

Research Article

General Minimum Lower-Order Confounding Designs with Multi-Block Variables

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Blocking the inhomogeneous units of experiments into groups is an efficient way to reduce the influence of systematic sources on the estimations of treatment effects. In practice, there are two types of blocking problems. One considers only a single block variable and the other considers multi-block variables. The present paper considers the blocking problem of multi-block variables. Theoretical results and systematical construction methods of optimal blocked 2^{n-m} designs with $(N/4) + 1 \leq n \leq 5N/16$ are developed under the prevalent general minimum lower-order confounding (GMC) criterion, where $N = 2^{n-m}$.

1. Introduction

The regular 2^{n-m} factorial experiment has played an important role in engineering, manufacturing industry, agriculture and medicine, and so on. It allows efficient and economic experimentation to estimate treatment effects. When the size of the experimental units is large, the inhomogeneity will cause unwanted variance to the estimations of treatment effects. To reduce such bad influence, a crucial way is to partition the experimental units into blocks.

There are two kinds of blocking problems as pointed out in [1]. One is called the single block variable problem which considers only a single block variable, and the other is called the multi-block variable problem which considers two or more block variables. In the last decades, choosing optimal blocked 2^{n-m} designs with a single block variable has been well investigated; for example, the authors in [2–9] studied the blocked 2^{n-m} designs under different minimum aberration criteria; Chen et al. and Zhao et al. [10, 11] explored the blocked 2^{n-m} designs under the clear effects criterion; Zhang and Mukerjee, Zhao et al., and Zhao and Zhao [12–14] explored the constructions of the blocked 2^{n-m} designs under the general minimum lower-order confounding (GMC) criterion proposed in [15]; and Zhao et al.

[16–18] gave construction methods of the blocked 2^{n-m} designs under another GMC criterion proposed in [19].

Compared to the large body of work on the blocking problem of single block variable, the studies on choosing optimal blocked 2^{n-m} designs with multi-block variables are relatively rare. However, it has been recognized that the blocking problem of multi-block variables can arise quite naturally in many practical situations. For example, in the agricultural context, Bisgaard [1] pointed that when designs are laid out in rectangular schemes, both row and column inhomogeneity effects probably exist in the soil. Another example of multi-block variables is from [20]. Considering the comparison of two gasoline additives by testing them on two cars with two drivers over two days, the “cars,” “drivers,” and “days” are three block variables which should be taken into account when performing experiments.

Under the clear effects criterion, Zhao and Zhao [21] proposed an algorithm for finding optimal blocked 2^{n-m} designs with multi-block variables. Under the minimum aberration criterion, Zhao and Zhao [22] developed some rules for constructing optimal blocked 2^{n-m} designs with multi-block variables. Zhang et al. [23] extended the idea of GMC criterion to the case of multi-block variable problem and developed the blocked GMC criterion, called

the B^2 -GMC criterion. Inheriting the advantage of the GMC criterion, the B^2 -GMC designs are particularly preferable when some prior information on the importance ordering of treatment effects is present. By computer search, Zhang et al. [23] tabulated some B^2 -GMC designs with small n and N , where $N = 2^{n-m}$. When n or N is large, computer search becomes computationally expensive. Zhao et al. [24] and Zhao and Zhao [25] completed the constructions of B^2 -GMC designs with $5N/16 + 1 \leq n \leq N - 1$. This paper aims at providing theories and systematical construction methods of the B^2 -GMC designs with $N/4 + 1 \leq n \leq 5N/16$.

The rest of the paper is organized as follows. Section 2 reviews doubling theory and B^2 -GMC criterion. Section 3 provides theoretical results and construction methods of B^2 -GMC designs. Section 4 gives concluding remarks. Some useful lemmas are deferred to Appendix.

2. Preliminaries

2.1. Doubling Theory. Let $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_t)$ be a matrix with entries 1 or -1 . Denote $\mathbf{J}_0 = (1, 1)'$ and $\mathbf{J}_1 = (1, -1)'$, where $'$ denotes transpose. Define

$$D(\mathbf{X}) = ((\mathbf{J}_0, \mathbf{J}_1) \otimes \mathbf{x}_1, \dots, (\mathbf{J}_0, \mathbf{J}_1) \otimes \mathbf{x}_t), \quad (1)$$

as a *double* of \mathbf{X} , where \otimes is the Kronecker product. Let $D^q(\mathbf{X})$ denote the design obtained by repeatedly doubling $\mathbf{X}q$ times, i.e., $D^q(\mathbf{X}) = D(D^{q-1}(\mathbf{X}))$. When $\mathbf{X} = \mathbf{1}$, we write $D^q(\mathbf{1}) = (\mathbf{1}_{2^q}, \mathbf{1}_{2^q}, \mathbf{2}_{2^q}, \mathbf{1}_{2^q}\mathbf{2}_{2^q}, \dots, \mathbf{1}_{2^q}\mathbf{2}_{2^q}\mathbf{3}_{2^q}, \dots, \mathbf{q}_{2^q})$, where the subscript 2^q means the dimension of a column, $\mathbf{1}_{2^q}$ is a column of 1's,

$$\begin{aligned} \mathbf{1}_{2^q} &= (1, \dots, 1, -1, \dots, -1) \\ \mathbf{2}_{2^q} &= (1, \dots, 1, -1, \dots, -1, \dots, 1, \dots, 1, -1, \dots, -1) \\ &\vdots \\ \mathbf{q}_{2^q} &= (1, -1, 1, -1, \dots, 1, -1, 1, -1), \end{aligned} \quad (2)$$

are q independent columns, and the other columns are the component-wise products of some of these q independent columns. For example, $\mathbf{1}_{2^q}\mathbf{2}_{2^q}$ is the component-wise product of the columns $\mathbf{1}_{2^q}$ and $\mathbf{2}_{2^q}$. Write $D^q(\mathbf{1}) = (\mathbf{I}_{2^q}, D^q(\cdot))$; then, $D^q(\cdot)$ is just the regular two-level saturated design with columns arranged in Yates order. As some subdesigns in $D^q(\cdot)$, denote $\mathbf{H}_0^q = \emptyset$, $\mathbf{H}_1^q = \mathbf{1}_{2^q}$, $\mathbf{H}_r^q = (\mathbf{H}_{r-1}^q, \mathbf{r}_{2^q}, \mathbf{r}_{2^q}\mathbf{H}_{r-1}^q)$ for $r = 2, 3, \dots, q$, where \emptyset denotes the empty set, the superscript q of \mathbf{H}_r^q refers to that \mathbf{H}_r^q is a subdesign of $D^q(\cdot)$, and $\mathbf{r}_{2^q}\mathbf{H}_{r-1}^q = (\mathbf{r}_{2^q}\mathbf{d}_{2^q}; \mathbf{d}_{2^q} \in \mathbf{H}_{r-1}^q)$, i.e., \mathbf{H}_r^q consists of the first $2^r - 1$ columns of $D^q(\cdot)$. Especially, $\mathbf{H}_q^q = D^q(\cdot)$.

2.2. B^2 -GMC Criterion. Before introducing the B^2 -GMC criterion, we first review some principles in the multi-block variable problem. Let b_1, b_2, \dots, b_s denote the s block variables which cause the inhomogeneity of the experimental units. Suppose that the block variable b_j partitions the $N (= 2^{n-m})$ experimental units into 2^{l_j} blocks; then, l_j independent columns are needed to carry out this blocking plan.

Denote S_j as the set of the l_j independent columns related to the block variable b_j . The block columns should follow the following rules:

- (i) The l_j block columns in $S_j (j = 1, 2, \dots, s)$ are independent of each other.
- (ii) A block column from S_j is not necessarily independent of the block columns from S_i with $j \neq i$.

In this paper, we focus on the case where each block variable is at two levels, i.e., $l_j = 1$.

The effect hierarchy principle for blocked designs with multi-block variables is as follows (see [23]):

- (i) The lower-order treatment factorial effects are more likely to be important than the higher-order ones, and the treatment factorial effects of the same order are equally likely to be important.
- (ii) The lower-order block factorial effects are more likely to be important than the higher-order ones, and the block factorial effects of the same order are equally likely to be important.
- (iii) All the interactions between treatment factors and block factors are negligible.

Since each variable or factor is assigned to one column of the design matrix when an experiment is carried out, we do not differentiate the variable, factor, and column. Based on the effect hierarchy principle and weak assumption that the effects involving three or more factors are usually not important and negligible, Zhang et al. [23] proposed the B^2 -GMC criterion which pays attention to only the confounding among main treatment effects and the two-factor interactions of treatment factors (2fi's for short). For the same reason, a common assumption in blocking problem is that only the main effects of block variables and the interactions of any two block factors are potentially significant, and if a treatment effect is confounded with a potentially significant block effect, the treatment effect cannot be estimated. Thus, the confounding between the main effects of treatment factors and any potentially significant block effect is not allowed.

Denote $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ as a $2^{n-m}; 2^s$ design, where \mathbf{D}_t consists of n treatment factors corresponding to a regular 2^{n-m} design and \mathbf{D}_b consists of s block factors each of which can partition the 2^{n-m} runs into 2 blocks. Denote $\#_1 C_2^{(p)}(\mathbf{D})$ as the number of main treatment effects which are aliased with p 2fi's but not with any potentially significant block effects, where $p = 0, 1, 2, \dots, P$ and $P = n(n-1)/2$. Similarly, $\#_2 C_2^{(p)}(\mathbf{D})$ denotes the number of 2fi's which are aliased with the other p 2fi's but not with any potentially significant block effects, where $p = 0, 1, \dots, P$. Denote

$$\#_1 C_2(\mathbf{D}) = \left(\#_1 C_2^{(0)}(\mathbf{D}), \#_1 C_2^{(1)}(\mathbf{D}), \dots, \#_1 C_2^{(P)}(\mathbf{D}) \right), \quad (3)$$

$$\#_2 C_2(\mathbf{D}) = \left(\#_2 C_2^{(0)}(\mathbf{D}), \#_2 C_2^{(1)}(\mathbf{D}), \dots, \#_2 C_2^{(P)}(\mathbf{D}) \right),$$

$$\# C(\mathbf{D}) = \left(\#_1 C_2(\mathbf{D}), \#_2 C_2(\mathbf{D}) \right). \quad (4)$$

A 2^{n-m} ; 2^s blocked design $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is called a B^2 -GMC design if it sequentially maximizes (4). Let $\#_1 C_2^{(p)}(\mathbf{D}_t)$ be the number of main effects which are aliased with p 2fi's of \mathbf{D}_t and $\#_2 C_2^{(p)}(\mathbf{D}_t)$ be the number of 2fi's which are aliased with the other p 2fi's of \mathbf{D}_t . Let

$$\begin{aligned} \#_1 C_2(\mathbf{D}_t) &= \left(\#_1 C_2^{(0)}(\mathbf{D}_t), \#_1 C_2^{(1)}(\mathbf{D}_t), \dots, \#_1 C_2^{(p)}(\mathbf{D}_t) \right), \\ \#_2 C_2(\mathbf{D}_t) &= \left(\#_2 C_2^{(0)}(\mathbf{D}_t), \#_2 C_2^{(1)}(\mathbf{D}_t), \dots, \#_2 C_2^{(p)}(\mathbf{D}_t) \right), \\ \# C(\mathbf{D}_t) &= \left(\#_1 C_2(\mathbf{D}_t), \#_2 C_2(\mathbf{D}_t) \right). \end{aligned} \quad (5)$$

A 2^{n-m} design \mathbf{D}_t is called a GMC design if \mathbf{D}_t sequentially maximizes (5).

Let $q = n - m$. Constructing a B^2 -GMC design is to choose \mathbf{D}_t and \mathbf{D}_b from $D^q(\cdot)$ such that (4) is sequentially maximized. In the following, without causing confusions, we omit the subscript of a column and the superscript of a design when they are taken from $D^q(\cdot)$. For example, we use \mathbf{a} , \mathbf{H}_r , and \mathbf{H}_q instead of \mathbf{a}_{2q} , \mathbf{H}_r^q , and \mathbf{H}_q^q , respectively. Denote

$$U(\mathbf{D}_b) = \{ \boldsymbol{\gamma} \in \mathbf{H}_q; \boldsymbol{\gamma} \in \mathbf{D}_b \text{ or } \boldsymbol{\gamma} = \mathbf{a}\mathbf{b} \text{ with } \mathbf{a}, \mathbf{b} \in \mathbf{D}_b \}, \quad (6)$$

and then $U(\mathbf{D}_b)$ consists of all the potentially significant block effects. As previously stated, the confounding between main treatment effects and potentially significant block effects is not allowed. This requires $\mathbf{D}_t \cap U(\mathbf{D}_b) = \emptyset$, and thus $\#_1 C_2^{(p)}(\mathbf{D}) = \#_1 C_2^{(p)}(\mathbf{D}_t)$ for $p = 0, 1, 2, \dots, P$.

For $\mathbf{D}_t \subset \mathbf{H}_q$ and $\boldsymbol{\gamma} \in \mathbf{H}_q$, define

$$B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = \#\{(\mathbf{d}_1, \mathbf{d}_2): \mathbf{d}_1, \mathbf{d}_2 \in \mathbf{D}_t, \mathbf{d}_1 \mathbf{d}_2 = \boldsymbol{\gamma}\}, \quad (7)$$

where $\#$ denotes the cardinality of a set and $\mathbf{d}_1 \mathbf{d}_2$ stands for the two-factor interaction of \mathbf{d}_1 and \mathbf{d}_2 . Thus, $B_2(\mathbf{D}_t, \boldsymbol{\gamma})$ is the number of 2fi's of \mathbf{D}_t appearing in the alias set that contains $\boldsymbol{\gamma}$.

Isomorphism introduced by Tang and Wu [26] is a useful concept which helps narrow down the search of the optimal blocked designs here. An isomorphism ϕ is a one-to-one mapping from \mathbf{H}_q to \mathbf{H}_q such that $\phi(\mathbf{x}\mathbf{y}) = \phi(\mathbf{x})\phi(\mathbf{y})$ for every $\mathbf{x} \neq \mathbf{y} \in \mathbf{H}_q$. The 2^{n-m} designs \mathbf{D}_t and \mathbf{D}_t^* are isomorphic if there exists an isomorphism ϕ that maps \mathbf{D}_t onto \mathbf{D}_t^* . The 2^{n-m} ; 2^s designs $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ and $\mathbf{D}^* = (\mathbf{D}_t^*; \mathbf{D}_b^*)$ are isomorphic if there exists an isomorphism ϕ that maps \mathbf{D}_t onto \mathbf{D}_t^* and \mathbf{D}_b onto \mathbf{D}_b^* .

3. Constructions of B^2 -GMC Designs

3.1. B^2 -GMC 2^{n-m} ; 2^s Designs with $n = (N/4) + 1$. A design is of MaxC2 (see [27]) if it has resolution IV and maximum number of clear 2fi's, where a resolution R design has no c -factor interaction confounded with any other interaction involving less than $R - c$ factors (see [28]), and a 2fi is called clear if it is not aliased with any main treatment effect and other 2fi's. Cheng and Zhang [29] showed that a 2^{n-m} design with $n = N/4 + 1$ is a MaxC2 design if and only if it is a GMC design. They also pointed out that, up to isomorphism, the GMC 2^{n-m} design with $n = N/4 + 1$ can be uniquely expressed as $\mathbf{S}_{N/4+1} = (\mathbf{q} - \mathbf{1}, \mathbf{q}, \mathbf{q}(\mathbf{q} - \mathbf{1}))_{\mathbf{H}_{q-2}}$. It is easy to obtain that

$$B_2(\mathbf{S}_{N/4+1}, \boldsymbol{\gamma}) = \begin{cases} 1, & \text{for } \boldsymbol{\gamma} \in \{(\mathbf{q} - \mathbf{1})\mathbf{H}_{q-2}, \mathbf{q}\mathbf{H}_{q-2}, \mathbf{q}(\mathbf{q} - \mathbf{1})\}, \\ \frac{N}{8} - 1, & \text{for } \boldsymbol{\gamma} \in \mathbf{H}_{q-2}, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Therefore,

$$\#_1 C_2^{(p)}(\mathbf{S}_{N/4+1}) = \begin{cases} \frac{N}{4} + 1, & \text{for } p = 0, \\ 0, & \text{otherwise,} \end{cases} \quad (9)$$

$$\#_2 C_2^{(p)}(\mathbf{S}_{N/4+1}) = \begin{cases} 2\left(\frac{N}{4} - 1\right) + 1, & \text{for } p = 0, \\ \left(\frac{N}{4} - 1\right)\left(\frac{N}{8} - 1\right), & \text{for } p = \frac{N}{8} - 2, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Lemma 1 is a straightforward extension of Lemma A.1, in Appendix, introduced from [24, 25].

Lemma 1. Suppose \mathbf{D}_b is any s -projection of $\{\mathbf{H}_r, \boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_r\}$ with $2^k \leq s \leq 2^{k+1} - 1$ for some k ($0 \leq k \leq r - 1$), where $\boldsymbol{\gamma}$ is independent of the columns of \mathbf{H}_r . We have

- (i) If $\mathbf{D}_b \cap \{\boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_r\} = \emptyset$, then $\#\{U(\mathbf{D}_b) \cap \mathbf{H}_r\} \geq 2^{k+1} - 1$ and the equality holds when \mathbf{D}_b has $k + 1$ independent columns.
- (ii) If $\mathbf{D}_b \cap \{\boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_r\} \neq \emptyset$, then $\#\{U(\mathbf{D}_b) \cap \mathbf{H}_r\} \geq 2^k - 1$ and the equality holds when \mathbf{D}_b has $k + 1$ independent columns.
- (iii) If $\mathbf{D}_b \cap \{\boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_r\} \neq \emptyset$, then $\#\{U(\mathbf{D}_b) \cap \{\boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_r\}\} \geq 2^k$ and the equality holds when \mathbf{D}_b has $k + 1$ independent columns.
- (iv) If $\mathbf{D}_b \subset \mathbf{H}_{k+1}$, then $U(\mathbf{D}_b) = \mathbf{H}_{k+1}$.
- (v) If $\mathbf{D}_b \subset \{\mathbf{H}_k, \boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_k\}$, then $U(\mathbf{D}_b) = \{\mathbf{H}_k, \boldsymbol{\gamma}, \boldsymbol{\gamma}\mathbf{H}_k\}$.

Lemma 2 provides a necessary condition for a 2^{n-m} ; 2^s design $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ with $n = N/4 + 1$ to be a B^2 -GMC design.

Lemma 2. Suppose $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a 2^{n-m} ; 2^s design with $n = N/4 + 1$ and $2^k \leq s \leq 2^{k+1} - 1$ for some k ($0 \leq k \leq q - 3$); then, $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC design only if $\mathbf{D}_t = \mathbf{S}_{N/4+1}$.

Proof. Let $\tilde{\mathbf{D}} = (\mathbf{S}_{N/4+1}; \tilde{\mathbf{D}}_b)$ be a 2^{n-m} ; 2^s design and $\tilde{\mathbf{D}}_b \in \mathbf{H}_{q-2}$. From (8)–(10), we can obtain

$$\#_1 C_2(\tilde{\mathbf{D}}) = \left(\frac{N}{4} + 1, 0, \dots, 0 \right), \quad (11)$$

which is sequentially maximized by $\tilde{\mathbf{D}}$, and $\#_2 C_2^{(0)}(\tilde{\mathbf{D}}) = 2(N/4 - 1) + 1 = N/2 - 1$. Suppose that any $\mathbf{D} = (\mathbf{S}_{N/4+1}; \mathbf{D}_b)$ is not a B^2 -GMC 2^{n-m} ; 2^s design for $n = N/4 + 1$. Then,

by the definition of B^2 -GMC criterion, there should be a 2^{n-m} : 2^s design $\mathbf{D}^* = (\mathbf{D}_t^*; \mathbf{D}_b^*)$ outperforming $\tilde{\mathbf{D}} = (\mathbf{S}_{N/4+1}; \tilde{\mathbf{D}}_b)$ in terms of (4). This leads to

$$\#_1 C_2(\mathbf{D}^*) = \left(\frac{N}{4} + 1, 0, \dots, 0 \right), \quad (12)$$

and $\#_2 C_2^{(0)}(\mathbf{D}^*) \geq (N/2) - 1$. Clearly,

$$\#_1 C_2(\mathbf{D}_t^*) = \#_1 C_2(\mathbf{D}^*) = \left(\frac{N}{4} + 1, 0, \dots, 0 \right), \quad (13)$$

and $\#_2 C_2^{(0)}(\mathbf{D}_t^*) \geq \#_2 C_2^{(0)}(\mathbf{D}^*) \geq N/2 - 1$ noting that \mathbf{D}_t^* is the unblocked part of \mathbf{D}^* and \mathbf{D}_t^* has $n = N/4 + 1$ columns.

In fact, the inequality in the formula $\#_2 C_2^{(0)}(\mathbf{D}_t^*) \geq N/2 - 1$ is not valid. Recall that the MaxC2 design $\mathbf{S}_{N/4+1}$ has the largest number, $N/2 - 1$, of clear 2fi's among all the 2^{n-m} designs with $n = N/4 + 1$. Therefore, \mathbf{D}_t^* has at most $N/2 - 1$ clear 2fi's, i.e., $\#_2 C_2^{(0)}(\mathbf{D}_t^*) = N/2 - 1$. Wu and Wu [30] showed that a 2^{n-m} design with $n = N/4 + 1$ is a MaxC2 design if and only if this design has $N/2 - 1$ clear 2fi's. This obtains that $\mathbf{D}_t^* = \mathbf{S}_{N/4+1}$ up to isomorphism. \square

With Lemma 2, Theorem 1 provides the constructions of B^2 -GMC designs with $n = N/4 + 1$.

Theorem 1. Suppose $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a 2^{n-m} : 2^s design with $n = N/4 + 1$ and $2^k \leq s \leq 2^{k+1} - 1$ for some $k (0 \leq k \leq q - 3)$; then, $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC design if $\mathbf{D}_t = \mathbf{S}_{N/4+1}$ and \mathbf{D}_b is any s -projection of \mathbf{H}_{k+1} .

Proof. By Lemma 2, if $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC 2^{n-m} : 2^s design with $n = N/4 + 1$, then $\mathbf{D}_t = \mathbf{S}_{N/4+1}$ up to isomorphism. Thus, $\#_1 C_2(\mathbf{D}) = (N/4 + 1, 0, \dots, 0)$ which is sequentially maximized by $\mathbf{D} = (\mathbf{S}_{N/4+1}; \mathbf{D}_b)$. Let $u_1 = \#\{U(\mathbf{D}_b) \cap \{(\mathbf{q} - \mathbf{1})\mathbf{H}_{q-2}, \mathbf{q}\mathbf{H}_{q-2}, \mathbf{q}(\mathbf{q} - \mathbf{1})\}\}$ and $u_2 = \#\{U(\mathbf{D}_b) \cap \mathbf{H}_{q-2}\}$; then, from (10), we have

$$\#_2 C_2^{(p)}(\mathbf{D}) = \begin{cases} 2\left(\frac{N}{4} - 1\right) + 1 - u_1, & \text{for } p = 0, \\ \left(\frac{N}{4} - 1 - u_2\right)\left(\frac{N}{8} - 1\right), & \text{for } p = \frac{N}{8} - 2, \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

If \mathbf{D} is a B^2 -GMC design, then \mathbf{D} must sequentially minimize (u_1, u_2) . There are two different ways to choose \mathbf{D}_b from $\mathbf{H}_q \setminus \mathbf{S}_{N/4+1}$:

- (i) $\mathbf{D}_b \cap \{(\mathbf{q} - \mathbf{1})\mathbf{H}_{q-2}, \mathbf{q}\mathbf{H}_{q-2}, \mathbf{q}(\mathbf{q} - \mathbf{1})\} = \emptyset$.
- (ii) $\mathbf{D}_b \cap \{(\mathbf{q} - \mathbf{1})\mathbf{H}_{q-2}, \mathbf{q}\mathbf{H}_{q-2}, \mathbf{q}(\mathbf{q} - \mathbf{1})\} \neq \emptyset$.

It is not hard to verify that \mathbf{D}_b in case (ii) results in $u_1 > 0$ while \mathbf{D}_b in case (i) gives $u_1 = 0$. Therefore, if $\mathbf{D} = (\mathbf{S}_{N/4+1}; \mathbf{D}_b)$ is a B^2 -GMC design, then \mathbf{D}_b must be of case (i). Recall that $\mathbf{D}_b \cap \mathbf{S}_{N/4+1} = \emptyset$, and we have $\mathbf{D}_b \subset \mathbf{H}_{q-2}$. By Lemma 1 (i), if $\mathbf{D}_b \subset \mathbf{H}_{q-2}$, then $u_2 = \#\{U(\mathbf{D}_b) \cap \mathbf{H}_{q-2}\} \geq 2^{k+1} - 1$ and the equality holds when

$\mathbf{D}_b \subset \mathbf{H}_{k+1}$ up to isomorphism. This completes the proof. \square

The following example illustrates the construction method in Theorem 1.

Example 1. Consider the construction of the B^2 -GMC 2^{9-4} : 2^5 design. Here, $n = 9, m = 4, q = 5, N = 32$, and $s = 5$ which leads to $k = 2$. According to Theorem 1, let $\mathbf{D}_t = (4, 5, 45\mathbf{H}_3)$ and \mathbf{D}_b be any 5-projection of \mathbf{H}_3 , say $\mathbf{D}_b = (1, 2, 12, 3, 13)$. Then, $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC 2^{9-4} : 2^5 design.

3.2. B^2 -GMC Designs with $N/4 + 1 < n \leq 5N/16$. To sequentially maximize (4), the first part $\#_1 C_2(\mathbf{D})$ of (4) should be first maximized. Recall that $\#_1 C_2(\mathbf{D}) = \#_1 C_2(\mathbf{D}_t)$. If \mathbf{D}_t has resolution IV, then $\#_1 C_2(\mathbf{D})$ must be maximized. According to [31], when $N/4 + 1 < n \leq 5N/16$, the \mathbf{D}_t with resolution IV must be an n -projection of some second-order saturated (SOS) designs. In the following, we first review the concept of SOS design.

A 2^{n-m} design is called an SOS design if all of its degrees of freedom can be used to estimate only the main treatment effects and 2fi's. In terms of coding theory and projective geometry, Davydov and Tombak [32] showed that, given N , only the SOS designs of $N/4 + 1, N(2^{w-2} + 1)/2^w$ with $w \geq 4$ and $N/2$ factors exist. Block and Mee [31] further showed that an SOS design of $N(2^{w-2} + 1)/2^w$ factors can be obtained by doubling some smaller SOS $2^{(2^{w-2}+1)-(2^{w-2}+1-w)}$ designs $q - w$ times. Zhang and Cheng [33] showed that the SOS design of $N/2$ factors can be uniquely represented by $\mathbf{S}_{N/2} = (\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1})$ up to isomorphism. Let

$$\mathcal{L}(w) = \left\{ D^{q-w}(\mathbf{Y}) : \mathbf{Y} \text{ is an SOS } 2^{(2^{w-2}+1)-(2^{w-2}+1-w)} \text{ design} \right\}, \quad (15)$$

denote the collection of all the SOS designs obtained by doubling some SOS $2^{(2^{w-2}+1)-(2^{w-2}+1-w)}$ designs $q - w$ times. Especially, in $\mathcal{L}(w)$, we denote the SOS design obtained by doubling the MaxC2 $2^{(2^{w-2}+1)-(2^{w-2}+1-w)}$ design $\Phi(w) = ((\mathbf{w} - \mathbf{1})_{2^w}, \mathbf{w}_{2^w}, \mathbf{w}_{2^w}(\mathbf{w} - \mathbf{1})_{2^w}, \mathbf{H}_{q-w}^w)$ as $\mathbf{S}_{N(2^{w-2}+1)/2^w}$.

With a little algebra, it is easy to verify that

$$\begin{aligned} D^{q-w}(\mathbf{I}_{2^w}) &= (\mathbf{I}_{2^q}, \mathbf{H}_{q-w}^q) = (\mathbf{I}, \mathbf{H}_{q-w}) \\ D^{q-w}(\mathbf{1}_{2^w}) &= (\mathbf{q} - \mathbf{w} + \mathbf{1})_{2^q}(\mathbf{I}_{2^q}, \mathbf{H}_{q-w}^q) \\ &= (\mathbf{q} - \mathbf{w} + \mathbf{1})(\mathbf{I}, \mathbf{H}_{q-w}) \\ D^{q-w}(\mathbf{2}_{2^w}) &= (\mathbf{q} - \mathbf{w} + \mathbf{2})_{2^q}(\mathbf{I}_{2^q}, \mathbf{H}_{q-w}^q) \\ &= (\mathbf{q} - \mathbf{w} + \mathbf{2})(\mathbf{I}, \mathbf{H}_{q-w}) \\ &\vdots \\ D^{q-w}((\mathbf{w} - \mathbf{1})_{2^w}) &= (\mathbf{q} - \mathbf{1})_{2^q}(\mathbf{I}_{2^q}, \mathbf{H}_{q-w}^q) = (\mathbf{q} - \mathbf{1})(\mathbf{I}, \mathbf{H}_{q-w}) \\ D^{q-w}(\mathbf{w}_{2^w}) &= \mathbf{q}_{2^q}(\mathbf{I}_{2^q}, \mathbf{H}_{q-w}^q) = \mathbf{q}(\mathbf{I}, \mathbf{H}_{q-w}), \end{aligned} \quad (16)$$

where $\mathbf{I}_{2^{q-w}} \otimes \mathbf{1}_{2^w} = (\mathbf{q} - \mathbf{w} + \mathbf{1})_{2^q}, \mathbf{I}_{2^{q-w}} \otimes \mathbf{2}_{2^w} = (\mathbf{q} - \mathbf{w} + \mathbf{2})_{2^q}, \dots, \mathbf{I}_{2^{q-w}} \otimes (\mathbf{w} - \mathbf{1})_{2^w} = (\mathbf{q} - \mathbf{1})_{2^q}$, and $\mathbf{I}_{2^{q-w}} \otimes \mathbf{w}_{2^w} = \mathbf{q}_{2^q}$. Then,

$$D^{q-w}((\mathbf{w}-\mathbf{1})_{2^w} \mathbf{w}_{2^w} \mathbf{H}_{w-2}^w) = \mathbf{q}(\mathbf{q}-\mathbf{1})(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}). \quad (17)$$

Therefore, $\mathbf{S}_{N(2^{w-2}+1)/2^w}$ can be expressed as

$$\begin{aligned} \mathbf{S}_{N(2^{w-2}+1)/2^w} &= D^{q-w}(\Phi(w)) \\ &= ((\mathbf{q}-\mathbf{1}), (\mathbf{q}-\mathbf{1})\mathbf{H}_{q-w}, \mathbf{q}, \mathbf{q}\mathbf{H}_{q-w}, \mathbf{q}(\mathbf{q}-\mathbf{1})) \\ &\quad (\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}). \end{aligned} \quad (18)$$

The following lemma provides a necessary condition for a 2^{n-m} : 2^s design $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ with $N/4 + 1 < n \leq 5N/16$ to be a B^2 -GMC design.

Lemma 3. Suppose $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a 2^{n-m} : 2^s design with $N(2^{w-1} + 1)/2^{w+1} < n \leq N(2^{w-2} + 1)/2^w$ for $w \geq 4$ and $2^k \leq s \leq 2^{k+1} - 1$ for some $k(0 \leq k \leq q-3)$; then, $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC design only if $\mathbf{D}_t \subset \mathbf{S}_{N(2^{w-2}+1)/2^w}$.

Proof. As discussed in the first paragraph of this section, if $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC 2^{n-m} : 2^s design with $N(2^{w-1} + 1)/2^{w+1} < n \leq N(2^{w-2} + 1)/2^w$, then \mathbf{D}_t must be an n -projection of some SOS designs in $\mathcal{L}(v)$ with $4 \leq v \leq w$ or $\mathbf{S}_{N/2}$.

Let \mathbf{P}_1 be an n -projection of $\mathbf{S}_{N/2}$. According to Theorem 3 in [33], we obtain

$$\#_2 C_2^{(p)}(\mathbf{P}_1) = \begin{cases} 0, & \text{for } p < n - \frac{N}{4} - 1, \\ \frac{N}{4} \left(n - \frac{N}{4} \right), & \text{for } p = n - \frac{N}{4} - 1, \\ 0, & \text{for } n - \frac{N}{4} - 1 < p < \frac{N}{8} - 1. \end{cases} \quad (19)$$

Let $\mathbf{X}_v = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2^{v-2}+1})$ be any SOS $2^{(2^{v-2}+1)-(2^{v-2}+1-v)}$ design but not a MaxC2 design; then, $D^{q-v}(\mathbf{X}_v) \in \mathcal{L}(v)$. Denote p_i as the number of clear 2fi's of \mathbf{X}_v involving \mathbf{a}_i for $i = 1, 2, \dots, 2^{v-2} + 1$ and c as the total number of clear 2fi's of \mathbf{X}_v . Let \mathbf{P}_v be an n -projection of $D^{q-v}(\mathbf{X}_v)$ and $\overline{\mathbf{P}}_v = D^{q-v}(\mathbf{X}_v) \setminus \mathbf{P}_v$. Then, $\#\overline{\mathbf{P}}_v = N(2^{v-2} + 1)/2^v - n \leq N/2^v$. Applying equations (23) and (24) in the proof of Theorem 3.1 of [29], among all the n -projections of $D^{q-v}(\mathbf{X}_v)$, \mathbf{P}_v sequentially maximizes $\#_2 C_2$ in (5), only if $\overline{\mathbf{P}}_v \subset D^{q-v}(\mathbf{a}_j)$, where \mathbf{a}_j is the column such that $p_j = \max\{p_1, p_2, \dots, p_{2^{v-2}+1}\}$. For a \mathbf{P}_v with $\overline{\mathbf{P}}_v \subset D^{q-v}(\mathbf{a}_j)$, according to equation (24) in the proof of Theorem 3.1 of [29], it is obtained that

$$\#_2 C_2^{(p)}(\mathbf{P}_v) = \begin{cases} 0, & \text{for } 0 \leq p < n - \frac{N}{4} - 1, \\ \frac{p_j N}{2^v} (n - N/4), & \text{for } p = n - \frac{N}{4} - 1, \\ 0, & \text{for } n - \frac{N}{4} - 1 < p < \frac{N}{2^v} - 1, \\ (c - p_j) \left(\frac{N}{2^v} \right)^2, & \text{for } p = \frac{N}{2^v} - 1. \end{cases} \quad (20)$$

For a MaxC22 $^{(2^{v-2}+1)-(2^{v-2}+1-v)}$ design, it has $2^{v-1} - 1$ clear 2fi's and $p_j = 2^{v-2}$. Since \mathbf{X}_v is not a MaxC2 design, we have

$$\begin{aligned} p_j &= 2^{v-2}, \\ c - p_j &< 2^{v-1} - 2^{v-2} - 1, \end{aligned} \quad (21)$$

or

$$\begin{aligned} p_j &< 2^{v-2}, \\ c &< 2^{v-1} - 1. \end{aligned} \quad (22)$$

Let \mathbf{Q}_v be an n -projection of $\mathbf{S}_{N(2^{v-2}+1)/2^v}$ obtained by doubling $\Phi(v)$, the MaxC22 $^{(2^{v-2}+1)-(2^{v-2}+1-v)}$ design as mentioned above. With a column permutation, rewrite $D^q(\cdot)$ as

$$\begin{aligned} D_{RC}^q(\cdot) &= (\mathbf{H}_{q-v}, \mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}, (\mathbf{q}-\mathbf{1})(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}), \\ &\quad \mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}), \mathbf{q}(\mathbf{q}-\mathbf{1})(\mathbf{I}, \mathbf{H}_{q-v}), \mathbf{S}_{N(2^{v-2}+1)/2^v}), \end{aligned} \quad (23)$$

in a re-changed Yates order. Write $\Phi(v)$ as $\Phi(v) = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{2^{v-2}+1})$, where $\mathbf{s}_1 = (\mathbf{v}-\mathbf{1})_{2^v}$, $\mathbf{s}_2 = \mathbf{v}_{2^v}$ and $(\mathbf{s}_3, \dots, \mathbf{s}_{2^{v-2}+1}) = \mathbf{v}_{2^v}(\mathbf{v}-\mathbf{1})_{2^v} \mathbf{H}_{v-2}^v$. Applying equations (23) and (24) in the proof of Theorem 3.1 of [29], among all the n -projections of $\mathbf{S}_{N(2^{v-2}+1)/2^v}$, \mathbf{Q}_v sequentially maximizes $\#_2 C_2$ in (3), only if $\overline{\mathbf{Q}}_v \subset D^{q-v}(\mathbf{s}_i)$ with $i = 1$ or 2. In the following, we investigate $\#_2 C_2^{(p)}(\mathbf{Q}_v)$ with $\overline{\mathbf{Q}}_v \subset D^{q-v}(\mathbf{s}_1)$ for which the following analysis and final conclusion are the same as that for $\#_2 C_2^{(p)}(\mathbf{Q}_v)$ with $\overline{\mathbf{Q}}_v \subset D^{q-v}(\mathbf{s}_2)$. For \mathbf{Q}_v with $\overline{\mathbf{Q}}_v \subset D^{q-v}(\mathbf{s}_1)$, from equations (18) and (23), it is obtained that

$$B_2(\mathbf{Q}_v, \gamma) = \begin{cases} 0, & \text{for } \gamma \in \mathbf{S}_{N(2^{v-2}+1)/2^v}, \\ n - \frac{N}{4}, & \text{for } \gamma \in \{\mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}), \mathbf{q}(\mathbf{q}-1)(\mathbf{I}, \mathbf{H}_{q-v})\}, \\ \frac{N}{2^v}, & \text{for } \gamma \in (\mathbf{q}-1)(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}), \\ \frac{N}{8} - \frac{N}{2^v}, & \text{for } \gamma \in \mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}, \\ B_2(\overline{\mathbf{Q}}_v, \gamma) + \frac{N}{8}, & \text{for } \gamma \in \mathbf{H}_{q-v}, \end{cases} \quad (24)$$

where the second equality can be easily verified by noting that

$$\begin{aligned} & \{\mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-v}), \mathbf{q}(\mathbf{q}-1)(\mathbf{I}, \mathbf{H}_{q-v})\} \\ &= \{D^{q-v}(\mathbf{s}_1 \mathbf{s}_2), D^{q-v}(\mathbf{s}_1 \mathbf{s}_3), \dots, D^{q-v}(\mathbf{s}_1 \mathbf{s}_{2^{v-2}+1})\}. \end{aligned} \quad (25)$$

Therefore,

$$\# {}_2 C_2^{(p)}(\mathbf{Q}_v) = \begin{cases} 0, & \text{for } 0 \leq p < n - \frac{N}{4} - 1, \\ \frac{N}{4}(n - N/4), & \text{for } p = n - \frac{N}{4} - 1, \\ 0, & \text{for } n - \frac{N}{4} - 1 < p < \frac{N}{2^v} - 1, \\ \frac{N}{2^v}(N/4 - N/2^v), & \text{for } p = \frac{N}{2^v} - 1, \\ 0, & \text{for } \frac{N}{2^v} - 1 < p < \frac{N}{8} - \frac{N}{2^v} - 1, \\ \left(\frac{N}{4} - \frac{N}{2^v}\right)\left(\frac{N}{8} - \frac{N}{2^v}\right), & \text{for } p = \frac{N}{8} - \frac{N}{2^v} - 1. \end{cases} \quad (26)$$

Denote $\mathbf{D}^* = (\mathbf{D}_t^*; \mathbf{D}_b^*)$, where $\mathbf{D}_t^* \subset \mathbf{S}_{N(2^{w-2}+1)/2^w}$, $\mathbf{S}_{N(2^{w-2}+1)/2^w} \setminus \mathbf{D}_t^* \subset D^{q-w}(\mathbf{s}_1)$ and $\mathbf{D}_b^* \subset \mathbf{H}_{k+1}$. Since $k \leq q-3$,

we have $U(\mathbf{D}_b^*) \subset \mathbf{H}_{q-2}$. Therefore, from (24) and (26), we obtain

$$\# {}_2 C_2^{(p)}(\mathbf{D}^*) = \begin{cases} 0, & \text{for } 0 \leq p < n - \frac{N}{4} - 1, \\ \frac{N}{4}(n - N/4), & \text{for } p = n - \frac{N}{4} - 1, \\ 0, & \text{for } n - \frac{N}{4} - 1 < p < \frac{N}{2^w} - 1, \\ \frac{N}{2^w}(N/4 - N/2^w), & \text{for } p = \frac{N}{2^w} - 1. \end{cases} \quad (27)$$

Comparing (27) with (19), (20) for $4 \leq v \leq w$, and (26) for $4 \leq v \leq w - 1$, it is obtained that if $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC design, \mathbf{D} should not be worse than \mathbf{D}^* in terms of (3). Therefore, if $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ is a B^2 -GMC design, \mathbf{D}_t should be an n -projection of $\mathbf{S}_{N(2^{w-2}+1)/2^w}$. This completes the proof. \square

The lemma below is an extension of Lemma A.2, in Appendix, introduced from [34].

Lemma 4. Suppose \mathbf{D}_t consists of the last n , $N(2^{w-1} + 1)/2^{w+1} < n \leq N(2^{w-2} + 1)/2^w$, columns of $\mathbf{S}_{N(2^{w-2}+1)/2^w}$; then, $B_2(\mathbf{D}_t, \gamma_1) \geq B_2(\mathbf{D}_t, \gamma_2)$ if γ_1 is ahead of γ_2 in \mathbf{H}_{q-w} .

Proof. Suppose \mathbf{O} consists of the last $n - N/4$ columns of $D^{q-w}((\mathbf{w} - \mathbf{1})_{2^w})$. Then, $\mathbf{D}_t = \{\mathbf{O}, \mathbf{S}_{N(2^{w-2}+1)/2^w} \setminus D^{q-w}(\mathbf{w} - \mathbf{1})\}$. From (16) and (18), for any $\gamma \in \mathbf{H}_{q-w}$, $B_2(\mathbf{D}_t, \gamma) = B_2(\mathbf{O}, \gamma) + N/8$. Straightforwardly, from Lemma A.2, it is obtained that $B_2(\mathbf{O}, \gamma_1) \geq B_2(\mathbf{O}, \gamma_2)$ if γ_1 is ahead of γ_2 in \mathbf{H}_{q-w} . This completes the proof. \square

With Lemmas 3 and 4, Theorem 2 provides the construction methods of B^2 -GMC 2^{n-m} : 2^s designs with $N/4 + 1 < n \leq 5N/16$.

Theorem 2. Let $\mathbf{D} = (\mathbf{D}_t; \mathbf{D}_b)$ be a 2^{n-m} : 2^s design with $N(2^{w-1} + 1)/2^{w+1} < n \leq N(2^{w-2} + 1)/2^w$ for $w \geq 4$ and $2^k \leq s \leq 2^{k+1} - 1$ for some $0 \leq k \leq q - 3$. Suppose $2^r \leq N(2^{w-2} + 1)/2^w - n \leq 2^{r+1} - 1$ for some $0 \leq r \leq q - w - 1$; then, \mathbf{D} is a B^2 -GMC design if \mathbf{D}_t consists of the last n columns of $\mathbf{S}_{N(2^{w-2}+1)/2^w}$ and

- (a) \mathbf{D}_b is any s -projection of $\{\mathbf{H}_k, \mathbf{q} - \mathbf{1}, (\mathbf{q} - \mathbf{1})\mathbf{H}_k\}$ when $k \leq r \leq q - w - 1$.
- (b) \mathbf{D}_b is any s -projection of \mathbf{H}_{k+1} when $r + 1 \leq k \leq q - w - 1$.
- (c) \mathbf{D}_b is any s -projection of \mathbf{H}_{q-2} when $q - w - 1 < k \leq q - 3$.

Proof. By Lemma 3 and its proof, if $\mathbf{D}^* = (\mathbf{D}_t^*; \mathbf{D}_b^*)$ is a B^2 -GMC 2^{n-m} : 2^s design, there should be $\mathbf{D}_t^* \subset \mathbf{S}_{N(2^{w-2}+1)/2^w}$ and $\overline{\mathbf{D}}_t^* = \mathbf{S}_{N(2^{w-2}+1)/2^w} \setminus \mathbf{D}_t^* \subset D^{q-w}((\mathbf{w} - \mathbf{1})_{2^w})$. Substituting v with w in (24), we obtain

$$B_2(\mathbf{D}_t^*, \gamma) = \begin{cases} 0, & \text{for } \gamma \in \mathbf{S}_{(N(2^{w-2}+1)/2^w)}, \\ n - \frac{N}{4}, & \text{for } \gamma \in \{\mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}), \mathbf{q}(\mathbf{q} - \mathbf{1})(\mathbf{I}, \mathbf{H}_{q-w})\}, \\ \frac{N}{2^w}, & \text{for } \gamma \in (\mathbf{q} - \mathbf{1})(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}), \\ \frac{N}{8} - \frac{N}{2^w}, & \text{for } \gamma \in \mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}, \\ B_2(\overline{\mathbf{D}}_t^*, \gamma) + \frac{N}{8}, & \text{for } \gamma \in \mathbf{H}_{q-w}. \end{cases} \quad (28)$$

Let

$$\begin{aligned} u_1 &= \#\{U(\mathbf{D}_b^*) \cap \{\mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}), \mathbf{q}(\mathbf{q} - \mathbf{1})(\mathbf{I}, \mathbf{H}_{q-w})\}\}, \\ u_2 &= \#\{U(\mathbf{D}_b^*) \cap (\mathbf{q} - \mathbf{1})(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w})\}, \\ u_3 &= \#\{U(\mathbf{D}_b^*) \cap (\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w})\}, \end{aligned} \quad (29)$$

and then

$$\#_2 C_2^{(p)}(\mathbf{D}^*) = \begin{cases} \left(n - \frac{N}{4}\right)\left(\frac{N}{4} - u_1\right), & \text{for } p = n - \frac{N}{4} - 1, \\ \frac{N}{2^w}\left(\frac{N}{4} - \frac{N}{2^w} - u_2\right), & \text{for } p = \frac{N}{2^w} - 1, \\ \left(\frac{N}{8} - \frac{N}{2^w}\right)\left(\frac{N}{4} - \frac{N}{2^w} - u_3\right), & \text{for } p = \frac{N}{8} - \frac{N}{2^w} - 1, \\ f(\mathbf{D}_t^*, p), & \text{for } p \geq \frac{N}{8} - 1, \\ 0, & \text{otherwise,} \end{cases} \quad (30)$$

where $f(\mathbf{D}_t^*, p) = \#_2 C_2^{(p)}(\mathbf{D}_t^*) - (p+1)\#\{\boldsymbol{\gamma} : \boldsymbol{\gamma} \in \mathbf{H}_{q-w} \cap U(\mathbf{D}_b^*), B_2(\mathbf{D}_t^*, \boldsymbol{\gamma}) = p+1\}$. Therefore, \mathbf{D}^* sequentially maximizes $\#_2 C_2$ among all the possible 2^{n-m} : 2^s designs, only if \mathbf{D}^* sequentially minimizes

$$(u_1, u_2, u_3). \quad (31)$$

For ease of presenting, let $\mathbf{A} = \{\mathbf{q}(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}), \mathbf{q}(\mathbf{q}-\mathbf{1})(\mathbf{I}, \mathbf{H}_{q-w})\}$, $\mathbf{B} = (\mathbf{q}-\mathbf{1})(\mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w})$, and $\mathbf{C} = \mathbf{H}_{q-2} \setminus \mathbf{H}_{q-w}$.

For (a), when $k \leq q-w-1$, there are two different ways to choose \mathbf{D}_b^* from \mathbf{H}_q : (i) $\mathbf{D}_b^* \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} = \emptyset$ and (ii) $\mathbf{D}_b^* \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \neq \emptyset$. Recall that if $\mathbf{D}^* = (\mathbf{D}_t^* : \mathbf{D}_b^*)$ is a B^2 -GMC design, then $\mathbf{D}_t^* \subset \mathbf{S}_{N(2^{w-2}+1)2^w}$ with $\overline{\mathbf{D}_t^*} = \mathbf{S}_{N(2^{w-2}+1)2^w} \setminus \mathbf{D}_t^* \subset D^{q-w}((\mathbf{w}-\mathbf{1})_{2^w})$. Case (i) implies that $\mathbf{D}_b^* \subset \{\mathbf{H}_{q-w}, \overline{\mathbf{D}_t^*}\}$, and thus $\mathbf{D}_b^* \subset \{\mathbf{H}_{q-w}, \mathbf{q}-\mathbf{1}, (\mathbf{q}-\mathbf{1})\mathbf{H}_{q-w}\}$. Therefore, choosing \mathbf{D}_b^* according to case (i) results in $u_1 = u_2 = u_3 = 0$. If \mathbf{D}_b^* is chosen according to case (ii), then $U(\mathbf{D}_b^*) \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \neq \emptyset$ which results in $u_1 > 0$, $u_2 > 0$, or $u_3 > 0$. Clearly, if \mathbf{D}^* is a B^2 -GMC design, \mathbf{D}_b^* should be

chosen as (i). In the following, we consider only $\mathbf{D}_b^* \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} = \emptyset$, i.e., $\mathbf{D}_b^* \subset \{\mathbf{H}_{q-w}, (\mathbf{q}-\mathbf{1}), (\mathbf{q}-\mathbf{1})\mathbf{H}_{q-w}\}$.

For $\mathbf{D} = (\mathbf{D}_t : \mathbf{D}_b)$ in (a), according to (v) in Lemma 1, if $\mathbf{D}_b \subset \{\mathbf{H}_k, (\mathbf{q}-\mathbf{1}), (\mathbf{q}-\mathbf{1})\mathbf{H}_k\}$ with $k \leq q-w-1$, then $U(\mathbf{D}_b) = \{\mathbf{H}_k, (\mathbf{q}-\mathbf{1}), (\mathbf{q}-\mathbf{1})\mathbf{H}_k\}$. Denote \mathbf{b} as the last column of \mathbf{H}_k in Yates order, and $B_2(\mathbf{D}_t, \mathbf{b}) = b$; then, $b \geq N/8$. According to Lemma 4, for $\boldsymbol{\gamma} \in \mathbf{H}_{q-w}$, if $\boldsymbol{\gamma}$ is ahead of \mathbf{b} in \mathbf{H}_{q-w} , then $B_2(\mathbf{D}_t, \boldsymbol{\gamma}) \geq b$ by (18). For any $\boldsymbol{\gamma} \in \{(\mathbf{q}-\mathbf{1}), (\mathbf{q}-\mathbf{1})\mathbf{H}_k\}$, we have $B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = 0$. For any $\boldsymbol{\gamma}$ such that $B_2(\mathbf{D}_t, \boldsymbol{\gamma}) < b$, we have $\boldsymbol{\gamma} \in \mathbf{H}_q \setminus \mathbf{H}_k$. Therefore, for $p \leq b-2$,

$$\begin{aligned} \#_2 C_2^{(p)}(\mathbf{D}) &= (p+1)\#\{\boldsymbol{\gamma} \in \mathbf{H}_q : \boldsymbol{\gamma} \notin U(\mathbf{D}_b), B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= (p+1)\#\{\boldsymbol{\gamma} \in \mathbf{H}_q : B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= \#_2 C_2^{(p)}(\mathbf{D}_t), \end{aligned} \quad (32)$$

and for $p \geq b$,

$$\begin{aligned} \#_2 C_2^{(p)}(\mathbf{D}) &= (p+1)\#\{\boldsymbol{\gamma} \in \mathbf{H}_q : \boldsymbol{\gamma} \notin U(\mathbf{D}_b), B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= (p+1)\#\{\boldsymbol{\gamma} \in \mathbf{H}_q : B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} - (p+1)\#\{\boldsymbol{\gamma} \in \mathbf{H}_k : B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= 0. \end{aligned} \quad (33)$$

From (33), it is obtained that

$$\begin{aligned} & \left(\#_2 C_2^{(0)}(\mathbf{D}), \dots, \#_2 C_2^{(N/8-1)}(\mathbf{D}), \#_2 C_2^{(N/8)}(\mathbf{D}), \dots, \#_2 C_2^{(b-2)}(\mathbf{D})\right) \\ &= \left(\#_2 C_2^{(0)}(\mathbf{D}_t), \dots, \#_2 C_2^{(N/8-1)}(\mathbf{D}_t), \#_2 C_2^{(N/8)}(\mathbf{D}_t), \dots, \#_2 C_2^{(b-2)}(\mathbf{D}_t)\right). \end{aligned} \quad (34)$$

Note that \mathbf{D}_t is a GMC design [29]; then, \mathbf{D} maximizes

$$\left(\#_2 C_2^{(0)}(\cdot), \dots, \#_2 C_2^{(N/8-1)}(\cdot), \#_2 C_2^{(N/8)}(\cdot), \dots, \#_2 C_2^{(b-2)}(\cdot)\right), \quad (35)$$

among all the possible 2^{n-m} : 2^s designs.

Suppose \mathbf{D} is not a B^2 -GMC design; then, there should be a $\mathbf{D}^* = (\mathbf{D}_t^* : \mathbf{D}_b^*)$ which outperforms $\mathbf{D} = (\mathbf{D}_t : \mathbf{D}_b)$ in terms of (3). This implies that there exists some $p_1 \geq b-1$ such that

$$\begin{aligned} & \left(\#_2 C_2^{(0)}(\mathbf{D}^*), \dots, \#_2 C_2^{(N/8-1)}(\mathbf{D}^*), \#_2 C_2^{(N/8)}(\mathbf{D}^*), \dots, \#_2 C_2^{(b-2)}(\mathbf{D}^*) \right) \\ & = \left(\#_2 C_2^{(0)}(\mathbf{D}), \dots, \#_2 C_2^{(N/8-1)}(\mathbf{D}), \#_2 C_2^{(N/8)}(\mathbf{D}), \dots, \#_2 C_2^{(b-2)}(\mathbf{D}) \right), \end{aligned} \quad (36)$$

$$\#_2 C_2^{(p_1-1)}(\mathbf{D}^*) > \#_2 C_2^{(p_1-1)}(\mathbf{D}). \quad (37)$$

Recalling the definitions of $\#_2 C_2^{(p)}(\mathbf{D})$ and $B_2(\mathbf{D}_t, \boldsymbol{\gamma})$, we have

$$\begin{aligned} \sum_{p=0}^P \frac{\#_2 C_2^{(p)}(\mathbf{D})}{(p+1)} &= \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in \mathbf{H}_q: \boldsymbol{\gamma} \notin U(\mathbf{D}_b), B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in (\mathbf{H}_q \setminus \mathbf{S}_{N(2^{w-2}+1)/2^w}): B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &\quad - \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in U(\mathbf{D}_b): B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = p+1\} \\ &= \#(\mathbf{H}_q \setminus \mathbf{S}_{N(2^{w-2}+1)/2^w}) - \#\mathbf{H}_k \\ &= 2^q - \frac{N(2^{w-2}+1)}{2^w} - 2^k, \end{aligned} \quad (38)$$

where the second and third equalities are due to $B_2(\mathbf{D}_t, \boldsymbol{\gamma}) = 0$ for any $\boldsymbol{\gamma} \in \mathbf{S}_{N(2^{w-2}+1)/2^w}$. Similarly,

$$\begin{aligned} \sum_{p=0}^P \frac{\#_2 C_2^{(p)}(\mathbf{D}^*)}{(p+1)} &= \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in \mathbf{H}_q: \boldsymbol{\gamma} \notin U(\mathbf{D}_b^*), B_2(\mathbf{D}_t^*, \boldsymbol{\gamma}) = p+1\} \\ &= \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in (\mathbf{H}_q \setminus \mathbf{S}_{N(2^{w-2}+1)/2^w}): B_2(\mathbf{D}_t^*, \boldsymbol{\gamma}) = p+1\} \\ &\quad - \sum_{p=0}^P \#\{\boldsymbol{\gamma} \in U(\mathbf{D}_b^*): B_2(\mathbf{D}_t^*, \boldsymbol{\gamma}) = p+1\}, \\ &= \#(\mathbf{H}_q \setminus \mathbf{S}_{N(2^{w-2}+1)/2^w}) - \#U(\mathbf{D}_b^*), \\ &= 2^q - \frac{N(2^{w-2}+1)}{2^w} - \#U(\mathbf{D}_b^*). \end{aligned} \quad (39)$$

From (33) and (38), we have

$$\begin{aligned} \sum_{p=0}^P \frac{\#_2 C_2^{(p)}(\mathbf{D})}{(p+1)} &= \sum_{p=0}^{b-2} \frac{\#_2 C_2^{(p)}(\mathbf{D})}{(p+1)} + \frac{\#_2 C_2^{(b-1)}(\mathbf{D})}{b} \\ &= 2^q - \frac{N(2^{w-2}+1)}{2^w} - 2^k. \end{aligned} \quad (40)$$

By (36) and (37), it is obtained that

$$\begin{aligned} \sum_{p=0}^p \frac{\# {}_2C_2^{(p)}(\mathbf{D}^*)}{(p+1)} &\geq \sum_{p=0}^{b-2} \frac{\# {}_2C_2^{(p)}(\mathbf{D}^*)}{(p+1)} + \frac{\# {}_2C_2^{(b-1)}(\mathbf{D}^*)}{b} \\ &> \sum_{p=0}^{b-2} \frac{\# {}_2C_2^{(p)}(\mathbf{D})}{(p+1)} + \frac{\# {}_2C_2^{(b-1)}(\mathbf{D})}{b}. \end{aligned} \quad (41)$$

Then, according to (39)–(41), we obtain $\#U(\mathbf{D}_b^*) < 2^k - 1$ which contradicts Lemma 1 (i) and (ii).

For (b), similar to (a), when $k \leq q - w - 1$, there are two different ways to choose \mathbf{D}_b^* from \mathbf{H}_q : (i) $\mathbf{D}_b^* \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} = \emptyset$ and (ii) $\mathbf{D}_b^* \cap \{\mathbf{A}, \mathbf{B}, \mathbf{C}\} \neq \emptyset$. With a similar argument to (a), case (i) results in $u_1 = u_2 = u_3 = 0$ while case (ii) results in $u_1 > 0, u_2 > 0$ or $u_3 > 0$. Therefore, if \mathbf{D}^* is a B^2 -GMC design, \mathbf{D}_b^* should be chosen according to case (i), i.e., $\mathbf{D}_b^* \subset \{\mathbf{H}_{q-w}, (\mathbf{q} - \mathbf{1}), (\mathbf{q} - \mathbf{1})\mathbf{H}_{q-w}\}$. When choosing \mathbf{D}_b^* according to (i), there should be $\mathbf{D}_b^* \cap \{\mathbf{q} - \mathbf{1}, (\mathbf{q} - \mathbf{1})\mathbf{H}_{q-w}\} = \emptyset$, i.e., $\mathbf{D}_b^* \subset \mathbf{H}_{q-w}$; otherwise, $\mathbf{D}_b^* \cap U(\mathbf{D}_b^*) \neq \emptyset$ noting that $\#\{U(\mathbf{D}_b^*) \cap (\mathbf{q} - \mathbf{1}, (\mathbf{q} - \mathbf{1})\mathbf{H}_{q-w})\} \geq 2^k > 2^{r+1} - 1$ as shown in Lemma 1 (iii). The remainder of the proof is similar to that of (a).

For (c), when $k > q - w - 1$, there are two different ways to choose \mathbf{D}_b^* from \mathbf{H}_q : (i) $\mathbf{D}_b^* \cap \{\mathbf{B}, \mathbf{C}\} = \emptyset$ and (ii) $\mathbf{D}_b^* \cap \{\mathbf{B}, \mathbf{C}\} \neq \emptyset$. With a similar argument to (a), it is not hard to verify that case (ii) results in $u_1 > 0$ or $u_2 > 0$ while case (i) results in $u_1 = u_2 = 0$. Therefore, if \mathbf{D}^* is a B^2 -GMC design, \mathbf{D}_b^* should be an s -projection of $\mathbf{H}_{q-2} = \{\mathbf{B}, \mathbf{C}\}$. The remainder of the proof is similar to that of (a). \square

Remark 1. Theorem 2 shows that when constructing the B^2 -GMC designs with $N/4 + 1 < n \leq 5N/16$, we can first partition the range $(N/4 + 1, 5N/16)$ into $q - 4$ sequential subranges as $(N(2^{w-1} + 1)/2^{w+1}, N(2^{w-2} + 1)/2^w)$ with $w = 4, 5, \dots, q - 1$ and then obtain the B^2 -GMC designs according to Theorem 2.

In the following, an example is provided to illustrate the construction method in Theorem 2 and Remark 1.

Example 2. Consider the constructions of B^2 -GMC 2^{19-13} : 2^s designs for $s = 1, 3, 4$. Here, $n = 19, m = 13, q = 6$, and $N = 64$. The values of the parameters N and n satisfy $N/4 + 1 < n \leq 5N/16$, i.e., $n \in (17, 20]$. From Remark 1, we partition the range $(17, 20]$ into two subranges $(17, 18]$ because $w = 5$ and $(18, 20]$ because $w = 4$. Since $19 \in (18, 20]$, according to Theorem 2, \mathbf{D}_t should be the last 19 columns of $\mathbf{S}_{5N/16}$, where $\mathbf{S}_{5N/16} = D^{q-4}(\Phi(4)) = (5, 5\mathbf{H}_2, 6, 6\mathbf{H}_2, 56(\mathbf{H}_4 \setminus \mathbf{H}_2))$ from (18). Therefore, $\mathbf{D}_t = (5\mathbf{H}_2, 6, 6\mathbf{H}_2, 56(\mathbf{H}_4 \setminus \mathbf{H}_2))$. Note that $5N/16 - n = 20 - 19 = 1$; then, $r = 0$. Next, we choose \mathbf{D}_b for $s = 1, 3, 4$.

For $s = 1$, we have $k = 0$. According to Theorem 2 (a), $\mathbf{D}_b = \mathbf{5}$ as $\mathbf{H}_0 = \emptyset$.

For $s = 3$, we have $k = 1$. According to Theorem 2 (b), \mathbf{D}_b should be a 3-projection of \mathbf{H}_2 , i.e., $\mathbf{D}_b = \mathbf{H}_2 = (\mathbf{1}, \mathbf{2}, \mathbf{12})$.

For $s = 4$, we have $k = 2$. According to Theorem 2 (c), \mathbf{D}_b should be a 4-projection of \mathbf{H}_4 . Without loss of generality, we choose $\mathbf{D}_b = (\mathbf{1}, \mathbf{2}, \mathbf{12}, \mathbf{3})$.

4. Concluding Remarks

The regular 2^{n-m} designs have wide applications in engineering, manufacturing industry, agriculture and medicine, and so on. When the size of experimental units is large, the inhomogeneity of experimental units results in unwanted variances of estimations of treatment effects. An essential way to solve this problem is to partition the experimental units into blocks.

There are two kinds of blocking problems as pointed out in [1]. One is called the single block variable problem which considers only a single block variable, and the other is called multi-block variable problem which considers two or more block variables. As stated in Section 1, experiments which involve multi-block variables are more widely concerned in practice than those which involve only a single block variable. However, due to the complexity of multi-block variable problem, the studies on constructing optimal designs with multi-block variables are relatively rare.

In this paper, we aim at exploring the theories and constructions of optimal blocked 2^{n-m} designs with multi-block variables. The prevalent B^2 -GMC criterion is adopted. This criterion is preferable when there is some prior knowledge on the importance ordering of the treatment effects. The systematical construction methods of the B^2 -GMC 2^{n-m} : 2^s designs with $N/4 + 1 \leq n \leq 5N/16$ are developed. The construction methods are concise and easy to implement as indicated by the examples provided.

Appendix

Lemma A.1 is some result from Lemmas A.2, A.4, and A.5 in [21] and Lemma 1 in [25].

Lemma A.1. Suppose \mathbf{O} is any s -projection of \mathbf{H}_q with $2^k \leq s \leq 2^{k+1} - 1$ and $k \leq q - 2$.

- (i) If $\mathbf{O} \cap \{\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1}\} = \emptyset$, then $\#\{U(\mathbf{O}) \cap \mathbf{H}_{q-1}\} \geq 2^{k+1} - 1$, and when \mathbf{O} has $k + 1$ independent columns, the equality holds.
- (ii) If $\mathbf{O} \cap \{\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1}\} \neq \emptyset$, then $\#\{U(\mathbf{O}) \cap \mathbf{H}_{q-1}\} \geq 2^k - 1$, and when \mathbf{O} has $k + 1$ independent columns, the equality holds.
- (iii) If $\mathbf{O} \cap \{\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1}\} \neq \emptyset$, then $\#\{U(\mathbf{O}) \cap \{\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1}\}\} \geq 2^k$, and when \mathbf{O} has $k + 1$ independent columns, the equality holds.
- (iv) If $\mathbf{O} \subset \mathbf{H}_{k+1}$, then $U(\mathbf{O}) = \mathbf{H}_{k+1}$.
- (v) If $\mathbf{O} \subset \{\mathbf{H}_k, \mathbf{q}, \mathbf{q}\mathbf{H}_k\}$, then $U(\mathbf{O}) = \{\mathbf{H}_k, \mathbf{q}, \mathbf{q}\mathbf{H}_k\}$.

Lemma A.2 is some result of Lemma 1 in [34].

Lemma A.2. Suppose \mathbf{O} consists of the last n columns of $\{\mathbf{q}, \mathbf{q}\mathbf{H}_{q-1}\}$; then, $B_2(\mathbf{O}, \boldsymbol{\gamma}_1) \geq B_2(\mathbf{O}, \boldsymbol{\gamma}_2)$ if $\boldsymbol{\gamma}_1$ is ahead of $\boldsymbol{\gamma}_2$ in \mathbf{H}_{q-1} .

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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