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GENERAL NATURAL RIEMANNIAN ALMOST PRODUCT AND PARA-HERMITIAN STRUCTURES ON TANGENT BUNDLES

Simona-Luiza Druţă-Romaniuc

Abstract. We find the almost product (locally product) structures of general natural lift type on the tangent bundle of a Riemannian manifold. We get the conditions under which the tangent bundle endowed with such a structure and with a general natural lifted metric is a Riemannian almost product (locally product) or an (almost) para-Hermitian manifold. We give a characterization of the general natural (almost) para-Hermitian structures, which are (almost) para-Kählerian on the tangent bundle.

1. Introduction

Geometric structures on (the total space of) fiber bundles have been object of much study since the middle of the last century. The latest papers in this field are related to natural fiber bundles over manifolds (e.g. see [1, 14-16, 21-28, 32]). The natural lifts of the metric g, from a Riemannian manifold (M, g) to (the total space of) its tangent or cotangent bundles, induce new (pseudo) Riemannian structures, with interesting geometric properties, some of them studied by the present author (e.g. in [9, 10]). To avoid a possible confusion, we mention that in the sequel the total space of the tangent bundle will be frequently called tangent bundle.

Maybe the best known Riemannian metric on the tangent bundle is that introduced by Sasaki in 1958 (see [31]), but in most cases the study of some geometric properties of the tangent bundle endowed with this metric led to the flatness of the base manifold. In the next years, the authors were interested in finding other lifted structures on the tangent bundles, with quite interesting properties. The results in [14-16], concerning the natural lifts, allowed the extension of the Sasaki metric to the metrics of general natural lifted type (see [25]), leading to interesting geometric structures studied in the last years (see [1, 26-28]), and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics (e.g. see [6]).

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Many authors considered almost product structures and almost para-Hermitian structures (called also almost hyperbolic Hermitian structures) on the tangent and cotangent bundles (see [4, 5, 7, 8, 12, 13, 18, 29, 33]).

The theory of Riemannian almost product structures was initiated by K. Yano in [33]. In 1983, A. M. Naveira made a classification of these structures, with respect to their covariant derivatives, obtaining 36 classes (see [24]). Having in mind these results, M. Staikova and K. Gribachev obtained, in 1992, a classification of the Riemannian almost product structures, for which the trace vanishes (see [30]). The basic class corresponds to the nonintegrable almost product structures, studied in some recent articles, such as [17].

In 1988, C. Bejan gave a classification of the almost para-Hermitian structures. She obtained 36 classes, up to duality, and the characterizations of some of them (see [3]). A classification à la Gray-Hervella was given in 1991, by P. M. Gadea and J. Muñoz Masqué, in [11], where 136 classes, up to duality, were obtained.

The present paper aims to determine the almost product structures P of general natural lift type on the tangent bundle TM of a Riemannian manifold M and the conditions under which the tangent bundle endowed with a structure of this type and with a general natural lifted metric G is a Riemannian almost product (locally product) manifold or an (almost) para-Hermitian manifold.

The best known class of almost para-Hermitian structures is that of the almost para-Kähler structures (see [2]), characterized by the closure of the associated 2-form. In the last section of this paper we shall characterize the general natural (almost) para-Kähler structures on the tangent bundle of a Riemannian manifold.

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r, being always $\{1, \dots, n\}$.

2. Preliminary Results

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. The total space TM has a structure of $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using the local charts on TM induced from the local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates x^i, y^j, for $i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates y^j, $j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U,φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of *M-tensor field* on it (see [20]), called by R. Miron and his collaborators *distinguished tensor field* or *d-tensor field* (e.g. see [6] and [19]).

Denote by $\dot{\nabla}$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM defined by $\dot{\nabla}$ (see [34]). The set of vector fields $\{\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1},\dots,\frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}$, $\Gamma^h_{0i} = y^k\Gamma^h_{ki}$, and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\delta_i,\partial_j\}_{i,j=\overline{1,n}}$, defines a local frame on TM, adapted to the direct sum decomposition (2.1).

Consider the energy density of the tangent vector y with respect to the Riemannian metric g,

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U). \tag{2.2}$$

Obviously, we have $t \in [0,\infty)$ for all $y \in TM$.

We shall use the following lemma, which may be proved easily.

Lemma 2.1. *If $n > 1$ and u, v are smooth functions on TM such that*
$ug_{ij} + vg_{0i}g_{0j} = 0$, *or* $u\delta^j_i + vy^jg_{0i} = 0$,
on the domain of any induced local chart on TM, then $u = 0$, $v = 0$. We used the notation $g_{0i} = y^hg_{hi}$.

3. Almost Product Structures of General Natural Lifted Type on TM

In this section we shall construct an $(1,1)$-tensor field P, obtained as general natural lift of the metric g, from the base manifold to the tangent bundle TM, and we shall get the conditions under which P defines an almost product structure on the tangent bundle.

Let us recall some general definitions concerning almost product and almost paracomplex manifolds.

An *almost product structure* J on a differentiable manifold M is an $(1,1)$-tensor field on M such that $J^2 = I$. The pair (M,J) is called an *almost product manifold*. An integrable almost product manifold is usually called a *locally product manifold*.

An *almost paracomplex manifold* is an almost product manifold (M, J), such that the two eigenbundles associated to the two eigenvalues $+1$ and -1 of J, respectively, have the same rank. Equivalently, a splitting of the tangent bundle TM into the Whitney sum of two subbundles $T^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure* on M.

Authors like C. Bejan, V. Cruceanu, A. Heydari, S. Ianuş, S. Ishihara, I. Mihai, C. Nicolau, V. Oproiu, L. Ornea, N. Papaghiuc, E. Peyghan, and K. Yano, considered almost product structures on the tangent and cotangent bundles of a manifold M.

Let ∇ be a linear connection on M and denote by X^V and X^H the vertical and horizontal lift of the vector field $X \in \mathcal{T}_0^1(M)$ to the tangent bundle TM. The simplest almost product structures on TM, denoted by P and Q are defined by the relations

$$P(X^V) = X^V, \quad P(X^H) = -X^H, \tag{3.1}$$

$$Q(X^V) = X^H, \quad Q(X^H) = X^V. \tag{3.2}$$

The structure P is a paracomplex structure if and only if ∇ has vanishing curvature, while Q is paracomplex if and only if ∇ has both vanishing torsion and curvature. These structures have been extended to the case of a nonlinear connection, and to the specific case of a nonlinear connection defined by a Finsler, Lagrange or Hamilton structure.

Let us introduce a natural 1st order lift of g to TM, by the relation:

$$\begin{cases} PX^H_y = a_1(t)X^V_y + b_1(t)g_{\tau(y)}(X,y)y^V_y - a_4(t)X^H_y - b_4(t)g_{\tau(y)}(X,y)y^H_y, \\ PX^V_y = a_3(t)X^V_y + b_3(t)g_{\tau(y)}(X,y)y^V_y + a_2(t)X^H_y + b_2(t)g_{\tau(y)}(X,y)y^H_y, \end{cases} \tag{3.3}$$

$\forall X \in \mathcal{T}_0^1(TM)$, $\forall y \in TM$, where a_α and b_α are smooth functions of the energy density t, for $\alpha = \overline{1,4}$.

Remark that in the case when $a_3(t) = a_4(t) = 1$, and the other coefficients vanish, we have the structure given by (3.1), and when the only nonzero coefficients are $a_1(t) = a_2(t) = 1$, we get the structure defined by (3.2).

In the following, all the computations are done by using the expressions with respect to the adapted frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$ on TM. Thus (3.3) becomes

$$P\delta_i = ({}^1P)^j_i\partial_j - ({}^3P)^j_i\delta_j, \; P\partial_i = ({}^4P)^j_i\partial_j + ({}^2P)^j_i\delta_j, \tag{3.4}$$

where the $M-$tensor fields involved as coefficients have the forms

$$({}^\alpha P)^j_i = a_\alpha(t)\delta^j_i + b_\alpha(t)y^j g_{0i}, \forall\alpha = \overline{1,4}. \tag{3.5}$$

The matrix associated to P with respect to the adapted frame $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$ is

$$P = \begin{pmatrix} -({}^3P)^j_i & ({}^1P)^j_i \\ ({}^2P)^j_i & ({}^4P)^j_i \end{pmatrix}. \tag{3.6}$$

In the following theorem we give the conditions under which the above $(1,1)-$tensor field P is an almost product structure on the tangent bundle.

Theorem 3.1. *The natural tensor field P of type $(1,1)$ on TM, given by* (3.3), *defines an almost product structure on TM, if and only if $a_4 = a_3,\ b_4 = b_3$, and the coefficients $a_1,\ a_2,\ a_3,\ b_1,\ b_2$ and b_3 are related by*

$$a_1a_2 = 1 - a_3^2\ , \quad (a_1+2tb_1)(a_2+2tb_2) = 1-(a_3+2tb_3)^2. \tag{3.7}$$

Proof. We have to solve the following system

$$\begin{cases} ({}^3P)^j_i({}^3P)^i_l + ({}^1P)^j_i({}^2P)^i_l = \delta^j_l,\ -({}^3P)^j_i({}^1P)^i_l + ({}^1P)^j_i({}^4P)^i_l = 0, \\ -({}^2P)^j_i({}^3P)^i_l + ({}^4P)^j_i({}^2P)^i_l = 0, ({}^2P)^j_i({}^1P)^i_l + ({}^4P)^j_i({}^4P)^i_l = \delta^j_l. \end{cases}$$

Replacing (3.5) and taking into account the relation $y^ig_{0i} = 2t$, the second and third equation become, respectively

$$-a_1(a_3-a_4)\delta^j_l + [b_1(a_4-a_3)-(b_3-b_4)(a_1+2tb_1)]y^jg_{0l} = 0,$$

$$-a_2(a_3-a_4)\delta^j_l + [b_2(a_4-a_3)-(b_3-b_4)(a_2+2tb_2)]y^jg_{0l} = 0.$$

Using Lemma 2.1, both relations lead to $a_3 = a_4$ and $b_3 = b_4$. Then, the first and fourth equations in the studied system take the same form

$$(a_1a_2+a_3^2-1)\delta^j_l + [b_1(a_2+2tb_2)+a_1b_2+a_3b_3+b_3(a_3+2tb_3)]y^jg_{0l} = 0. \tag{3.8}$$

This relation holds if and only if the two coefficients involved are zero. From the vanishing condition of the first one, we have the first relation in (3.7). We obtain also the second relation in the statement of the theorem if to the first coefficient we add the second one, multiplied by $2t$. Thus the theorem is proved.

Remark 3.2. If we take $a_3 = b_1 = b_2 = b_3 = 0$ and some particular values for a_1 and a_2, such that $a_1a_2 = 1$, we obtain the almost product structures studied in [12] and [29].

Now we characterize the general natural locally product structures on the tangent bundle, i.e. the integrable general natural almost product structures on TM.

Theorem 3.3. *Let (M,g) be an n-dimensional connected Riemannian manifold, with $n>2$. The general natural almost product structure P on TM is integrable (i.e. P is a locally product structure on TM) if and only if (M,g) has constant sectional curvature c, and the coefficients $b_1,\ b_2,\ b_3$ are given by:*

$$b_1 = \frac{a_1^2 a_1' + a_1 c + 3a_1 a_3^2 c - 2a_1^2 a_2' ct + 2a_2 c^2 t - 2a_2 a_3^2 c^2 t}{a_1(a_1 - 2a_1' t + 2a_2 ct + 4a_2' ct^2)},$$

$$b_2 = \frac{1}{2a_3}[2a_2' a_3 - a_2 a_3' - \frac{(a_2+2a_2' t)(2a_1 a_2 a_3 c - a_1^2 a_3' - 4a_1' a_2 a_3 ct - 2a_3' ct - 6a_3^2 a_3' ct)}{a_1^2 - 2a_1 a_1' t + 2a_1 a_2 ct - 4a_1' a_2 ct^2 - 8a_3 a_3' ct^2}],$$

$$b_3 = \frac{a_1^2 a_3' - 2a_1 a_2 a_3 c + 4a_1' a_2 a_3 ct + 2a_3' ct + 6a_3^2 a_3' ct}{a_1^2 - 2a_1 a_1' t + 2a_1 a_2 ct - 4a_1' a_2 ct^2 - 8a_3 a_3' ct^2}.$$

Proof. The integrability of an almost product structure P on TM is characterized by the vanishing condition of its Nijenhuis tensor field N_P, defined by

$$N_P(X,Y) = [PX, PY] - P[PX, Y] - P[X, PY] + [X, Y],$$

for all the vector fields X and Y on TM.

From the condition $N_P(\partial_i, \partial_j) = 0$ we obtain, by using Lemma 2.1, that the horizontal component of this Nijenhuis bracket vanishes if and only if

$$a_2' = \frac{a_2 a_3' + 2a_3 b_2 - a_2 b_3}{2(a_3 + tb_3)}, \tag{3.9}$$

and the vertical component vanishes if and only if

$$\begin{aligned}&(a_1 a_2' - a_1 b_2 - 2a_3' b_3 t)(\delta_i^h g_{0j} - \delta_j^h g_{0i}) - a_2^2 y^k R_{kij}^h \\ &+ a_2 b_2 y^k y^l (g_{0i} R_{kjl}^h - g_{0j} R_{kil}^h) = 0.\end{aligned} \tag{3.10}$$

Since the curvature of the base manifold does not depend on y, we differentiate with respect to y^k in (3.10). From the value of this derivative at $y = 0$, we get

$$R_{kij}^h = c(\delta_i^h g_{kj} - \delta_j^h g_{ki}), \tag{3.11}$$

where

$$c = \frac{a_1(0)}{a_2^2(0)}(a_2'(0) - b_2(0)),$$

which is a function depending on $x^1, ..., x^n$ only. According to Schur's theorem, c must be a constant when $n > 2$ and M is connected.

Now, from the vanishing conditions of the vertical and horizontal components of $N_P(\delta_i, \delta_j)$ we obtain the expressions of a_1' and a_3'

$$a_1' = \frac{a_1 b_1 - c(1 + 3a_3^2 + 4ta_3 b_3)}{a_1 + 2tb_1}, \qquad a_3' = \frac{a_1 b_3 + 2ca_2(a_3 + tb_3)}{a_1 + 2tb_1}. \tag{3.12}$$

Replacing the obtained value of a_3' in (3.9), and using the relations (3.7) we have

$$a_2' = \frac{2a_3 b_3 - a_2 b_1 - ca_2^2}{a_1 + 2tb_1}. \tag{3.13}$$

The vertical component of the mixed Nijenhuis bracket $N_P(\partial_i, \delta_j)$ vanishes when we replace the values of a_1' and a_3' from (3.12) and a_2' from (3.13). The same values fulfill also the relation

$$a_1 a_2' + a_1' a_2 = -2a_3 a_3', \tag{3.14}$$

obtained by differentiating the first of the relations (3.7) with respect to t.

Solving the system given by (3.9) and (3.12), with respect to $b_1,\ b_2,\ b_3$, and taking into account the relation (3.14), we obtain the expressions in the statement, which satisfy the vanishing conditions of all the components of the Nijenhuis tensor field N_P. Thus the almost product structure P on TM is integrable.

Example 3.4. When a_3, b_1, b_2, b_3 vanish and $a_2 = \frac{1}{a_1} = a(L^2)$, where $L^2 = 2t$, the relation (3.13) becomes of the form $2a' = -ca^3$. Using the notations of A. Heydary and E. Peyghan, the characterization of the first locally product structure constructed in [12] is proved. In an analogous way, we can obtain the other locally product structures in the mentioned paper and in [29].

Remark 3.5. Considering $a_1 = \frac{1}{a_2} = \sqrt{2t},\ a_3 = b_1 = b_2 = b_3 = 0$, the relation (3.10) becomes

$$(\delta_i^h g_{kj} - \delta_j^h g_{ki})y^k + R_{kij}^h y^k = 0,$$

which implies that the base manifold has constant sectional curvature -1, and [29, Theorem 12] is proved, since all the other components of the Nijenhuis tensor vanish.

4. Riemannian Almost Product and Almost Para-hermitian Tangent Bundles

The theory of Riemannian almost product structures was initiated in 1965 by K. Yano and developed by A. M. Naveira, M. Staikova, K. Gribachev, D. Mekerov, etc. On the other hand, the almost para-Hermitian structures, classified by C. Bejan, then by P. M. Gadea and J. Muñoz Masqué, were studied by many authors.

A Riemannian manifold (M, g), endowed with an almost product structure J, satisfying the relation

$$g(JX, JY) = \varepsilon g(X, Y),\ \forall X, Y \in \mathcal{T}_0^1(M), \tag{4.1}$$

is called a *Riemannian almost product manifold* if $\varepsilon = 1$, or an *almost para-Hermitian manifold* if $\varepsilon = -1$.

In the sequel we shall obtain the conditions under which the tangent bundle TM, endowed with the almost product structure P determined in the previous section and with a metric G of general natural lift type, is a Riemannian almost product manifold, or a para-Hermitian manifold.

Recall the expression of the semi-Riemannian metric G of general natural lift type on TM, considered by V. Oproiu in [25]:

$$(4.2)\quad \begin{cases} G(X_y^H, Y_y^H) = c_1(t)g_{\tau(y)}(X,Y) + d_1(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^V) = c_2(t)g_{\tau(y)}(X,Y) + d_2(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ G(X_y^V, Y_y^H) = c_3(t)g_{\tau(y)}(X,Y) + d_3(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \end{cases}$$

$\forall X, Y \in \mathcal{T}_0^1(TM),\ \forall y \in TM$, where $c_\alpha,\ d_\alpha$, with $\alpha = \overline{1,3}$, are six smooth functions of the energy density on TM.

The conditions for G to be nondegenerate are assured if

$$c_1c_2 - c_3^2 \neq 0, \quad (c_1 + 2td_1)(c_2 + 2td_2) - (c_3 + 2td_3)^2 \neq 0.$$

The metric G is definite positive if

$$(4.3)\quad c_1+2td_1 > 0, \quad c_2+2td_2 > 0, \quad (c_1+2td_1)(c_2+2td_2)-(c_3+2td_3)^2 > 0.$$

Now we prove the following theorems.

Theorem 4.1. *The tangent bundle of a Riemannian manifold M, endowed with the metric G and the almost product structure P of general natural lift type, is a Riemannian almost product manifold if and only if the coefficients of G and P satisfy the relations*

$$a_1c_2 - a_2c_1 = 2a_3c_3,$$
$$(a_1 + 2tb_1)(c_2 + 2td_2) - (a_2 + 2tb_2)(c_1 + 2td_1) = 2(a_3 + 2tb_3)(c_3 + 2td_3).$$

If moreover, the conditions in Theorem 3.3 *hold, then (TM, G, P) is a Riemannian locally product manifold.*

Proof. With respect to the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$, for $\varepsilon = 1$, (4.1) has the form

$$(4.4)\quad G(P\delta_i, P\delta_j) = G(\delta_i, \delta_j),\ G(P\partial_i, P\partial_j) = G(\partial_i, \partial_j),\ G(P\partial_i, P\delta_j) = G(\partial_i, \delta_j).$$

Replacing in (4.4) the expressions of the almost product structure P and of the metric G, and using Lemma 2.1, we have that the coefficients of g_{ij} and $g_{0i}g_{0j}$ vanish. From the vanishing conditions of the coefficients of g_{ij} we get the following homogeneous linear system in $c_1,\ c_2,\ c_3$

$$(4.5)\quad \begin{cases} (a_3^2 - 1)c_1 + a_1^2c_2 - 2a_1a_3c_3 = 0, \\ a_2^2c_1 + (a_3^2 - 1)c_2 + 2a_2a_3c_3 = 0, \\ -a_2c_1 + a_1c_2 - 2a_3c_3 = 0. \end{cases}$$

It is easy to see that the rank of the system (4.5) is 1, due to the first relation in (3.7), then the first relation in the statement holds.

The vanishing conditions for the coefficients of $g_{0i}g_{0j}$ in (4.4) lead to a more complicated system. Multiplying the equations of the new system by $2t$, and then adding the corresponding equations from (4.5), we obtain:

$$(4.6)\qquad \begin{cases} [(a_3+2tb_3)^2-1]S_1+(a_1+2tb_1)^2S_2-2(a_1+2tb_1)(a_3+2tb_3)S_3=0,\\ (a_2+2td_2)^2S_1+[(a_3+2tb_3)^2-1]S_2+2(a_2+2tb_2)(a_3+2tb_3)S_3=0,\\ -(a_2+2tb_2)S_1+(a_1+2tb_1)S_2-2(a_3+2tb_3)S_3=0, \end{cases}$$

where we denoted by $S_1,\ S_2,\ S_3$, the unknowns c_1+2td_1, c_2+2td_2, c_3+2td_3.

Since the second relation in (3.7) holds, the rank of the above system is equal to 1, and the second relation in the statement of the theorem is true.

Remark 4.2. If we consider $a_1=\frac{1}{a_2}=\sqrt{2t},\ c_3=\sqrt{\frac{2}{t}}$, and the other coefficients in the definitions of P and G vanish, the systems (4.5) and (4.6) are satisfied, and we obtain the results in [29, Theorem 11]. Taking Remark 3.5 into account, [29, Theorem 12] is also proved.

Theorem 4.3. *The family of general natural Riemannian metrics G on TM such that (TM,G,P) is an almost para-Hermitian manifold, is given by* (4.2)*, provided that its coefficients are related to the coefficients of the almost product structure P of general natural lift type by the following proportionality relations*

$$(4.7)\qquad \frac{c_1}{a_1}=-\frac{c_2}{a_2}=\frac{c_3}{a_3}=\lambda,\qquad \frac{c_1+2td_1}{a_1+2tb_1}=-\frac{c_2+2td_2}{a_2+2tb_2}=\frac{c_3+2td_3}{a_3+2tb_3}=\lambda+2t\mu,$$

where the proportionality coefficients $\lambda>0$ and $\lambda+2t\mu>0$ are functions depending on t. If moreover, the base manifold is a space form, and $b_1,\ b_2,\ b_3$ have the expressions given in Theorem 3.3, *then (TM,G,P) is a para-Hermitian manifold.*

Proof. We use the local adapted frame $\{\delta_i,\partial_j\}_{i,j=1,\dots,n}$. The metric G is almost para-Hermitian with respect to the almost product structure P if and only if the following relations are fulfilled

$$(*) G(P\delta_i,P\delta_j)=-G(\delta_i,\delta_j),\ G(P\partial_i,P\partial_j)=-G(\partial_i,\partial_j),\ G(P\partial_i,P\delta_j)=-G(\partial_i,\delta_j).$$

By using Lemma 2.1 we obtain that the coefficients of g_{ij} and $g_{0i}g_{0j}$ in $(*)$ must vanish. Imposing this condition for the coefficients of g_{ij} it follows that the parameters $c_1,\ c_2$, and c_3, in the definition of the metric G, satisfy the homogeneous linear system

$$(4.8)\qquad \begin{cases} (a_3^2+1)c_1+a_1^2c_2-2a_1a_3c_3=0,\\ a_2^2c_1+(a_3^2+1)c_2+2a_2a_3c_3=0,\\ -a_2a_3c_1+a_1a_3c_2+2a_1a_2c_3=0. \end{cases}$$

The rank of the system (4.8) is 2, and its nontrivial solutions satisfy the first relation in (4.7).

From the vanishing condition for the coefficients of $g_{0i}g_{0j}$ in $(*)$, we obtain a much more complicated system, fulfilled by $d_1,\ d_2,\ d_3$. In order to get a certain similitude with the above system (4.8), fulfilled by $c_1,\ c_2$, and c_3, we multiply the new equations by $2t$ and add the equations of the system (4.8), respectively. The new system may be written in the following form

$$\begin{cases} [(a_3+2tb_3)^2+1]S_1+(a_1+2tb_1)^2S_2-2(a_1+2tb_1)(a_3+2tb_3)S_3=0,\\ (a_2+2tb_2)^2S_1+[(a_3+2tb_3)^2+1]S_2+2(a_2+2tb_2)(a_3+2tb_3)S_3=0,\\ -(a_2+2tb_2)(a_3+2tb_3)S_1+(a_1+2tb_1)(a_3+2tb_3)S_2+2(a_1+2tb_1)(a_2+2tb_2)S_3=0, \end{cases}$$

where the new unknowns are S_1, S_2, and S_3, defined in the previous proof.

The nonzero solutions of the above system satisfy the second relation in (4.7).

Moreover, the conditions (4.3) are fulfilled, due to the properties (3.7) of the coefficients of the almost product structure P.

Finally, the explicit expressions of the coefficients d_1, d_2, and d_3, obtained from (4.7), are

(4.9) $$d_1=\lambda b_1+\mu(a_1+2tb_1), d_2=-\lambda b_2-\mu(a_2+2tb_2), d_3=\lambda b_3+\mu(a_3+2tb_3).$$

Example 4.4. In the case when $a_1=a_2=1$ and $a_3=0$, it follows from (4.8) that $c_1=-c_2=1$. If the other coefficients involved in the definitions (3.3) and (4.2) vanish, we obtain the almost para-Hermitian structure considered in [7].

Remark 4.5. When $a_1=\frac{1}{a_2}=\sqrt{2t},\ c_1=-2,\ c_2=\frac{1}{t}$, and the other coefficients in the definitions of P and G vanish, the two systems in the above proof are satisfied, and thus [29, Theorem 16] is proved. Taking into account Remark 3.5, we obtain the result stated in [29, Theorem 18].

5. Almost para-Kähler Structures of General Natural Lift Type on TM

Considering the 2-form Ω defined by the almost para-Hermitian structure (G,P) on TM,

$$\Omega(X,Y)=G(X,PY),$$

for all the vector fields X,Y on TM, we obtain the following result:

Proposition 5.1. *The expression of the* 2*-form* Ω *associated to the general natural almost para-Hermitian structure* (G,P) *on the tangent bundle is given by*

$$\Omega\left(X^V_y,Y^V_y\right)=0,\quad \Omega\left(X^H_y,Y^H_y\right)=0,$$

$$\Omega\left(X^H_y,Y^V_y\right)=-\Omega\left(X^V_y,Y^H_y\right)=\lambda g_{\tau(y)}(X,Y)+\mu g_{\tau(y)}(y,X)g_{\tau(y)}(y,Y),$$

for every tangent vector fields $X,Y\in\mathcal{T}^1_0(M)$*, and every tangent vector* $y\in TM$.

In the local adapted frame $\{\delta_i,\partial_j\}_{i,j=1,\dots,n}$ *on* TM *we have*

(5.1) $$\Omega=(\lambda g_{ij}+\mu g_{0i}g_{0j})dx^i\wedge Dy^j,$$

where $Dy^i=dy^i+\Gamma^i_{0h}dx^h$ *is the absolute differential of* y^i.

Next, by calculating the exterior differential of Ω, we obtain:

Theorem 5.2. *The almost para-Hermitian structure* (G, P) *of general natural lift type on* TM *is almost para-Kählerian if and only if*

$$\mu = \lambda'.$$

Proof. The differential of Ω is

$$d\Omega = (d\lambda g_{ij} + \lambda dg_{ij} + d\mu g_{0i}g_{0j} + \mu dg_{0i}g_{0j} + \mu g_{0i}dg_{0j}) \wedge dx^i \wedge Dy^j$$
$$-(\lambda g_{ij} + \mu g_{0i}g_{0j})dx^i \wedge dDy^j.$$

We first obtain the expressions of $d\lambda,\ d\mu,\ dg_{0i}$ and dDy^i:

$$d\lambda = \lambda' g_{0h}Dy^h, \quad d\mu = \mu' g_{0h}Dy^h, \quad dg_{0i} = g_{hi}Dy^h + g_{0h}\Gamma^h_{ik}dx^k,$$
$$dDy^h = \frac{1}{2}R^h_{0ik}dx^i \wedge dx^k + \Gamma^h_{ik}Dy^i \wedge dx^k.$$

By substituting these relations into $d\Omega$, using the properties of the exterior product, the symmetry of g_{ij} and Γ^h_{ik}, and the Bianchi identities, we get

$$d\Omega = \frac{1}{2}(\mu - \lambda')(g_{ij}g_{0k} - g_{0i}g_{jk})Dy^k \wedge Dy^i \wedge dx^j.$$

Hence the structure (G, P) on TM is almost para-Kählerian (i.e. $d\Omega = 0$) if and only if $\mu = \lambda'$.

Remark 5.3. The family of general natural almost para-Kählerian structures on TM depends on five coefficients, $a_1,\ a_3,\ b_1,\ b_3$, and λ, which must satisfy the supplementary conditions $a_1 > 0,\ a_1 + 2tb_1 > 0,\ \lambda > 0,\ \lambda + 2t\mu > 0$.

Combining the results from the theorems 3.1, 3.3 and 5.2 we may state

Theorem 5.4. *An almost para-Hermitian structure* (G, P) *of general natural lift type on* TM *is para-Kählerian if and only if the almost product structure* P *is integrable (see* Theorem 3.3*) and* $\mu = \lambda'$.

Remark 5.5. The family of general natural para-Kählerian structures on TM depends on three parameters, $a_1,\ a_3$, and λ, which must satisfy the supplementary conditions $a_1 > 0,\ a_1 + 2tb_1 > 0,\ \lambda > 0,\ \lambda + 2t\lambda' > 0$, b_1 being given in Theorem 3.3.

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REFERENCES

1. M. T. K. Abbassi and M. Sarih, On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds, *Diff. Geom. Appl.*, **22** (2005), 19-47.

2. D. V. Alekseevsky, C. Medori and A. Tomassini, Homogeneous Para-Kähler Einstein manifolds, *Russ. Math. Surv.*, **64(1)** (2009), 1-43.

3. C. Bejan, A classification of the almost parahermitian manifolds, *Proc. Conference on Diff. Geom. and Appl.*, Dubrovnik, 1988, pp. 23-27.

4. C. Bejan, Almost parahermitian structures on the tangent bundle of an almost para-cohermitian manifold, *Proc. Fifth Nat. Sem. Finsler and Lagrange spaces*, Brasov, 1988, pp. 105-109.

5. C. Bejan and L. Ornea, An example of an almost hyperbolic Hermitian manifold, *Int. J. Math. Math. Sci.*, **21(3)** (1998), 613-618.

6. I. Bucătaru and R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Editura Academiei Române, Bucureşti, 2007.

7. M. Capursi and A. Palombella, On the almost hermitian structures associated to a generalized Lagrange space, *Proc. 4th Nat. Sem. Finsler and Lagrange spaces, Brasov, 1986*, Soc. Şti. Mat. R. S. Rom., Bucharest, 1986, pp. 115-128.

8. V. Cruceanu, *Selected Papers*, Editura PIM, Iaşi, 2006.

9. S. L. Druţă, Classes of general natural almost anti-Hermitian structures on the cotangent bundles, *Mediterr. J. Math.*, DOI: 10.1007/s00009-010-0075-7.

10. S. L. Druţă, Kaehler-Einstein structures of general natural lifted type on the cotangent bundles, *Balkan J. Geom. Appl.*, **14** (2009), 30-39.

11. P. M. Gadea and J. Muñoz Masqué, Classification of almost para-Hermitian manifolds, *Rend. Mat. Appl.*, **11(7)** (1991), 377-396.

12. A. Heydari and E. Peyghan, A characterization of the infinitesimal conformal transformations on tangent bundles, *Bull. Iranian Math. Soc.*, **34(2)** (2008), 59-70.

13. I. Ianuş, Some almost product structures on manifolds with linear connections, *Kodai Math. J.*, **23** (1971), 303-310.

14. J. Janyška, Natural 2-forms on the tangent bundle of a Riemannian manifold, *Rend. Circ. Mat. Palermo, Serie II, Supplemento, The Proceedings of the Winter School Geometry and Topology Srnì-January 1992*, **32** (1993), 165-174.

15. I. Kolář, P. Michor and J. Slovak, *Natural Operations in Differential Geometry*, Springer Verlag, Berlin, vi, 1993, p. 434.

16. O. Kowalski and M. Sekizawa, Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles - a classification, *Bull. Tokyo Gakugei Univ.*, **40(4)** (1988), 1-29.

17. D. Mekerov, On Riemannian almost product manifolds with nonintegrable structure, *J. Geom.*, **89** (2008), 119-129.

18. I. Mihai and C. Nicolau, Almost product structures on the tangent bundle of an almost paracontact manifold, *Demonstratio Math.*, **15** (1982), 1045-1058.

19. R. Miron and M. Anastasiei, *The Geometry of Lagrange Space: Theory and Application*, Kluwer, Dordrecht, 1994.

20. K. P. Mok, E. M. Patterson and Y. C. Wong, Structure of symmetric tensors of type (0,2) and tensors of type (1,1) on the tangent bundle, *Trans. Amer. Math. Soc.*, **234** (1977), 253-278.

21. M. I. Munteanu, Cheeger Gromoll type metrics on the tangent bundle, *Proceedings of Fifth International Symposium BioMathsPhys, Iaşi, June 2006, U.A.S.V.M. Ion Ionescu de la Brad* **49(2)** (2006) 257-268.

22. M. I. Munteanu, Old and new structures on the tangent bundle, *Proceedings of the Eighth International Conference on Geometry, Integrability and Quantization, June 9-14, 2006, Varna, Bulgaria*, (I. M. Mladenov and M. de León Ed.), Sofia, 2007, pp. 264-278.

23. M. I. Munteanu, Some aspects on the geometry of the tangent bundles and, tangent sphere bundles of a Riemannian manifold, *Mediterr. J. Math.* **5(1)** (2008), 43-59.

24. A. M. Naveira, A classification of Riemannian almost product manifolds, *Rend. Math.*, **3** (1983), 577-592.

25. V. Oproiu, A generalization of natural almost Hermitian structures on the tangent bundles, *Math. J. Toyama Univ.*, **22** (1999), 1-14.

26. V. Oproiu and N. Papaghiuc, Classes of almost anti-Hermitian structures on the tangent bundle of a Riemannian manifold (I), *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Math.* (*N.S.*), **50** (2004), 175-190.

27. V. Oproiu and N. Papaghiuc, General natural Einstein-Kähler structures on tangent bundles, *Differential Geom. Appl.*, **27(3)** (2009), 384-392.

28. V. Oproiu and N. Papaghiuc, Some classes of almost anti-Hermitian structures on the tangent bundle, *Mediterr. J. Math.*, **1(3)** (2004), 269-282.

29. E. Peyghan, A. Razavi and A. Heydari, *Product and anti-Hermitian structures on the tangent space*, arXiv:0710.3825, math.DG, 2007.

30. M. Staikova and K. Gribachev, Canonical connections and conformal invariants on Riemannian almost product manifolds, *Serdica Math. J.*, **18** (1992), 150-161.

31. S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, *Tôhoku Math. J.*, **10** (1958), 338-354.

32. M. Tahara, L. Vanhecke and Y. Watanabe, New structures on tangent bundles, *Note Mat. (Lecce)*, **18** (1998), 131-141.

33. K. Yano, *Differential Geometry of Complex and Almost Complex Spaces*, Pergamon Press, Oxford, 1965.

34. K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc., New York, 1973.

Simona-Luiza Druţă-Romaniuc
Department of Sciences
"Al. I. Cuza" University of Iaşi
54 Lascar Catargi Street
RO-700107 Iaşi
E-mail: simonadruta@yahoo.com