New general lower and upper bounds under minimum-error quantum state discrimination

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Abstract

For the optimal success probability under minimum-error discrimination between $r \geq 2$ arbitrary quantum states prepared with any a priori probabilities, we find new general analytical lower and upper bounds and specify the relations between these new general bounds and the general bounds known in the literature. We also present the example where the new general analytical bounds, lower and upper, on the optimal success probability are tighter than most of the general analytical bounds known in the literature. The new upper bound on the optimal success probability explicitly generalizes to r > 2 the form of the Helstrom bound. For r = 2, each of our new bounds, lower and upper, reduces to the Helstrom bound.

1 Introduction

Different aspects of quantum state discrimination are discussed in the literature ever since the seminal papers of Helstrom and Holevo [1, 2, 3, 4, 5] and are now presented in many textbooks and reviews, see, for example, in [6, 7, 8] and references therein.

Let a sender prepare a quantum system described in terms of a complex Hilbert space \mathcal{H} in one of $r \geq 2$ quantum states $\rho_1, ..., \rho_r$, pure or mixed, with probabilities $q_1, ..., q_r$, $\sum_i q_i = 1, q_i > 0$, and send this quantum system in an initial state

$$\rho = \sum_{i=1,\dots,r} q_i \rho_i, \quad \sum_{i=1,\dots,r} q_i = 1, \quad q_i > 0,$$
(1)

to a receiver. For discriminating between states $\rho_1, ..., \rho_r$, a receiver performs a measurement described by a POV measure

$$M_{r} = \left\{ M_{r}(i), \quad i = 1, ..., r; \quad \sum_{i=1,...,r} M_{r}(i) = \mathbb{I}_{\mathcal{H}} \right\}$$
(2)

and the success probability to take under this measurement the proper decision equals to

$$\mathbf{P}^{success}_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}(\mathbf{M}_r) = \sum_{i=1,\dots,r} q_i \mathrm{tr}\left\{\rho_i \mathbf{M}_r(i)\right\},\tag{3}$$

correspondingly, the error probability

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{error}(\mathbf{M}_r) = \sum_{i=1,...,r} q_i \text{tr} \{\rho_i(\mathbb{I} - \mathbf{M}_r(i))\}$$
(4)
= 1 - P_{\rho_1,...,\rho_r|q_1,...,q_r}^{success}(\mathbf{M}_r).

Denote by $\mathfrak{M}_r = {\{M_r\}, r \ge 2, \text{ the set of all possible POV measures (2). Under$ the maximum likelihood (the minimum error) state discrimination strategy, the optimalsuccess probability and the optimal error probability are given by

$$P_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}^{opt.success} := \max_{\mathcal{M}_r \in \mathfrak{M}_r} P_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}^{success}(\mathcal{M}_r),$$
(5)

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.error}: = \min_{M_r \in \mathfrak{M}_r} P_{\rho_1,...,\rho_r|q_1,...,q_r}^{error}(M_r) = 1 - P_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.success},$$
(6)

and are attained at some extreme point of the convex set \mathfrak{M}_r .

The alternative expressions for the optimal error probability (6) are presented in Theorem 1 and Corollary 1 of [9].

The following general statement was first formulated and proved by Holevo in [3, 4].

Theorem 1 Under the maximum likelihood (the minimum error) state discrimination strategy, a POV measure $M_r^{(opt)} \in \mathfrak{M}_r$ is optimal if and only if there exists a self-adjoint trace class operator Λ_0 such that: (i) $(\Lambda_0 - q_i\rho_i) M_r^{(opt)}(i) = 0$; (ii) $\Lambda_0 \ge q_i\rho_i$, for all i = 1, ..., r. Herewith,

$$\mathbf{P}_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}^{opt.success} = \mathrm{tr}\{\Lambda_0\}, \qquad \Lambda_0 = \sum_{i=1,\dots,r} q_i \rho_i \mathbf{M}_r^{(opt)}(i).$$
(7)

For r = 2, the success probability (3) admits the Helstrom upper bound [1, 2, 5]

$$P_{\rho_1,\rho_2|q_1,q_2}^{success}(\mathbf{M}_2) \le \frac{1}{2} \left(1 + \|q_1\rho_1 - q_2\rho_2\|_1 \right), \tag{8}$$

which is attained, so that, for r = 2, the optimal success probability is given by [1, 2, 5]

$$P_{\rho_1,\rho_2|q_1,q_2}^{opt.success} = \frac{1}{2} \left(1 + \|q_1\rho_1 - q_2\rho_2\|_1 \right), \tag{9}$$

where $\|\cdot\|_1$ is the trace norm.

For an arbitrary r > 2, a precise general expression for the optimal success probability (5) is not known, however, there were introduced and studied [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] several general upper and lower bounds on the optimal success probability $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.success}$, expressed via different characteristics of quantum states. As proved by Qiu&Li [17], in some cases, the general lower bound on the optimal error probability (correspondingly, the upper bound on the optimal success probability $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.success}$), introduced by them in [17], is tighter than the other general lower bounds known in the literature.

Computation of bounds on $P^{opt.success}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ within semidefinite programming was considered recently in [21].

In the present article, for the optimal success probability (5), we find (Theorems 2, 3) for all $r \ge 2$ the new general lower and upper bounds and specify (Propositions 1, 2) the relation of our bounds to the general lower and upper bounds known in literature. For r = 2, each of the new general bounds, lower and upper, reduces to the Helstrom bound in (8), and this proves in the other way the Helstrom result (9).

2 New lower bounds

Taking into account that $M_r(j) = \mathbb{I}_{\mathcal{H}} - \sum_{i \neq j} M_r(i)$, we rewrite the right-hand side of expression (3) in either of *j*-th representations:

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{success}(\mathbf{M}_r) = \sum_{i=1,...,r} q_i \operatorname{tr}\{\rho_i \mathbf{M}_r(i)\}$$
(10)
= $q_j + \sum_{i=1,...,r} \operatorname{tr}\{(q_i\rho_i - q_j\rho_j) \mathbf{M}_r(i)\}, \quad j = 1,...,r,$

for every POV measure M_r . Summing up the left-hand and the right-hand sides of (10) over all j = 1, ..., r, for any POV measure $M_r \in \mathfrak{M}_r$, we also come to the following representation for the success probability

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{success}(\mathbf{M}_r) = \frac{1}{r} \left(1 + \sum_{i,j=1,...,r} \operatorname{tr} \left\{ (q_i \rho_i - q_j \rho_j) \, \mathbf{M}_r(i) \right\} \right).$$
(11)

Recall that a self-adjoint (Hermitian) bounded operator X on \mathcal{H} admits the decomposition:

$$X = X^{(+)} - X^{(-)}, \quad X^{(\pm)} \ge 0,$$

$$X^{(+)} = \sum_{\lambda_k > 0,} \lambda_k \mathcal{E}_X(\lambda_k), \quad X^{(-)} = \sum_{\lambda_k \le 0,} |\lambda_k| \mathcal{E}_X(\lambda_k),$$
(12)

where $E_X(\lambda_k)$ the spectral projections of a Hermitian operator X. If a bounded operator X is trace class, then operators $X^{(\pm)} \geq 0$ are also trace class and

$$||X||_{1} := \operatorname{tr} |X|, \quad |X| = X^{(+)} + X^{(-)}, \tag{13}$$
$$||X^{(\pm)}||_{1} = \operatorname{tr} \{X^{(\pm)}\}.$$

From relations [7] $|tr\{W\}| \leq ||W||_1$ and $||AB||_1 \leq ||A||_1 ||B||_0$, valid for all trace-class operators W, A and all bounded operators B, it follows that if $X, Y \geq 0$ (hence, $tr\{XY\} \geq 0$), then

$$0 \le \operatorname{tr}\{XY\} \le \|X\|_1 \, \|Y\|_0, \tag{14}$$

where notation $\left\|\cdot\right\|_{0}$ means the operator norm.

Definition (5) and relations (10)–(14) imply.

Theorem 2 (New lower bounds) For any number $r \ge 2$ of arbitrary quantum states $\rho_1, ..., \rho_r$ prepared with probabilities $q_1, ..., q_r$, the optimal success probability (5) admits the lower bounds

$$P_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}^{opt.success} \ge \mathfrak{L}_{1,new}^{(r)} := \max_{j=1,\dots,r} \left\{ q_j + \frac{1}{r-1} \sum_{i=1,\dots,r} \left\| (q_i\rho_i - q_j\rho_j)^{(+)} \right\|_1 \right\}$$
(15)

$$= \frac{1}{2(r-1)} + \frac{1}{2(r-1)} \max_{j=1,\dots,r} \left\{ \sum_{i=1,\dots,r} \|q_i\rho_i - q_j\rho_j\|_1 + q_j(r-2) \right\}$$
(16)

$$\geq \mathfrak{L}_{2,new}^{(r)} := \frac{1}{r} \left(1 + \frac{1}{r-1} \sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \right).$$
(17)

For r = 2, each of these new lower bounds reduces to the Helstrom bound in (8).

Proof. Let $E_{(q_i\rho_i-q_j\rho_j)}(\lambda_k)$ be the spectral projections of the Hermitian operator $(q_i\rho_i-q_j\rho_j)$ on \mathcal{H} and

$$\mathbf{P}_{ij}^{(+)} := \sum_{\lambda_k > 0} \mathbf{E}_{(q_i \rho_i - q_j \rho_j)}(\lambda_k), \quad i \neq j,$$
(18)

denote the orthogonal projection on the proper subspace of operator $(q_i\rho_i - q_j\rho_j)$, corresponding to its positive eigenvalues. Note that by (13):

$$q_{i} - q_{j} = \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(+)} \right\|_{1} - \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(-)} \right\|_{1},$$
(19)
$$\|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1} = \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(+)} \right\|_{1} + \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(-)} \right\|_{1}.$$

Introduce the POV measures $M_r^{(j)}$, j = 1, ..., r, each with the elements

$$M_{r}^{(j)}(i) = \frac{1}{r-1} P_{ij}^{(+)}, \quad i \neq j.$$

$$M_{r}^{(j)}(j) = \mathbb{I}_{\mathcal{H}} - \frac{1}{r-1} \sum_{i=1,\dots,r, i \neq j} P_{ij}^{(+)}$$

$$= \frac{1}{r-1} \sum_{i=1,\dots,r, i \neq j} (\mathbb{I}_{\mathcal{H}} - P_{ij}^{(+)}) \ge 0.$$
(20)

From the *j*-th representation in (10) and relations (19) it follows that, for the *j*-th POV measure (20), we have:

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{success}(\mathbf{M}_r^{(j)}) = q_j + \sum_{i=1,...,N} \operatorname{tr}\{(q_i\rho_i - q_j\rho_j) \,\mathbf{M}_r^{(j)}(i)\}$$
(21)
$$= q_j + \frac{1}{r-1} \sum_{i=1,...,r} \left\| (q_i\rho_i - q_j\rho_j)^{(+)} \right\|_1,$$

$$j = 1, ..., r.$$

For the optimal success probability (5), equalities (21) imply

$$P_{\rho_{1},...,\rho_{r}|q_{1},...,q_{r}}^{opt.success} \ge q_{j} + \frac{1}{r-1} \sum_{i=1,...,r} \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(+)} \right\|_{1}, \qquad (22)$$
$$\forall j = 1,...,r,$$

hence,

$$P_{\rho_1,\dots,\rho_r|q_1,\dots,q_r}^{opt.success} \ge \max_{j=1,\dots,r} \left\{ q_j + \frac{1}{r-1} \sum_{i=1,\dots,r} \left\| (q_i\rho_i - q_j\rho_j)^{(+)} \right\|_1 \right\}.$$
 (23)

Since by (19)

$$\left\| (q_i \rho_i - q_j \rho_j)^{(+)} \right\|_1 = \frac{1}{2} (q_i - q_j) + \frac{1}{2} \left\| q_i \rho_i - q_j \rho_j \right\|_1,$$
(24)

the expression in the right-hand side of (22) is otherwise equal to

$$q_{j} + \frac{1}{r-1} \sum_{i=1,\dots,r} \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(+)} \right\|_{1}$$

$$= \frac{1}{2(r-1)} + \frac{1}{2(r-1)} \left\{ \sum_{i=1,\dots,r} \left\| q_{i}\rho_{i} - q_{j}\rho_{j} \right\|_{1} + q_{j}(r-2) \right\}.$$
(25)

This and relation (23) imply the lower bounds (15) and (16). Summing up the lefthand and the right-hand sides of (22) over all j = 1, ..., r and taking into account $\sum_{j=1,...,r} q_j = 1$, relations (19) and

$$(q_i\rho_i - q_j\rho_j)^+ = (q_j\rho_j - q_i\rho_i)^{(-)}, \qquad (26)$$

we derive

$$\mathbf{P}_{\rho_{1},\dots,\rho_{r}|q_{1},\dots,q_{r}}^{opt.success} \ge \frac{1}{r} \left(1 + \frac{1}{r-1} \sum_{1 \le i < j \le r} \|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1} \right),$$
(27)

that is, the lower bound (17). Furthermore, since, for any positive numbers α_j , j = 1, ..., r, their sum

$$\sum_{j=1,\dots,r} \alpha_j \le r \max_{j=1,\dots,r} \alpha_j,\tag{28}$$

we have

$$\max_{j=1,\dots,r} \left\{ q_j + \frac{1}{r-1} \sum_{i=1,\dots,N} \left\| (q_i \rho_i - q_j \rho_j)^{(+)} \right\|_1 \right\}$$

$$\geq \frac{1}{r} \left(1 + \frac{1}{r-1} \sum_{1 \le i < j \le r} \left\| q_i \rho_i - q_j \rho_j \right\|_1 \right).$$
(29)

Relations (23), (25) and (29) prove the statement of Theorem 2. \blacksquare

Consider now the relation of the new lower bounds (15)-(17) to the known general lower bounds on $\mathbb{P}^{opt.success}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$

$$\mathfrak{L}_{1}^{(r)} := \max_{j=1,\dots,r} q_{j} \ge \frac{1}{r},\tag{30}$$

$$\mathfrak{L}_2^{(r)} := 1 - \sum_{1 \le i < j \le r} \sqrt{q_i q_j} F_{ij}, \qquad (31)$$

$$\mathfrak{L}_{3}^{(r)} := \left(\operatorname{tr}\left[\sqrt{\sum_{i=1,\dots,r} q_{i}^{2} \rho_{i}^{2}} \right] \right)^{2}, \qquad (32)$$

where (a) bound (30) follows from item (ii) and relation (7) in Theorem 1; (b) bound (31) was introduced by Barnum&Knill in [10] and further studied by Audenaert&Mosonyi in [20]; (c) bound (32) was introduced by Tyson in [19]. Here, $F_{ij} := \left\| \sqrt{\rho_i} \sqrt{\rho_j} \right\|_1$ is the pairwise fidelity.

Proposition 1 (i) The new lower bound (15) is tighter

$$\mathfrak{L}_{1,new}^{(r)} \ge \mathfrak{L}_{1}^{(r)} \tag{33}$$

than the known lower bound (30) for any number $r \geq 2$ of arbitrary states $\rho_1, ..., \rho_r$ and probabilities $q_1, ..., q_r$.

(ii) For r = 2, the new lower bounds (15), (17) are equal to the Helstrom bound and $\mathfrak{L}_{1,new}^{(2)} = \mathfrak{L}_{2,new}^{(2)} \ge \mathfrak{L}_{2}^{(2)}$ for all states ρ_1, ρ_2 and probabilities q_1, q_2 . (iii) For any r > 2, the new lower bound (17) is tighter than the lower bound (31)

$$\mathfrak{L}_{2,new}^{(r)} \ge \mathfrak{L}_2^{(r)} \tag{34}$$

if

$$\sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \le \frac{(r-1)^2}{r+1}$$
(35)

and otherwise if

$$\frac{(r-1)^2}{r+1} < \sum_{1 \le i < j \le r} \left\| q_i \rho_i - q_j \rho_j \right\|_1 \le r-1.$$
(36)

Proof. Due to the structure of the new lower bound (15), relation (33) is obvious. In order to prove (ii) and (iii), consider the difference

$$\mathfrak{L}_{2,new}^{(r)} - \mathfrak{L}_{2}^{(r)} = \frac{1}{r} + \frac{1}{r(r-1)} \sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 - 1 + \sum_{1 \le i < j \le r} \sqrt{q_i q_j} F(\rho_i, \rho_j).$$
(37)

By Lemma 5 in [17]

$$\sqrt{q_i q_j} F(\rho_i, \rho_j) \ge \frac{1}{2} (q_i + q_j) - \frac{1}{2} \| q_i \rho_i - q_j \rho_j \|_1.$$
(38)

Substituting this relation into (37) and taking into account

$$\sum_{1 \le i < j \le r} (q_i + q_j) = r - 1, \tag{39}$$

we derive

$$\mathfrak{L}_{2,new}^{(r)} - \mathfrak{L}_{2}^{(r)} \ge \frac{(r-2)(r-1)}{2r} - \left(\frac{1}{2} - \frac{1}{r(r-1)}\right) \sum_{1 \le i < j \le r} \|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1} \qquad (40)$$
$$= \frac{(r-2)(r-1)}{2r} \left(1 - \frac{r+1}{(r-1)^{2}} \sum_{1 \le i < j \le r} \|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1}\right).$$

This proves relations (34), (35), also, the left-hand side inequality of (36). The right-hand side inequality of (36) follows from the relation

$$\sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \le \sum_{1 \le i < j \le r} (q_i + q_j) = r - 1,$$
(41)

where we took into account (39).

3 New upper bound

From relations (12)–(14) and inequality $\|\mathbf{M}_r(i)\|_0 \leq 1$ it follows that, for every POV measure \mathbf{M}_r , in representation (10)

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{success}(\mathbf{M}_r) = q_k + \sum_{i=1,...,r} \operatorname{tr}\{(q_i\rho_i - q_k\rho_k)\mathbf{M}_r(i)\}$$

$$\leq q_m + \sum_{i=1,...,r} \left\| (q_i\rho_i - q_m\rho_m)^{(+)} \right\|_1, \quad k, m = 1,...,r,$$
(42)

and in representation (11)

$$P_{\rho_{1},...,\rho_{r}|q_{1},...,q_{r}}^{success}(\mathbf{M}_{r}) = \frac{1}{r} \left(1 + \sum_{i,j=1,...,r} \operatorname{tr} \left\{ \left(q_{i}\rho_{i} - q_{j}\rho_{j} \right) \mathbf{M}_{r}(i) \right\} \right)$$

$$\leq \frac{1}{r} \left(1 + \sum_{i,j=1,...,r} \left\| \left(q_{i}\rho_{i} - q_{j}\rho_{j} \right)^{(+)} \right\|_{1} \right).$$
(43)

Relation (42) immediately implies the upper bound

$$P_{\rho_{1},...,\rho_{r}|q_{1},...,q_{r}}^{opt.success} \leq Q_{4}^{(r)} := \min_{j=1,...,r} \left\{ q_{j} + \sum_{i=1,...,r} \left\| (q_{i}\rho_{i} - q_{j}\rho_{j})^{(+)} \right\|_{1} \right\}$$

$$= \frac{1}{2} + \frac{1}{2} \min_{j=1,...,r} \left\{ \sum_{i=1,...,r} \|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1} - q_{j}(r-2) \right\},$$

$$(44)$$

which agrees due to the relation $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.success} = 1 - P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.error}$ with the lower bound L_4 by Qiu&Li on $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.error}$ introduced in [17]. Here, in order to derive the expression in the second line of (44) we took into account relation (24).

For convenience in comparison, we take for the upper bound (44) and the below upper bounds (54)-(56) on the optimal success probability the numeration similar to that for the lower bounds $L_n^{(r)}$ in [17] on the optimal error probability with the obvious correspondence $Q_n^{(r)} = 1 - L_n^{(r)}$.

The following theorem introduces a new upper bound on the optimal success probability $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.success}$ and establishes its relation to the upper bound (44).

Theorem 3 (New upper bound) For any number $r \ge 2$ of arbitrary quantum states $\rho_1, ..., \rho_{r_r}$ prepared with probabilities $q_1, ..., q_r$, the optimal success probability (5) admits the upper bound

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.success} \le Q_{new}^{(r)} := \frac{1}{r} \left(1 + \sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \right),$$
(45)

explicitly generalizing to r > 2 the form of the Helstrom upper bound in (8) and relating to the upper bound (44) as

$$Q_4^{(r)} \le Q_{new}^{(r)}.$$
 (46)

Proof. In view of (5), relations (43) and (26) immediately imply the upper bound (45). Also, by (42)

$$P_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.success} \le q_j + \sum_{i=1,...,r} \left\| (q_i\rho_i - q_j\rho_j)^{(+)} \right\|_1, \quad j = 1,...,r.$$
(47)

Summing up the left-hand and the right-hand sides of this relation over j = 1, ..., r, and taking into account (26), we derive

$$r \mathbf{P}_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.success} \leq \sum_{j=1,...,r} q_j + \sum_{i=1,...,r} \left\| (q_i\rho_i - q_j\rho_j)^{(+)} \right\|_1$$

$$= 1 + \sum_{1 \leq i < j \leq r} \|q_i\rho_i - q_j\rho_j\|_1.$$
(48)

Since, for any positive numbers α_j , j = 1, ..., r, their sum

$$\sum_{j=1,\dots,r} \alpha_j \ge r \min_{j=1,\dots,r} \alpha_j,\tag{49}$$

we have

$$\min_{j=1,\dots,r} \left\{ q_j + \sum_i \left\| (q_i \rho_i - q_j \rho_j)^{(+)} \right\|_1 \right\}$$

$$\leq \frac{1}{r} \left(1 + \sum_{1 \leq i < j \leq r} \left\| q_i \rho_i - q_j \rho_j \right\|_1 \right),$$
(50)

that is, relation (46). This proves the statement of Theorem 3. \blacksquare

We stress that if r is rather large, then the calculation of bound (45) is easier than finding the maximum in bound (44).

Remark 1 From relations (24) and $||q_i\rho_i - q_j\rho_j||_1 \le q_i + q_j$, it follows $||(q_i\rho_i - q_j\rho_j)^{(+)}||_1 \le q_i$, therefore,

$$\sum_{i=1,\dots,r} \left\| (q_i \rho_i - q_j \rho_j)^{(+)} \right\|_1 \le \sum_{i=1,\dots,r,\ i \neq j} q_i = 1 - q_j.$$
(51)

Relations (26), (39) and (51) imply

$$\sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 = \sum_{i,j=1,\dots,r,} \left\| (q_i \rho_i - q_j \rho_j)^{(+)} \right\|_1 \le r - 1,$$
(52)

which agrees with (41). Therefore, the new upper bound in Theorem 3 is nontrivial (i. e. not more than one). Also, by relation (42) specified for the POV measure (20), the bounds in Theorem 2 and the upper bound (44) are consistent in the sense

$$\max_{k=1,\dots,r} \left\{ q_k + \frac{1}{r-1} \sum_{i=1,\dots,r} \left\| (q_i \rho_i - q_k \rho_k)^{(+)} \right\|_1 \right\}$$

$$\leq \min_{m=1,\dots,r} \left\{ q_m + \sum_{i=1,\dots,r} \left\| (q_i \rho_i - q_m \rho_m)^{(+)} \right\|_1 \right\}$$
(53)

for all $r \geq 2$.

Besides relation (46) of the new upper bound (45) to the upper bound (44) introduced

in [17], consider also its relation to the general upper bounds on $P_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}^{opt.success}$:

$$Q_2^{(r)} := \frac{1}{2} \left(1 + \frac{1}{r-1} \sum_{1 \le i < j \le r} \left\| q_i \rho_i - q_j \rho_j \right\|_1 \right),$$
(54)

$$Q_3^{(r)} := 1 - \sum_{1 \le i < j \le r} q_i q_j F_{ij}^2, \tag{55}$$

$$Q_5^{(r)} := \operatorname{tr}\left[\sqrt{\sum_{i=1,\dots,r} q_i^2 \rho_i^2}\right],\tag{56}$$

which follow from the lower bounds $L_n^{(r)}$ on the optimal error probability introduced, correspondingly: (i) by Qiu in [15]; (ii) by Montanaro in [16]; (iii) by Ogawa&Nagaoka in [18] for the equiprobable case and by Tyson [19] for a general case.

The detailed study of these general bounds and some other bounds for specific families of quantum states is presented in [20].

Proposition 2 (i) For any number $r \ge 2$, of arbitrary quantum states $\rho_1, ..., \rho_r$ and a priori probabilities $q_1, ..., q_r$, the new upper bound (45) and the upper bounds (54)–(55) satisfy the relations

$$Q_{new}^{(r)} \le Q_2^{(r)} \le 1 - \frac{1}{N-1} (1 - Q_3^{(r)}).$$
 (57)

(ii) In the equiprobable case, the new upper bound (45) is tighter than the known upper bound (55):

$$Q_{new}^{(r)} \le Q_3^{(r)}.$$
 (58)

Proof. (i) The first inequality in (57) follows from

$$Q_{2}^{(r)} - Q_{new}^{(r)}$$

$$= \frac{r-2}{2r} \left(1 - \frac{1}{r-1} \sum_{1 \le i < j \le r} \|q_{i}\rho_{i} - q_{j}\rho_{j}\|_{1} \right) \ge 0,$$
(59)

where we take into account estimate (52). To prove the second inequality in (57), we use estimate (55) in [16] and derive

$$1 - Q_2^{(r)} \ge \frac{1 - Q_3^{(r)}}{r - 1}.$$
(60)

(ii) For the equiprobable case $q_1 = \dots = q_r = \frac{1}{r}$, we consider the difference

$$Q_{new}^{(r)} - Q_3^{(r)} = \frac{1}{r} + \frac{1}{r^2} \sum_{1 \le i < j \le r} \|\rho_i - \rho_j\|_1 - 1 + \frac{1}{r^2} \sum_{1 \le i < j \le r} F^2(\rho_i, \rho_j)$$
(61)
$$= \frac{1}{r^2} \sum_{1 \le i < j \le r} \left(F^2(\rho_i, \rho_j) + \|\rho_i - \rho_j\|_1 \right) - \frac{r-1}{r}.$$

Taking into account relation (20) in [17] which implies

$$\sum_{1 \le i < j \le r} \left(F^2(\rho_i, \rho_j) + \|\rho_i - \rho_j\|_1 \right) \le \sum_{1 \le i < j \le r} 2 = r(r-1)$$
(62)

and substituting this into (61), we have

$$Q_{new}^{(r)} - Q_3^{(r)} \le \frac{r-1}{r} - \frac{r-1}{r} = 0.$$
 (63)

This and relation (59) prove the statement. \blacksquare

4 General relations

Theorem 2 and Theorem 3 imply. the following general relations.

Corollary 1 For any number $r \ge 2$ of arbitrary quantum states $\rho_1, ..., \rho_r$ prepared with any probabilities $q_1, ..., q_r$, the optimal success probability (5) admits the bounds:

$$\frac{1}{r} \left(1 + \frac{1}{(r-1)} \sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \right) \\
\leq \frac{1}{2(r-1)} + \frac{1}{2(r-1)} \max_{j=1,...,r} \left\{ \sum_{i=1,...,r} \|q_i \rho_i - q_j \rho_j\|_1 + q_j(r-2) \right\} \\
\leq P_{\rho_1,...,\rho_r|q_1,...,q_r}^{opt.success} \left\{ \sum_{i=1,...,r} \|q_i \rho_i - q_j \rho_j\|_1 - q_j(r-2) \right\} \\
\leq \frac{1}{2} + \frac{1}{2} \min_{j=1,...,r} \left\{ \sum_{i=1,...,r} \|q_i \rho_i - q_j \rho_j\|_1 - q_j(r-2) \right\}$$

$$\leq \frac{1}{r} \left(1 + \sum_{1 \le i < j \le r} \|q_i \rho_i - q_j \rho_j\|_1 \right).$$
(64)

In the first and the last lines of (64), the bounds are quite similar by its form to the Helstrom upper bound (8). For r = 2, each of the lower and all bounds in (64) reduces to the Helstrom bound in (8) and this proves in the other way the Helstrom result (9).

For the equiprobable case, bounds (64) take the forms.

Corollary 2 For any number $r \ge 2$, of arbitrary quantum states $\rho_1, ..., \rho_r$, prepared with equal probabilities $q_1 = ... = q_1 = \frac{1}{r}$, the optimal success probability (5) admits the

bounds:

$$\frac{1}{r} \left(1 + \frac{1}{r(r-1)} \sum_{1 \le i < j \le r} \|\rho_i - \rho_j\|_1 \right) \\
\leq \frac{1}{r} + \frac{1}{2r(r-1)} \max_{j=1,...,r} \left\{ \sum_{i=1,...,r} \|\rho_i - \rho_j\|_1 \right\} \\
\leq P_{\rho_1,...,\rho_r|\frac{1}{r},...,\frac{1}{r}}^{opt.success} \\
\leq \frac{1}{r} + \frac{1}{2r} \min_{j=1,...,r} \left\{ \sum_{i=1,...,r} \|\rho_i - \rho_j\|_1 \right\} \\
\leq \frac{1}{r} \left(1 + \frac{1}{r} \sum_{1 \le i < j \le r} \|\rho_i - \rho_j\|_1 \right).$$
(65)

4.1 Example

For the numerical comparison of the new lower bound (15)-(17) and the new upper bound (45) with the known lower bounds (30)-(32) and the known upper bounds (44), (54)-(56), we analyze the discrimination between the following three equiprobable qubit states

$$\rho_{1} = \frac{7}{8} |0\rangle \langle 0| + \frac{1}{8} |1\rangle \langle 1|, \quad \rho_{2} = \frac{5}{8} |0\rangle \langle 0| + \frac{3}{8} |1\rangle \langle 1|, \qquad (66)$$

$$\rho_{3} = \frac{3}{4} |0\rangle \langle 0| + \frac{1}{4} |1\rangle \langle 1|.$$

In this case, $\|\rho_1 - \rho_2\|_1 = \frac{1}{2}$, $\|\rho_1 - \rho_3\|_1 = \frac{1}{4}$, $\|\rho_2 - \rho_3\|_1 = \frac{1}{4}$ and

$$\sum_{1 \le i < j \le 3} \|\rho_i - \rho_j\|_1 = 1, \quad \min_{j=1,2,3} \left\{ \sum_{i=1,2,3} \|\rho_i - \rho_j\|_1 \right\} = \frac{1}{2}, \quad (67)$$
$$\max_{j=1,2,3} \left\{ \sum_{i=1,2,3} \|\rho_i - \rho_j\|_1 \right\} = \frac{3}{4}.$$

Therefore, in the case considered, the values of the new bounds (45), (15), (17) are equal to

$$\mathfrak{L}_{1,new} = \frac{19}{48} = 0.3958, \quad \mathfrak{L}_{2,new} = \frac{7}{18} \simeq 0.3889, \tag{68}$$
$$Q_{new} = \frac{4}{9} \simeq 0.4444,$$

while the values of the known bounds (44), (54), (30) are

$$Q_4 = \frac{5}{12} \simeq 0.4166, \quad Q_2 = \frac{7}{12} \simeq 0.5833, \quad \mathfrak{L}_1 = \frac{1}{3}.$$
 (69)

Note that since states (66) mutually commute and are of the form $\rho_i = \sum_{n=0,1} \lambda_n^{(i)} |n\rangle \langle n|$, i = 1, 2, 3, the optimal success probability [4]

$$\mathbf{P}_{\rho_1,\rho_2,\rho_3|\frac{1}{3},\frac{1}{3},\frac{1}{3}}^{opt.success} = \frac{1}{3} \sum_{n=0,1} \left(\max_{i=1,2,3} \lambda_n^{(i)} \right) = \frac{5}{12} = Q_4 \simeq 0.4166.$$
(70)

For the calculation of the lower bounds (31), (32) and the upper bounds (55), (56) for equiprobable states (66), we calculate fidelities $F_{ij} := \left\| \sqrt{\rho_i} \sqrt{\rho_j} \right\|_1$ for states (66):

$$F_{12} = \frac{\sqrt{35} + \sqrt{3}}{8} \simeq 0.9560, \quad F_{13} = \frac{\sqrt{42} + \sqrt{2}}{8} \simeq 0,9868, \tag{71}$$
$$F_{23} = \frac{\sqrt{30} + \sqrt{6}}{8} \simeq 0,9909,$$

and the trace

$$\operatorname{tr}\left[\sqrt{\sum_{i=1,2,3}\rho_i^2}\right] = \frac{\sqrt{110} + \sqrt{14}}{8} \simeq 1,7787.$$
(72)

Therefore,

$$Q_{3} = 1 - \frac{1}{9} \sum_{1 \le i < j \le 3} F_{ij}^{2} \simeq 0.6812, \qquad Q_{5} = \frac{1}{3} \operatorname{tr} \left[\sqrt{\sum_{i=1,2,3} \rho_{i}^{2}} \right] \simeq 0,5929, \tag{73}$$
$$\mathfrak{L}_{2} = 1 - \frac{1}{3} \sum_{1 \le i < j \le 3} F_{ij} \simeq 0,0221, \qquad \mathfrak{L}_{3} = (Q_{5})^{2} \simeq 0.3515.$$

From (68)–(73) it follows that, in the case considered,

$$P_{\rho_1,\rho_2,\rho_3|\frac{1}{3},\frac{1}{3},\frac{1}{3}}^{opt.success} = Q_4 < Q_{new} < Q_2 < Q_5 < Q_3$$
(74)

and

$$P_{\rho_{1},\rho_{2},\rho_{3}|\frac{1}{3},\frac{1}{3},\frac{1}{3}}^{opt.success} > \mathfrak{L}_{1,new} > \mathfrak{L}_{2,new} > \mathfrak{L}_{3} > \mathfrak{L}_{1} > \mathfrak{L}_{2},$$
(75)

that is, the values of the new lower bounds $\mathfrak{L}_{new,1}$, $\mathfrak{L}_{new,2}$ and the new upper bound Q_{new} are tighter than the values of the known lower bounds (30)–(32) and the known upper bounds (54)–(56), respectively.

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6 Conclusions

In the present article, for the optimal success probability (5), we find for all $r \ge 2$: (i) the new general lower bounds (Theorem 2) and specify their relation (Proposition 1) to the general lower bounds known in the literature; (ii) the new general upper bound (Theorem 3) and specify its relation (Proposition 2) to the known general upper bounds.

We also present the example where the values of the new general analytical bounds, lower and upper, on the optimal success probability are tighter than the values of the known lower bounds (30)–(32) and the known upper bounds (54)–(56), respectively.

The new upper bound (45) on the optimal success probability has the form explicitly generalizing to r > 2 the Helstrom bound in (8) and is easily calculated. For r = 2, each of our new bounds, lower and upper, reduces to the Helstrom bound in (8), and this proves in the other way the Helstrom result (9).

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