{j=0}^{m} a_{k_i-m+j,k_i} \frac{C_{j+1}^n}{\lambda_i^{j+1}} \\ &= a_{k_i-m-1,k_i} + \sum_{j=1}^{m+1} a_{k_i-m-1+j,k_i} \frac{C_j^n}{\lambda_i^j} \\ &= \sum_{j=0}^{m+1} a_{k_i-m-1+j,k_i} \frac{C_j^n}{\lambda_i^j}. \end{aligned} \tag{31}$$

From (31) and by induction it follows that (27) holds for every $i \in \{1, \ldots, l\}$ and $m \in \{1, \ldots, k_i - k_{i-1} - 1\}$, which can also be written as follows

$$x_n^{(k_i-m)} = \alpha_n^{(k_i-m)} x_n^{(k_i)}, \tag{32}$$

where

$$\alpha_n^{(k_i-m)} := \sum_{j=0}^{m} a_{k_i-m+j,k_i} \frac{C_j^n}{\lambda_i^j}, \tag{33}$$

for $i \in \{1, \ldots, l\}$ and $m \in \{1, \ldots, k_i - k_{i-1} - 1\}$.

On the other hand, from (20) we also have

$$\frac{x_{n+1}^{(k_i)}}{x_{n+1}^{(k_l)}} = \frac{\lambda_i x_n^{(k_i)}}{\lambda_l x_n^{(k_l)}}, \tag{34}$$

for $1 \le i \le l-1$, from which it follows that

$$x_n^{(k_i)} = \left(\frac{\lambda_i}{\lambda_l}\right)^n a_{k_i,k_l} x_n^{(k_l)}, \tag{35}$$

for $1 \le i \le l-1$.

From (32) and (35) it follows that

$$x_n^{(k_i-m)} = \alpha_n^{(k_i-m)} \left(\frac{\lambda_i}{\lambda_l}\right)^n a_{k_i,k_l} x_n^{(k_l)}, \tag{36}$$

for $1 \le i \le l-1$ and $m \in \{1, \ldots, k_i - k_{i-1} - 1\}$.

From the above analysis we have the following result.

Lemma 1. *Consider system (16), where J_i, $i = \overline{1,l}$, are Jordan blocks of orders ρ_i, $i = \overline{1,l}$, whose diagonal elements are nonzero numbers λ_i, $i = \overline{1,l}$. Let*

$$\beta_n^{(k_i-m)} = \begin{cases} \left(\frac{\lambda_i}{\lambda_l}\right)^n a_{k_i,k_l}, & 1 \le i \le l-1 \text{ and } m = 0; \\ \alpha_n^{(k_i-m)} \left(\frac{\lambda_i}{\lambda_l}\right)^n a_{k_i,k_l}, & 1 \le i \le l-1 \text{ and } 1 \le m \le k_i - k_{i-1} - 1; \\ \alpha_n^{(k_l-m)}, & 1 \le m \le k_l - k_{l-1} - 1, \\ 1, & i = l \text{ and } m = 0, \end{cases} \tag{37}$$

where k_i, $i = \overline{1,l}$ are defined in (17), $\alpha_n^{(k_i-m)}$, $1 \le i \le l$, $1 \le m \le k_i - k_{i-1} - 1$ are defined in (33), whereas $a_{s,t}$, $s,t \in \{1,2,\dots,k\}$, are defined in (21).

Then, for any solution $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ to the system such that $x_n^{(j)} \ne 0$, for every $n \in \mathbb{N}_0$, $j = \overline{1,k}$, the following equalities hold

$$x_n^{(j)} = \beta_n^{(j)} x_n^{(k_l)}, \tag{38}$$

for $j = \overline{1,k_l}$.

Employing (38) in the following consequence of (20)

$$x_{n+1}^{(k_l)} = \frac{\lambda_l x_n^{(k_l)}}{f\big(\sum_{j=1}^k t_{1j} x_n^{(j)}, \dots, \sum_{j=1}^k t_{kj} x_n^{(j)}\big)}, \quad n \in \mathbb{N}_0,$$

is obtained

$$x_{n+1}^{(k_l)} = \frac{\lambda_l x_n^{(k_l)}}{f\big(\big(\sum_{j=1}^k t_{1j}\beta_n^{(j)}\big) x_n^{(k_l)}, \dots, \big(\sum_{j=1}^k t_{kj}\beta_n^{(j)}\big) x_n^{(k_l)}\big)}, \quad n \in \mathbb{N}_0,$$

which due to the fact that $k_l = k$, is nothing but

$$x_{n+1}^{(k)} = \frac{\lambda_l x_n^{(k)}}{f\big(\big(\sum_{j=1}^k t_{1j}\beta_n^{(j)}\big) x_n^{(k)}, \dots, \big(\sum_{j=1}^k t_{kj}\beta_n^{(j)}\big) x_n^{(k)}\big)}, \quad n \in \mathbb{N}_0. \tag{39}$$

Remark 2. *Bearing in mind Remark 1, we see that the transition matrix could be chosen such that any of the Jordan blocks could go to the last position (lth one in our notations) in the corresponding Jordan matrix J, from which it follows that in formula (39) instead of λ_l can be any of the other characteristic values of matrix A. The form of formula (39) will be the same, but the values of sequences $(\beta_n^{(j)})_{n\in\mathbb{N}_0}$, $j = \overline{1,k}$, can be different.*

The transformation in (15), as well as its inverse one, maps the sets of k-dimensional Lebesgue measure zero to sets of measure zero, since T is a nonsingular matrix. Note also that the set

$$S := \big\{(w_1, w_2, \dots, w_k) \in \mathbb{C}^k : w_j = 0, \text{ for some } j \in \{1,2,\dots,k\}\big\},$$

has k-dimensional Lebesgue measure zero.

Using these two facts, it follows that the sets

$$S_n^1 := \big\{(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)}) \in \mathbb{C}^k : x_n^{(j)} = 0, \text{ for some } j \in \{1,2,\dots,k\}\big\},$$

and

$$S_n^2 := \big\{(y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k)}) \in \mathbb{C}^k : y_n^{(j)} = 0, \text{ for some } j \in \{1,2,\dots,k\}\big\}, \tag{40}$$

$$= \big\{(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)}) \in \mathbb{C}^k : \sum_{l=1}^k t_{jl} x_n^{(l)} = 0, \text{ for some } j \in \{1,2,\dots,k\}\big\}, \tag{41}$$

have measure zero for every $n \in \mathbb{N}_0$, and consequently the unions $\cup_{n\in\mathbb{N}_0} S_n^l$, $l = 1,2$.

From this and the condition in (10), it follows that all the solutions to the systems (8) and (16) are well-defined outside a set of k-dimensional Lebesgue measure zero.

Now we formulate and prove our main result.

Theorem 1. *Consider system (8), where f is a complex-valued function on $\mathbb{C}^k$ satisfying the conditions (9) and (10), $A = [a_{ij}]$ is a complex square matrix of order k, $T = [t_{ij}]$ is a nonsingular transition matrix which transforms the matrix A to its Jordan normal form J, whose blocks J_i, $i = \overline{1,l}$, correspond to the characteristic values λ_i, $i = \overline{1,l}$, $\alpha_n^{(k_i-m)}$, $1 \le i \le l$, $1 \le m \le k_i - k_{i-1} - 1$ are defined by (33), and $\beta_n^{(k_i-m)}$, $1 \le i \le l$, $0 \le m \le k_i - k_{i-1} - 1$ are defined by (37), where $a_{i,j}$, $1 \le i \le j \le k$ are some nonzero arbitrary constants.*

If the difference equation

$$z_{n+1} = \frac{\lambda_l z_n}{f\big(\big(\sum_{j=1}^k t_{1j}\beta_n^{(j)}\big)z_n, \dots, \big(\sum_{j=1}^k t_{kj}\beta_n^{(j)}\big)z_n\big)}, \quad n \in \mathbb{N}_0, \tag{42}$$

is solvable in closed form, for some $\lambda_l \neq 0$, then system (8) is also solvable in closed form.

Proof. First assume that (19) holds, that is, $\lambda_j \neq 0$ for $j = \overline{1,k}$. By using the change of variables (15), system (8) is transformed into system (16). From the analysis preceding the formulation of the theorem we see that for every solution to system (16) which has no zero component (so, for almost all initial values) the sequence $x_n^{(k)}$ is a solution to Equation (42). Since the equation is solvable, a closed-form formula for sequence $x_n^{(k)}$ can be found, from which along with Lemma 1 it follows that some closed-form formulas for sequences $x_n^{(j)}$, $1 \le j \le k-1$ can be found. By using the formulas in (15) we get some closed-form formulas for solutions to (8), from which the theorem follows in this case.

Now assume that zero is a characteristic value of matrix A of order s. Then, due to the comment in Remark 1 we may assume that

$$\lambda_{k-s+1} = \cdots = \lambda_k = 0, \tag{43}$$

which implies that

$$\lambda_j \neq 0, \quad j = \overline{1, k-s}.$$

In this case, one or several Jordan blocks correspond to the zero characteristic value. Then from (18) with $\lambda_i = 0$, we obtain

$$\begin{bmatrix} x_{n+1}^{(k_{i-1}+1)} \\ \vdots \\ x_{n+1}^{(k_i-1)} \\ x_{n+1}^{(k_i)} \end{bmatrix} = \frac{1}{f\big(\sum_{j=1}^k t_{1j}x_n^{(j)}, \dots, \sum_{j=1}^k t_{kj}x_n^{(j)}\big)} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_n^{(k_{i-1}+1)} \\ \vdots \\ x_n^{(k_i-1)} \\ x_n^{(k_i)} \end{bmatrix},$$

for every $n \in \mathbb{N}_0$, from which, for every (well-defined) solution to (16), it follows that

$$\begin{aligned} x_{n+1}^{(k_{i-1}+1)} &= \frac{x_n^{(k_{i-1}+2)}}{f\big(\sum_{j=1}^k t_{1j}x_n^{(j)}, \dots, \sum_{j=1}^k t_{kj}x_n^{(j)}\big)} \\ x_{n+1}^{(k_{i-1}+2)} &= \frac{x_n^{(k_{i-1}+3)}}{f\big(\sum_{j=1}^k t_{1j}x_n^{(j)}, \dots, \sum_{j=1}^k t_{kj}x_n^{(j)}\big)} \\ &\vdots \\ x_{n+1}^{(k_i-1)} &= \frac{x_n^{(k_i)}}{f\big(\sum_{j=1}^k t_{1j}x_n^{(j)}, \dots, \sum_{j=1}^k t_{kj}x_n^{(j)}\big)} \\ x_{n+1}^{(k_i)} &= 0, \end{aligned} \tag{44}$$

for $n \in \mathbb{N}_0$.

From (44) it is easily obtained that

$$x^{(k_{i-1}+1)}_{n+k_i-k_{i-1}} = x^{(k_{i-1}+2)}_{n+k_i-k_{i-1}-1} = \cdots = x^{(k_i-1)}_{n+2} = x^{(k_i)}_{n+1}, \quad n \in \mathbb{N}_0. \tag{45}$$

Since (45) holds for any Jordan block corresponding to the characteristic value $\lambda = 0$, we have that

$$x_n^{(k-s+1)} = \cdots = x_n^{(k)} = 0, \tag{46}$$

for large enough n.

Hence, if all the characteristic zeros of matrix A are equal to zero, then we see that all the solutions will be eventually equal to zero.

If there is a nonzero characteristic value, then (20) holds when l is replaced by $l-1$, and condition (42) assumes that it holds when λ_l is replaced by λ_{l-1}, from which, as in the first case, it follows that some closed-form formulas can be found for $(x_n^{(j)})_{n\in\mathbb{N}_0}$, $j = \overline{1, k_{l-1}}$, from which along with (46) and (15) closed-form formulas for system (8) are found. □

Example 1. The corresponding k-dimensional extension of systems (1), (4) and (5) is the following

$$\begin{aligned} x_{n+1}^{(1)} &= \frac{a_{11}}{\prod_{j=1,j\neq 1}^k x_n^{(j)}} + \frac{a_{12}}{\prod_{j=1,j\neq 2}^k x_n^{(j)}} + \cdots + \frac{a_{1k}}{\prod_{j=1,j\neq k}^k x_n^{(j)}} \\ &\vdots \\ x_{n+1}^{(i)} &= \frac{a_{i1}}{\prod_{j=1,j\neq 1}^k x_n^{(j)}} + \frac{a_{i2}}{\prod_{j=1,j\neq 2}^k x_n^{(j)}} + \cdots + \frac{a_{ik}}{\prod_{j=1,j\neq k}^k x_n^{(j)}} \\ &\vdots \\ x_{n+1}^{(k)} &= \frac{a_{k1}}{\prod_{j=1,j\neq 1}^k x_n^{(j)}} + \frac{a_{k2}}{\prod_{j=1,j\neq 2}^k x_n^{(j)}} + \cdots + \frac{a_{kk}}{\prod_{j=1,j\neq k}^k x_n^{(j)}} \end{aligned} \tag{47}$$

for $n \in \mathbb{N}_0$.

Note that system (47) can be written in the form

$$\begin{aligned} x_{n+1}^{(1)} &= \frac{a_{11}x_n^{(1)} + a_{12}x_n^{(2)} + \cdots + a_{1k}x_n^{(k)}}{\prod_{j=1}^k x_n^{(j)}} \\ &\vdots \\ x_{n+1}^{(i)} &= \frac{a_{i1}x_n^{(1)} + a_{i2}x_n^{(2)} + \cdots + a_{ik}x_n^{(k)}}{\prod_{j=1}^k x_n^{(j)}} \\ &\vdots \\ x_{n+1}^{(k)} &= \frac{a_{k1}x_n^{(1)} + a_{k2}x_n^{(2)} + \cdots + a_{kk}x_n^{(k)}}{\prod_{j=1}^k x_n^{(j)}} \end{aligned} \tag{48}$$

for $n \in \mathbb{N}_0$.

Now note that from (48), it follows that

$$\begin{aligned}\frac{x_{n+1}^{(i)}}{a_{i1}x_n^{(1)}+a_{i2}x_n^{(2)}+\cdots+a_{ik}x_n^{(k)}} &= \frac{x_{n+1}^{(k)}}{a_{k1}x_n^{(1)}+a_{k2}x_n^{(2)}+\cdots+a_{kk}x_n^{(k)}}\\ &= \frac{1}{\prod_{j=1}^{k}x_n^{(j)}}\end{aligned}$$

for $i=\overline{1,k-1}$ and $n\in\mathbb{N}_0$.

Let

$$\widetilde{A}=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k}\\ a_{21} & a_{22} & \dots & a_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k1} & a_{k2} & \dots & a_{kk}\end{bmatrix},$$

where a_{ij} are the coefficients of system (48).

By using the change of variables

$$\begin{bmatrix} x_n^{(1)}\\ x_n^{(2)}\\ \vdots\\ x_n^{(k)}\end{bmatrix}=\begin{bmatrix} t_{11} & t_{12} & \dots & t_{1k}\\ t_{21} & t_{22} & \dots & t_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ t_{k1} & t_{k2} & \dots & t_{kk}\end{bmatrix}\cdot\begin{bmatrix} z_n^{(1)}\\ z_n^{(2)}\\ \vdots\\ z_n^{(k)}\end{bmatrix},$$

where $[t_{ij}]$ is a transition matrix for the matrix $\widetilde{A}$ and employing the above presented procedure we get that for solvability of system (47), it is enough to prove the solvability of the system

$$z_{n+1}=\frac{\lambda_l z_n^{1-k}}{\prod_{i=1}^{k}\left(\sum_{j=1}^{k}t_{ij}\beta_n^{(j)}\right)},\quad n\in\mathbb{N}_0,\tag{49}$$

where λ_l is a nonzero characteristic value of the matrix A, while $(\beta_n^{(j)})_{n\in\mathbb{N}_0}$ are the sequences defined in (37), for the case when all the characteristic zeros of A are different from zero. For the case when one of the characteristic values is equal to zero but is not of order k, then the situation is similar and λ_l is a nonzero characteristic value of the matrix A, but instead of k will be another integer number. When all the characteristic zeros of the matrix A are equal to zero, then from the proof of Theorem 1 we see that all the solutions will be eventually equal to zero.

Equation (49) is a special case of the following one

$$z_n=b_n z_{n-1}^{a_n},\quad n\in\mathbb{N}.\tag{50}$$

Various special cases of Equation (50) have appeared recently during investigation of solvability of some product-type systems (see, for example, Theorem 2.1 in [31], Theorem 1 and Theorem 2 in [32]).

Now we will generalize these results from [31,32], by showing the solvability of Equation (50) under some more general conditions, which includes Equation (49).

Lemma 2. *Assume that $(a_n)_{n\in\mathbb{N}}\subset\mathbb{Z}$, $(b_n)_{n\in\mathbb{N}}\subset\mathbb{C}$ and $z_0\in\mathbb{C}$. Then for every solution to Equation (50), the following equality holds*

$$z_n=\left(\prod_{j=k}^{n}b_j^{\prod_{i=j+1}^{n}a_i}\right)z_{k-1}^{\prod_{i=k}^{n}a_i},\tag{51}$$

for every $k,n\in\mathbb{N}$ such that $k\le n$.

Proof. If $n = 1$, then $k = 1$ and (51) is reduced to (50) for $n = 1$. Assume (51) holds for an $n_0 \in \mathbb{N}$ and $1 \le k \le n_0$. Then from (50) with $n = n_0 + 1$ and the hypothesis, we have

$$\begin{aligned} z_{n_0+1} =& b_{n_0+1} z_{n_0}^{a_{n_0+1}} = b_{n_0+1}\left(\left(\prod_{j=k}^{n_0} b_j^{\prod_{i=j+1}^{n_0} a_i}\right) z_{k-1}^{\prod_{i=k}^{n_0} a_i}\right)^{a_{n_0+1}} \\ =& \left(\prod_{j=k}^{n_0+1} b_j^{\prod_{i=j+1}^{n_0+1} a_i}\right) z_{k-1}^{\prod_{i=k}^{n_0+1} a_i}, \end{aligned} \tag{52}$$

for $k \le n_0$. If $k = n_0 + 1$, then (51) obviously holds. The inductive argument proves the lemma. □

If in (51) we chose $k = 1$, we have the following corollary.

Corollary 1. *Assume $(a_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$, $(b_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then the general solution to Equation (50) is*

$$z_n = \left(\prod_{j=1}^{n} b_j^{\prod_{i=j+1}^{n} a_i}\right) z_0^{\prod_{i=1}^{n} a_i}, \tag{53}$$

for every $n \in \mathbb{N}$.

From Corollary 1 and since $1 - k \in \mathbb{Z}$ it follows that Equation (49) is solvable in closed form when $\prod_{i=1}^{k}\left(\sum_{j=1}^{k} t_{ij}\beta_n^{(j)}\right) \ne 0$, for every $n \in \mathbb{N}_0$, from which the solvability of system (47) follows.

Remark 3. *Since for $k \le 4$ the characteristic polynomial*

$$P_k(\lambda) = \begin{vmatrix} a_{11}-\lambda & \cdots & \cdots & \cdots & a_{1k} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{ii}-\lambda & \cdots & a_{ik} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & \cdots & \cdots & a_{kk}-\lambda \end{vmatrix},$$

associated to the matrix $\widetilde{A}$ appearing in system (47) can be solved by well-known formulas ([49]), in this case the system is also practically solvable, as we have proved in [37,39]. The same fact was one of the main reasons for the solvability of the product-type difference equations and systems recently investigated in our papers [30–33].

If $k \ge 5$, then the system cannot be solved by using the method, since the polynomial $P_k(\lambda)$ need not be solvable by radicals. Nevertheless, the proof in Example 1 shows that the system is theoretically solvable.

Remark 4. *Note also that if the sequence $(a_n)_{n\in\mathbb{N}_0}$ in (50) is not a sequence of integers, then the equation need not have a unique solution. For example, the sequence defined by the following recurrence relation*

$$z_{n+1} = z_n^{\frac{1}{2}}, \quad n \in \mathbb{N}_0, \tag{54}$$

where $z_0 \in \mathbb{C} \setminus \{0\}$ is not uniquely defined.

Namely, if

$$z_0 = re^{i\theta}, \quad \theta \in [0, 2\pi),$$

then

$$\sqrt{z_0} = \sqrt{r}e^{i\frac{\theta}{2}+k\pi i}, \quad k \in \mathbb{Z},$$

which implies that

$$\sqrt{z_0} = \sqrt{r}e^{i\frac{\theta}{2}}$$

if k is even, and

$$\sqrt{z_0} = \sqrt{r}e^{i\frac{\theta}{2}+i\pi} = -\sqrt{r}e^{i\frac{\theta}{2}}$$

if k is odd, which are two different points due to condition $z_0 \neq 0$.

Repeating the procedure we get a binary tree whose branching points are 2^n different values of $\sqrt[2^n]{z_n}$, $n \in \mathbb{N}$. This shows that difference Equation (54) can have a continuum of solutions.

3. Conclusions

Motivated by some recent papers on solvability of some classes of rational systems of difference equations, here we present a quite general solvable system of difference equations which includes several ones in the recent papers. Our main result together with the analysis preceding it essentially explains what decides the solvability of the systems. It is interesting to note that a product-type difference equation appears and is one of the things that decides the solvability, which again shows the importance of product-type equations and systems which have been studied considerably recently. A natural example is given. The main result can be applied on many other concrete systems of difference equations and no doubt that experts in mathematics and other related branches of sciences will come across some systems whose solvability will be explained by our main result. Beside some applications it is expected that the main result can be extended to some other settings which can be another topic for further investigations in the area.

Acknowledgments: The work was supported by the Serbian Ministry of Education and Science projects III 44006 and III 41025.

Author Contributions: The paper is based on original ideas by this author. The author has solely contributed to the writing of this paper. He read and approved the manuscript.

Conflicts of Interest: The author declares that he has no conflicts of interest.

References

1. Brand, L. *Differential and Difference Equations*; John Wiley & Sons, Inc.: New York, NY, USA, 1966.
2. Fort, T. *Finite Differences and Difference Equations in the Real Domain*; Oxford University Press: London, UK, 1948.
3. Jordan, C. *Calculus of Finite Differences*; Chelsea Publishing Company: New York, NY, USA, 1956.
4. Krechmar, V.A. *A Problem Book in Algebra*; Mir Publishers: Moscow, Russia, 1974.
5. Levy, H.; Lessman, F. *Finite Difference Equations*; Dover Publications: New York, NY, USA, 1992.
6. Mitrinović, D.S.; Adamović, D.D. *Sequences and Series*; Naučna Knjiga: Beograd, Serbia, 1980. (In Serbian)
7. Mitrinović, D.S.; Kečkić, J.D. *Methods for Calculating Finite Sums*; Naučna Knjiga: Beograd, Serbia, 1984. (In Serbian)
8. Papaschinopoulos, G.; Stefanidou, G. Asymptotic behavior of the solutions of a class of rational difference equations. *Int. J. Differ. Equ.* **2010**, *5*, 233–249.
9. Stević, S. On the difference equation $x_n = x_{n-k}/(b + cx_{n-1} \cdots x_{n-k})$. *Appl. Math. Comput.* **2012**, *218*, 6291–6296.
10. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On the difference equation $x_n = a_n x_{n-k}/(b_n + c_n x_{n-1} \cdots x_{n-k})$. *Abstr. Appl. Anal.* **2012**, *2012*, Article 409237.
11. Andruch-Sobilo, A.; Migda, M. Further properties of the rational recursive sequence $x_{n+1} = ax_{n-1}/(b + cx_n x_{n-1})$. *Opusc. Math.* **2006**, *26*, 387–394.
12. Stević, S.; Diblik, J.; Iričanin. B.; Šmarda, Z. On a third-order system of difference equations with variable coefficients. *Abstr. Appl. Anal.* **2012**, *2012*, Article 508523.
13. Stević, S.; Diblik, J.; Iričanin. B.; Šmarda, Z. On the difference equation $x_{n+1} = x_n x_{n-k}/(x_{n-k+1}(a + bx_n x_{n-k}))$. *Abstr. Appl. Anal.* **2012**, *2012*, Article 108047.
14. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. Solvability of nonlinear difference equations of fourth order. *Electron. J. Differ. Equations* **2014**, *2014*, Article 264.
15. Berg, L.; Stević, S. On some systems of difference equations. *Appl. Math. Comput.* **2011**, *218*, 1713–1718.
16. Stević, S.; Diblik, J.; Iričanin. B.; Šmarda, Z. On some solvable difference equations and systems of difference equations. *Abstr. Appl. Anal.* **2012**, *2012*, Article 541761.

17. Stević, S.; Diblik, J.; Iričanin, B.; Šmarda, Z. On a solvable system of rational difference equations. *J. Differ. Equ. Appl.* **2014**, *20*, 811–825.
18. Fotiades, N.; Papaschinopoulos, G. On a system of difference equations with maximum. *Appl. Math. Comput.* **2013**, *221*, 684–690.
19. Papaschinopoulos, G.; Schinas, C.J. On the behavior of the solutions of a system of two nonlinear difference equations. *Comm. Appl. Nonlinear Anal.* **1998**, *5*, 47–59.
20. Papaschinopoulos, G.; Schinas, C.J. Invariants for systems of two nonlinear difference equations. *Differ. Equ. Dyn. Syst.* **1999**, *7*, 181–196.
21. Papaschinopoulos, G.; Schinas, C.J. Invariants and oscillation for systems of two nonlinear difference equations. *Nonlinear Anal. Theory Methods Appl.* **2001**, *46*, 967–978.
22. Papaschinopoulos, G.; Schinas, C.J. On the dynamics of two exponential type systems of difference equations. *Comput. Math. Appl.* **2012**, *64*, 2326–2334.
23. Papaschinopoulos, G.; Schinas, C.J.; Stefanidou, G. On a *k*-order system of Lyness-type difference equations. *Adv. Difference Equ.* **2007**, *2007*, Article 31272.
24. Stefanidou, S.; Papaschinopoulos, G.; Schinas, C.J. On a system of two exponential type difference equations. *Commun. Appl. Nonlinear Anal.* **2010**, *17*, 1–13.
25. Berezansky, L.; Braverman, E. On impulsive Beverton-Holt difference equations and their applications. *J. Differ. Equ. Appl.* **2004**, *10*, 851–868.
26. Diblik, J.; Halfarová, H. Explicit general solution of planar linear discrete systems with constant coefficients and weak delays. *Adv. Differ. Equ.* **2013**, *2013*, Article 50.
27. Iričanin, B.; Stević, S. Eventually constant solutions of a rational difference equation. *Appl. Math. Comput.* **2009**, *215*, 854–856.
28. Polya, G.; Szegö, G. *Aufgaben Und Lehrsätze Aus Der Analysis*; Verlag von Julius Springer: Berlin, Germany, 1925.
29. Stević, S. Existence of a unique bounded solution to a linear second order difference equation and the linear first order difference equation. *Adv. Differ. Equ.* **2017**, *2017*, Article 169.
30. Stević, S.; Iričanin. B.; Šmarda, Z. On a product-type system of difference equations of second order solvable in closed form. *J. Inequal. Appl.* **2015**, *2015*, Article 327.
31. Stević, S.; Iričanin. B.; Šmarda, Z. Solvability of a close to symmetric system of difference equations. *Electron. J. Differ. Equ.* **2016**, *2016*, Article 159.
32. Stević, S.; Iričanin. B.; Šmarda, Z. Two-dimensional product-type system of difference equations solvable in closed form. *Adv. Differ. Equ.* **2016**, *2016*, Article 253.
33. Stević, S. Solvable subclasses of a class of nonlinear second-order difference equations. *Adv. Nonlinear Anal.* **2016**, *5*, 147–165.
34. Stević, S. On a system of difference equations which can be solved in closed form. *Appl. Math. Comput.* **2013**, *219*, 9223–9228.
35. Brand, L. A sequence defined by a difference equation. *Amer. Math. Mon.* **1955**, *62*, 489–492.
36. Stević, S. Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences. *Electron. J. Qual. Theory Differ. Equ.* **2014**, *2014*, Article 67.
37. Stević, S. New class of solvable systems of difference equations. *Appl. Math. Lett.* **2017**, *63*, 137–144.
38. Abel, N.H. Mémoire sur les équations algébriques, ou l'on démontre l'impossibilité de la résolution de l'équation générale du cinquiṁe degré; In *Oeuvres Complètes de Niels Henrik Abel, I*, 2nd ed.; Sylow, L., Lie, S., Eds.; Grondahl & Son: Oslo, Norway, 1888; pp. 28–33.
39. Stević, S. A four-dimensional solvable system of difference equations in the complex domain. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, **2017**, doi:10.1007/s13398-017-0421-8.
40. Mitrinović, D.S. *Mathematical Induction, Binomial Formula, Combinatorics*; Gradjevinska Knjiga: Beograd, Serbia, 1980. (In Serbian)
41. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Nauka: Moscow, Russia, 1981. (In Russian)
42. Riordan, J. *Combinatorial Identities*; John Wiley & Sons Inc.: New York, NY, USA; London, UK; Sydney, Australia, 1968.
43. Yablonskiy, S.V. *Introduction to Discrete Mathematics*; Mir Publishers: Moscow, Russia, 1989.
44. Musielak, A.; Popenda, J. On the hyperbolic partial difference equations and their oscillatory properties. *Glas. Mat.* **1998**, *33*, 209–221.
45. Stević, S. Note on the binomial partial difference equation. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *2015*, Article 96.

46. Stević, S.; Iričanin. B.; Šmarda, Z. Boundary value problems for some important classes of recurrent relations with two independent variables. *Symmetry* **2017**, *9*, Article 323.
47. Beauregard, R.A.; Fraleigh, J.B. *A First Course In Linear Algebra: With Optional Introduction to Groups, Rings, and Fields*; Houghton Mifflin Co.: Boston, MA, USA, 1973.
48. Nering, E.D. *Linear Algebra and Matrix Theory*, 2nd ed.; Wiley: New York, NY, USA, 1970.
49. Faddeyev, D.K. *Lectures on Algebra*; Nauka: Moscow, Russia, 1984. (In Russian)