General normal cycles and Lipschitz manifolds of bounded curvature

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Abstract

Closed Legendrian $(d-1)$-dimensional locally rectifiable currents on the sphere bundle in $\mathbb{R}^d$ are considered and the associated index functions are studied. A topological condition assuring the validity of a local version of the Gauss-Bonnet formula is established. The case of lower-dimensional Lipschitz submanifolds in $\mathbb{R}^d$ and their associated normal cycles is examined in detail.

Keywords: Lipschitz manifold, normal cycle, curvature measure, Gauss-Bonnet formula, Principal kinematic formula

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1 Introduction

Lipschitz-Killing curvatures and measures have been extended from the convex ring (cf. Schneider [13], [14] and the references therein) to different other classes of singular sets in $\mathbb{R}^d$ or more general Riemannian manifolds. All these sets admit piecewise linear approximations in a certain sense and therefore the results of Cheeger, Müller and Schrader [4] become meaningful. These authors have shown that the curvatures of Riemannian polyhedra are intrinsic invariants and converge to those of a smooth manifold approximated by such polyhedra in an appropriate way.

The following general classes are known from the literature: Locally finite unions of sets with positive reach, whose intersections have also positive reach ([16], [11]), subanalytic sets (Fu [9]), Whitney stratified sets from some o-minimal structures (Bernig and Bröcker [3]). A survey on these and more general aspects of curvatures in singular spaces with many references and historical remarks can be found in Bernig [2]. In particular, the main related results from convex geometry, differential geometry and geometric measure theory are mentioned.

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One aim of the present paper is to extend results of [12] for d-dimensional Lipschitz manifolds of bounded curvature to the case of lower dimensional submanifolds of this type (a particular case are convex surfaces which were considered by Hug and Schätzle in [10]). Note that none of the classes of unions of sets with positive reach, of Whitney stratified sets (including the subanalytic and the semialgebraic sets) and of Lipschitz submanifolds with bounded curvature contains one of the others. It is an open problem to find some general description of geometric sets admitting Lipschitz-Killing curvatures.

A common approach to all the above classes is to investigate the associated (unit) normal cycles. In particular, the Lipschitz-Killing curvatures are obtained by integrating the universal Lipschitz-Killing curvature forms $\varphi_0, \varphi_1, \ldots, \varphi_{d-1}$ on the sphere bundle $\mathbb{R}^d \times S^{d-1}$ over these normal cycles.

The characteristic properties of normal cycles have been worked out in Fu [8], [9]: Given an integer "multiplicity function" ι_T on $\mathbb{R}^d \times S^{d-1}$, there is at most one current T with the following properties:

1. T is a $(d-1)$-dimensional locally rectifiable current on the sphere bundle $\mathbb{R}^d \times S^{d-1}$.
2. $\partial T = 0$ (T vanishes on the closed $(d-1)$-forms. Such currents are called *cycles.*)
3. $T \llcorner \alpha = 0$ for the canonical 1-form $\alpha(x,n) = \sum_{i=1}^{d} n_i \, dx_i$. (Such currents are called *Legendrian.*)
4. $d\,\omega_d\, T(g\varphi_0) = \int_{S^{d-1}} \sum_{x\in\mathbb{R}^d} g(x,n)\,\iota_T(x,n)\, d\mathcal{H}^{d-1}(n)$
   for the Gauss curvature form φ_0 on the sphere bundle and any smooth function g with compact support. (ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.)

Let X be the projection of the support $\operatorname{spt} T$ onto the first component of $\mathbb{R}^d \times S^{d-1}$. If $\iota_T(x,n)$ coincides with a local topological index function of X, then the basic set X was called by Fu *geometric* and T its *normal cycle* N_X. Note that all sets from the above classes admit such normal cycles, which has been proved with different methods.

In the present paper currents T with the properties 1-3 will be called *general normal cycles.* Our second aim is to work out some properties of general normal cycles: explicit representations of the currents and of associated curvature measures as well as Morse-type relationships.

In order to prepare these considerations, we summarize and complete in Section 1 some results from [12] for normal cycles of Lipschitz d-manifolds of bounded curvature.

In Section 2, for general normal cycles an abstract notion of generalized principal curvatures is introduced. As in the known special cases of geometric sets, the corresponding directions of principal curvatures form a "frame bundle", or equivalently, a simple unit $(d-1)$-vector field orienting T. By means of the associated multiplicity function we give explicit integral representations of T and of the associated general Lipschitz-Killing curvature measures (Theorem 3). The above property 4 for T is then a consequence.

In Section 3, this property is localized in form of an abstract Morse-type relationship. We also state a topological condition characterizing the multiplicity function as a geometric index function which can be determined locally. Under this assumption we obtain a Morse-type relationship for the Euler number of the underlying geometric set.

In Section 4, we turn back to Lipschitz d-manifolds of bounded curvature. We prove that their normal cycles may be approximated (in the flat topology) by those of small outer ε-neighborhoods. (For inner neighborhoods this was shown in [12].)

Section 5 contains the extension of normal cycles to lower dimensional Lipschitz submanifolds which are representable as finite intersections of certain d-dimensional Lipschitz-submanifolds of bounded curvature or their boundaries (Theorem 5). Then all results of Sections 2 and 3 are available, in particular, the Gauss-Bonnet formula. Additionally, we prove the Principal kinematic formula (Theorem 6) for this case.

2 Preliminaries

We start with introducing certain tangent and normal cones to subsets of $\mathbb{R}^d$ which can be found in [5] or [1]. Given a subset $X \subseteq \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we denote by $\mathrm{Tan}(X, x)$ the cone of tangent vectors to X at x in the usual sense, i.e., a nonzero vector u belongs to $\mathrm{Tan}(X, x)$ if there exists a sequence (x_n) of points from $X \setminus \{x\}$ converging to x such that $r_n(x_n - x) \to u$ for some positive numbers r_n. The *Clarke tangent cone* of X at x is defined as

$$C_X(x) = \liminf_{x_n \to x, x_n \in X} \mathrm{Tan}(X, x_n),$$

i.e., $u \in C_X(x)$ iff whenever $x_n \to x, x_n \in X$, there exist tangent vectors $u_n \in \mathrm{Tan}(X, x_n)$ with $u_n \to u$. The Clarke tangent cone $C_X(x)$ is always a closed convex cone and it is a subcone of $\mathrm{Tan}(X, x)$. The *Clarke normal cone* $\mathcal{N}_X(x)$ of X at x is defined as the dual cone to $C_X(x)$, i.e.,

$$\mathcal{N}_X(x) = \{v \in \mathbb{R}^d : \, v \cdot u \leq 0 \text{ for any } u \in C_X(x)\}.$$

Hence, $\mathcal{N}_X(x)$ is a closed convex cone as well. Note that $\mathcal{N}_X(x)$ is larger in general than the usual outer normal cone $\mathrm{Nor}(X, x)$ defined e.g. in [7] as the dual cone to $\mathrm{Tan}(X, x)$.

Let $\mathcal{M}_d$ denote the family of all d-dimensional Lipschitz manifolds in $\mathbb{R}^d$ with boundary (i.e., sets which are locally representable as subgraphs of Lipschitz functions, see [12]). By the Rademacher theorem, $\mathcal{H}^{d-1}$-almost all boundary points x of a given set $X \in \mathcal{M}_d$ are *regular* in the sense that the tangent cone $\mathrm{Tan}(X, x)$ is a halfspace in $\mathbb{R}^d$ and, hence, there exists a unique unit outer normal vector $n(x)$ to X at x. The Clarke normal cone $\mathcal{N}_X(x)$ can be expressed as the smallest closed convex cone containing all limits $\lim_i n(x_i)$, where x_i are regular boundary points of X converging to x. We shall denote the corresponding sphere bundle by

$$\mathcal{N}X = \{(x, n) : \, x \in \partial X, n \in \mathcal{N}_X(x) \cap S^{d-1}\}$$

and call it *unit normal bundle of* X. Note that if $\mathrm{reach}\, X > 0$ then for any $x \in X$, $C_X(x) = \mathrm{Tan}(X, x)$, $\mathcal{N}_X(x) = \mathrm{Nor}(X, x)$ and $\mathcal{N}X$ coincides with the usual unit normal bundle $\mathrm{nor}\, X$, see [12].

Given a set $A \subseteq \mathbb{R}^d$ and $\varepsilon > 0$, we denote by X_ε, $X_{-\varepsilon}$, the outer, inner ε-parallel set to A, i.e.,

$$A_\varepsilon = \{z \in \mathbb{R}^d : \, \mathrm{dist}\,(z, A) \leq \varepsilon\}, \quad A_{-\varepsilon} = \{z \in A : \, \mathrm{dist}\,(z, \mathbb{R}^d \setminus A) \geq \varepsilon\}.$$

Further, $\tilde{A} = \overline{\mathbb{R}^d \setminus A}$ denotes the closure of the complement to A.

It has been shown in [12] that for $X \in \mathcal{M}_d$, $\widetilde{X_\varepsilon}$ and $X_{-\varepsilon}$ have locally positive reach for sufficiently small ε. Consider the following conditions on a Lipschitz manifold $X \in \mathcal{M}_d$:

$$\limsup_{\varepsilon \to 0} \mathcal{H}^{d-1}(\mathrm{nor}\, X_{-\varepsilon}) < \infty, \tag{1}$$

$$\mathcal{N}X \text{ is locally } (\mathcal{H}^{d-1}, d-1) - \text{rectifiable}, \tag{2}$$

$$\mathcal{N}X \text{ is countably } (d-1) - \text{rectifiable}. \tag{3}$$

Note that $\mathcal{H}^{d-1}(\operatorname{nor} X_{-\varepsilon})$ equals the mass of the current $N_{X_{-\varepsilon}}$. Recall also that a countably $(d-1)$-rectifiable set is a countable union of Lipschitz images of bounded subsets of $\mathbb{R}^{d-1}$, and that a locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable set is a union of a countably $(d-1)$-rectifiable and $\mathcal{H}^{d-1}$-measurable set of locally finite $(d-1)$-dimensional measure with an $\mathcal{H}^{d-1}$-zero set.

Notation: We shall denote by $\mathcal{MB}_d$ the family of all Lipschitz manifolds $X \in \mathcal{M}_d$ which satisfy (1) and (2), and by $\mathcal{MB}_d^*$ the family of all sets $X \in \mathcal{M}_d$ satisfying (1) and (3).

It has been shown in [12] that for $X \in \mathcal{MB}_d$, the normal cycle N_X can be defined as

$$N_X = (F) \lim_{\varepsilon \to 0} N_{X_{-\varepsilon}}, \tag{4}$$

where $(F)\lim$ is the flat limit (limit of currents in the flat seminorms).

Remark 1 Condition (1) was called LBIC (locally bounded inner curvature) in [12]. The assumption (2) was missing in [12], nevertheless, it seems to be necessary in order to guarantee the property which is stated in the next proposition and which was used in [12] to show the uniqueness of the definition on N_X.

We recall that two d-dimensional Lipschitz manifolds $X, Y \in \mathcal{M}_d$ *osculate* if there exists a point $z \in X \cap Y$ and nonzero vectors $m \in \mathcal{N}_X(z)$, $n \in \mathcal{N}_Y(z)$ with $m + n = 0$.

Proposition 1 *If $X \in \mathcal{MB}_d$ then X and H do not osculate and $X \cap H \in \mathcal{MB}_d$ for almost all halfspaces H in $\mathbb{R}^d$.*

Proof: The locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable set $\mathcal{N}X$ can be written as a union of a countably $(d-1)$-rectifiable set with an $\mathcal{H}^{d-1}$-zero set. Clearly also the projection of an $\mathcal{H}^{d-1}$-zero set is into its second (normal) coordinate vector has $(d-1)$-dimensional measure zero, hence, this set does not "osculate" with almost all halfspaces H. Thus, we may assume that $\mathcal{N}X$ is countably $(d-1)$-rectifiable and the assertion follows from Proposition 2.

Proposition 2 *If $X, Y \in \mathcal{MB}_d^*$ then X, gY do not osculate and $X \cap gY \in \mathcal{MB}_d$ for almost all euclidean motions g.*

Proof: We shall show first that X, gY do not osculate for almost all euclidean motions g. The proof is based on the following lemma due to Federer:

Lemma 1 ([6, Lemma 6.3]) *If Z is a separable p-dimensional Riemannian manifold of class* 1 *and $\mu : Z \to \mathbb{R}^d$, $\nu : Z \to S^{d-1}$ are Lipschitz mappings, then*

$$\{(z, R) \in Z \times \mathrm{SO}(d) : \mu(z) + R\nu(z) = 0\}$$

is countably $p + (d-1)(d-2)/2$-rectifiable.

Note that the lemma still holds if Z is a locally p-rectifiable set, by definition of rectifiability. Applying the mentioned lemma to the product $Z = \mathcal{N}X \times \mathcal{N}Y$ (which is locally $(2d-2)$-rectifiable) and to the mappings $\mu(x,m,y,n) = m$, $\nu(x,m,y,n) = n$, $(x,m) \in \mathcal{N}X$, $(y,n) \in \mathcal{N}Y$, $\rho \in \mathrm{SO}(d)$, we obtain that the set

$$C := \{(x,m,y,n,\rho) \in \mathcal{N}X \times \mathcal{N}Y \times \mathrm{SO}(d) : \, m + \rho n = 0\}$$

is countably $\left(\frac{d(d+1)}{2} - 1\right)$-rectifiable, hence, its $\frac{d(d+1)}{2}$-dimensional measure is zero. Applying [7, §2.10.25] to the projection of C to $\mathrm{SO}(d)$, we get that

$$\mathcal{H}^d\big(\{(x,m,y,n) \in Z : \, m + \rho n = 0\}\big) = 0$$

for $\mathcal{H}^{\frac{d(d-1)}{2}}$-almost all $\rho \in \mathrm{SO}(d)$. Applying [7, §2.10.25] again, now to the mapping $(x,m,y,n) \mapsto x - y$, we obtain that for $\mathcal{H}^{\frac{d(d-1)}{2}}$-almost all $\rho \in \mathrm{SO}(d)$ and for $\mathcal{H}^d$-almost all $z \in \mathbb{R}^d$, the sets X and $z + \rho Y$ do not osculate, which proves the first statement.

Using the already proved assertion, we have that for almost all rotations ρ, the sets $X, \rho Y + z$ do not osculate for almost all translations $z \in \mathbb{R}^d$. Thus, applying [12, Lemma 5] to X and ρY, we get that (1) and (2) are satisfied by $X, \rho Y + z$ for almost all z. □

Summarizing, the following two basic results were shown in [12]:

Theorem 1 (Gauss-Bonnet formula, [12, Theorems 2,3]) *If $X \in \mathcal{MB}_d$ then the normal cycle N_X is correctly defined by* (4) *and if X is compact then the Gauss-Bonnet formula $N_X(\varphi_0) = \chi(X)$ holds.*

Recall that the k-th curvature measure of X is defined as $C_k(X;A) = (N_X \llcorner \mathbf{1}_A)(\varphi_k)$ for any bounded Borel measurable set $A \subseteq \mathbb{R}^d \times S^{d-1}$ (φ_k is the usual k-th curvature form). We denote by $\bar{C}_k(X;\cdot) = C_k(\cdot \times S^{d-1})$ the projections to $\mathbb{R}^d$, $0 \le k \le d-1$, and we set $\bar{C}_d(X;\cdot) = \mathcal{H}^d \llcorner X$.

Theorem 2 (Principal kinematic formula, [12, Theorem 4]) *If $X, Y \in \mathcal{MB}_d^*$ then $X \cap gY \in \mathcal{MB}_d$ for almost all euclidean motions g and for any $0 \le k \le d-1$ and bounded Borel subsets A, B of $\mathbb{R}^d$ we have,*

$$\int_{\mathcal{G}_d} \bar{C}_k\big(X \cap gY; A \cap gB\big)\mu_d(dg) = \sum_{\substack{1 \le r,s \le d \\ r+s=d+k}} c(d,r,s)\bar{C}_r(X;A)\bar{C}_s(Y;B),$$

where

$$c(d,r,s) = \frac{\Gamma((r+1)/2)\Gamma((s+1)/2)}{\Gamma((r+s-d+1)/2)\Gamma((d+1)/2)}.$$

3 Integral representation of general normal cycles

Let $T \in \mathcal{R}^{\mathrm{loc}}_{d-1}(\mathbb{R}^{2d})$ be a $(d-1)$-dimensional locally rectifiable current with the properties

$$\operatorname{spt} T \subseteq \mathbb{R}^d \times S^{d-1}, \tag{5}$$

$$\partial T = 0 \quad (T \text{ is a cycle}), \tag{6}$$

$$T \llcorner \alpha = 0 \quad (T \text{ is Legendrian}), \tag{7}$$

where α is the contact form in $\mathbb{R}^{2d}$ acting as $\langle (u,v), \alpha(x,n)\rangle = u\cdot n$ (cf. [9]). Such a current T will be called a *general normal cycle* in the sequel. By the rectifiability, T is representable by integration with respect to $\mathcal{H}^{d-1}$ over a locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable set $W(T) \subseteq \mathbb{R}^d \times S^{d-1}$, with integer multiplicity (cf. [7, \$4.1.28]). We denote by $\|T\|$ the (scalar) measure induced by T. Note that $\|T\|$ is equivalent (in the sense of absolute continuity) to $\mathcal{H}^{d-1} \llcorner W(T)$. Note also that the set $W(T) \subseteq \operatorname{spt} T$ is not determined uniquely but up to a difference of $\mathcal{H}^{d-1}$-measure zero.

We use the notation $\operatorname{Tan}^k(A,a)$ for the k-dimensional approximate tangent cone of a set A at a point a, see [7, §3.2.16]. Recall that if A is (locally) $(\mathcal{H}^k, k)$-rectifiable then $\operatorname{Tan}^k(A,a)$ is a k-dimensional subspace for $\mathcal{H}^k$-almost all $a \in A$ (see [7, §3.2.19]).

Proposition 3 *For $\|T\|$-almost all $(x,n) \in \operatorname{spt} T$, $\operatorname{Tan}^{d-1}(W(T),(x,n))$ is a $(d-1)$-dimensional linear subspace of $\mathbb{R}^{2d}$ and there exist vectors $a_1(x,n),\dots,a_{d-1}(x,n)$ in $\mathbb{R}^d$ such that $a_1(x,n),\dots, a_{d-1}(x,n), n$ form a positively oriented orthonormal basis of $\mathbb{R}^d$ and numbers $\kappa_1(x,n),\dots,\kappa_{d-1}(x,n) \in (-\infty,\infty]$ such that the vectors*

$$\left(\frac{1}{\sqrt{1+\kappa_i^2(x,n)}} a_i(x,n), \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i^2(x,n)}} a_i(x,n) \right), \quad i = 1,\dots,d-1,$$

form an orthonormal basis of the tangent space $\operatorname{Tan}^{d-1}(W(T),(x,n))$. (We set $\frac{1}{\sqrt{1+\infty^2}} = 0$ and $\frac{\infty}{\sqrt{1+\infty^2}} = 1$.)

In analogy to the case of sets with positive reach, we call the κ_i's *generalized principal curvatures* and the a_i's *generalized principal directions* of T at (x,n). (The corresponding uniqueness property is shown in Lemma 2 below.)

Proof: Denote for brevity $T(x,n) = \operatorname{Tan}^{d-1}(W(T),(x,n))$. By rectifiability, $T(x,n)$ is a $(d-1)$-dimensional subspace for $\mathcal{H}^{d-1}$-almost all $(x,n) \in W(T)$. Fix such an (x,n) and an $\varepsilon > 0$ such that the linear mapping $(u,v) \mapsto u + \varepsilon v$ is injective on $T(x,n)$. We shall show that the linear operator

$$L : u + \varepsilon v \mapsto v$$

is selfadjoint on $n^\perp$. Indeed, we have

$$v \cdot (u' + \varepsilon v') - (u + \varepsilon v) \cdot v' = u' \cdot v - u \cdot v'$$

and the last expression vanishes at $(u,v),(u',v') \in T(x,n)$ for $\|T\|$-almost all $(x,n) \in W(T)$ by (7) since

$$u' \cdot v - u \cdot v' = \langle (u,v) \wedge (u',v'), d\alpha(x,n)\rangle$$

and $T \llcorner d\alpha = 0$. (For the last equation observe that $T \llcorner \alpha = 0 = \partial T$ implies $T \llcorner d\alpha = \partial(T \llcorner \alpha) = 0$.). Thus, we can choose an orthonormal basis $\{a_i : i = 1,\dots,d-1\}$ of eigenvectors of L and the corresponding real eigenvalues λ_i satisfy $v_i = \lambda_i(u_i + v_i)$, where $a_i = u_i + \varepsilon v_i$, hence $((1-\lambda_i\varepsilon)a_i, \lambda_i a_i)$, $i = 1,\dots,d-1$, are basis vectors of $T(x,n)$. Setting $\kappa_i = \lambda_i/(1-\lambda_i\varepsilon)$ (which is defined as infinity if $\lambda_i\varepsilon = 1$), we get the assertion. □

Lemma 2 *The κ_i's are uniquely determined at $\|T\|$-a.a. (x,n), up to the order. Furthermore, the subspace spanned by principal directions $a_j(x,n)$ assigned to one given value of $\kappa_i(x,n)$ is uniquely determined.*

Proof: Assume that

$$\left(\frac{1}{\sqrt{1+\kappa_i^2}}a_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}}a_i\right), \quad i = 1, \dots, d-1,$$

and

$$\left(\frac{1}{\sqrt{1+(\kappa_i')^2}}a_i', \frac{\kappa_i'}{\sqrt{1+(\kappa_i')^2}}a_i'\right), \quad i = 1, \dots, d-1,$$

are two orthonormal bases of $T(x,n)$, where $\{a_i\}$, $\{a_i'\}$, are two orthonormal bases of $n^\perp$. Then there exist coefficients c_{ij} such that

$$\frac{1}{\sqrt{1+(\kappa_i')^2}}a_i' = \sum_j c_{ij}\frac{1}{\sqrt{1+\kappa_j^2}}a_j, \tag{8}$$

$$\frac{\kappa_i'}{\sqrt{1+(\kappa_i')^2}}a_i' = \sum_j c_{ij}\frac{\kappa_j}{\sqrt{1+\kappa_j^2}}a_j. \tag{9}$$

Fix some $1 \le i \le d-1$ and assume first that $\kappa_i' < \infty$. Multiplying (8) with κ_i', we get

$$\frac{\kappa_i'}{\sqrt{1+(\kappa_i')^2}}a_i' = \sum_j c_{ij}\frac{\kappa_i'}{\sqrt{1+\kappa_j^2}}a_j \tag{10}$$

and, comparing (9) and (10), we obtain that

$$c_{ij}\left(\frac{\kappa_j}{\sqrt{1+\kappa_j^2}} - \frac{\kappa_i'}{\sqrt{1+\kappa_j^2}}\right) = 0$$

for all j. Consequently, we have $\kappa_j < \infty$ and $c_{ij}\kappa_i' = c_{ij}\kappa_j$ for all j, hence, the alternative

$$c_{ij} = 0 \text{ or } \kappa_i' = \kappa_j \tag{11}$$

holds for any j.

Assume now that $\kappa_i' = \infty$. Then we have zero on the left hand side of (8) which implies that $c_{ij}/\sqrt{1+\kappa_j^2} = 0$, hence $c_{ij} = 0$ or $\kappa_j = \infty$, for all j. Thus (11) holds again for all j. It follows from (11) that the sets of eigenvalues $\{\kappa_i\}$ and $\{\kappa_i'\}$ coincide and that any a_i' is a linear combination of those a_j belonging to the same eigenvalue. From this property, the assertion follows. $\square$

We shall assume in the sequel that the principal directions are ordered in such a way that

$$\left\langle \Lambda^{d-1}(\pi_0 + \varepsilon\pi_1)a_T(x,n) \wedge n, \Omega_d \right\rangle > 0$$

for sufficiently small ε; here $\pi_0(x,n) = x$, $\pi_1(x,n) = n$ are the projections and Ω_d is the volume form in $\mathbb{R}^d$. We shall denote by

$$a_T(x,n) = a_1(x,n) \wedge \dots \wedge a_{d-1}(x,n)$$

a unit $(d-1)$-vector field orienting $W(T)$.
Recall that the *Lipschitz-Killing curvature forms* φ_k, $k=0,\dots,d-1$, on the sphere bundle $\mathbb{R}^d\times S^{d-1}$ are given by

$$\langle \xi_1\wedge\ldots\wedge\xi_{d-1},\varphi_k(n)\rangle = \mathcal{O}_{d-1-k}^{-1}\sum_{\substack{\varepsilon_i=0,1\\ \sum\varepsilon_i=d-1-k}}\left\langle\bigwedge_{i=1}^{d-1}\pi_{\varepsilon_i}(\xi_i)\wedge n,\Omega\right\rangle$$

where $\xi_1,\dots,\xi_{d-1}\in\mathbb{R}^d\times S^{d-1}$ and $\mathcal{O}_m=\mathcal{H}^m(S^m)$.

Proposition 3 yields the following representation of the current T.

Theorem 3 *For a normal cycle T we can write*

$$T(\cdot)=\int_{\mathbb{R}^d\times S^{d-1}} i_T(x,n)\langle a_T(x,n),\cdot\rangle\mathcal{H}^{d-1}(d(x,n)), \tag{12}$$

where i_T is an integer valued locally $\mathcal{H}^{d-1}$-integrable function on $\mathbb{R}^d\times S^{d-1}$. Applying this formula to the k-th Lipschitz-Killing curvature form φ_k ($k=0,\dots,d-1$), we get for any bounded Borel subset A of $\mathbb{R}^d\times S^{d-1}$

$$(T\llcorner\mathbf{1}_A)(\varphi_k)=\mathcal{O}_{d-1-k}^{-1}\int_A i_T\sum_{i_1<\cdots<i_{d-k-1}}\frac{\kappa_{i_1}\cdots\kappa_{i_{d-k-1}}}{\prod_{i=1}^{d-1}\sqrt{1+\kappa_i^2}}\,d\mathcal{H}^{d-1}. \tag{13}$$

Remark: The $(d-1)$-vector field a_T is defined only $\mathcal{H}^{d-1}$-almost everywhere on $W(T)$, but this is sufficient to determine the integral (12) since the index function i_X vanishes outside of $W(T)$.

The value $(T\llcorner\mathbf{1}_A)(\varphi_k)$ can be interpreted as *generalized k-th curvature measure* corresponding to the normal cycle T and applied to a set A.

Example: Consider a Lipschitz d-manifold $X\in\mathcal{MB}_d$. Then its normal cycle N_X satisfies assumptions (5)–(7) and $W(N_X)\subseteq\mathcal{N}X$. Hence, its generalized principal directions and generalized principal curvatures are defined by Proposition 3 and we get the following *integral representation of its normal cycle*:

$$N_X(\cdot)=\int_{\mathcal{N}X} i_X(x,n)\langle a_X(x,n),\cdot\rangle\mathcal{H}^{d-1}(d(x,n)), \tag{14}$$

where we write a_X for a_{N_X} and i_X for i_{N_X}.

Remark 2 Assumptions (5)–(7) are clearly not sufficient for the current T to have the geometrical properties of a normal cycle defining proper curvature measures. Consider e.g. a union $X=X_1\cup X_2$ of two touching balls. Then the current $T=N_{X_1}+N_{X_2}$ fulfills (5)–(7) but $T(\varphi_0)=2\neq 1=\chi(X)$, hence, the Gauss-Bonnet formula fails.

4 Index function

Let T be a general normal cycle as in the previous section. Recall that the integer-valued function i_T is determined $\mathcal{H}^{d-1}$-almost everywhere in $\mathbb{R}^d\times S^{d-1}$. We also consider the following modification of i_T which differs only by a possible change of sign:

$$\iota_T(x,n):=(-1)^{\lambda(x,n)}i_T(x,n),$$

where $\lambda(x,n)$ is the number of negative principal curvatures $\kappa_i(x,n)$ at (x,n). (Note that $(-1)^{\lambda(x,n)}$ is the sign of the determinant of the mapping L used in the proof of Proposition 3.) We shall call ι_T the *index function* corresponding to the general normal cycle T.

Proposition 4 *For the Gauss curvature form φ_0, any smooth function g on $\mathbb{R}^d \times S^{d-1}$ and current T as above we have*

$$\mathcal{O}_{d-1}T(g\varphi_0) = \int_{S^{d-1}} \sum_{x\in\mathbb{R}^d} g(x,n)\iota_T(x,n)\mathcal{H}^{d-1}(dn).$$

Proof: The Jacobian of the sphere map $h := \pi_1|_{W(T)}$ at $\mathcal{H}^{d-1}$-a.a. (x,n) is equal by Proposition 3 to

$$\mathcal{J}^{d-1}h(x,n) = \left|\prod_{i=1}^{d-1} \frac{\kappa_i(x,n)}{\sqrt{1+\kappa_i^2(x,n)}}\right|.$$

On the other hand, we have by the definition of φ_0

$$\mathcal{O}_{d-1}\langle a_T(x,n), \varphi_0(n)\rangle = (-1)^{\lambda(x,n)}\mathcal{J}^{d-1}h(x,n).$$

Therefore, the area theorem (Federer [7, §3.2.3]) applied to the mapping h implies

$$\begin{aligned}
\mathcal{O}_{d-1}T(g\varphi_0) &= \int_{\operatorname{spt} T} g(x,n)i_T(x,n)\langle a_T(x,n),\varphi_0(n)\rangle\,\mathcal{H}^{d-1}(d(x,n)) \\
&= \int_{S^{d-1}} \sum_{x\in\mathbb{R}^d} g(x,n)i_T(x,n)(-1)^{\lambda(x,n)}\,\mathcal{H}^{d-1}(d(x,n)) \\
&= \int_{S^{d-1}} \sum_{x\in\mathbb{R}^d} g(x,n)\iota_T(x,n)\,\mathcal{H}^{d-1}(d(x,n)),
\end{aligned}$$

which completes the proof. □

Remark 3 According to Proposition 4, for $\mathcal{H}^{d-1}$-a.a. $n \in S^{d-1}$, $\iota_T(x,n)$ coincides with the index function introduced by Fu [8, \$3.3.1.2a] and takes non-zero values at only finitely many x.

In order to establish the corresponding geometric interpretation, we consider intersections with closed halfspaces

$$\begin{aligned}
H_{v,t}(x) &:= \{y \in \mathbb{R}^d : (y-x)\cdot v \le t\}, \\
H_{v,t} &:= H_{v,t}(0),
\end{aligned}$$

$x \in \mathbb{R}^d$, $u \in S^{d-1}$, $t \in \mathbb{R}$.

Recall that the "connecting current" $\mathcal{J}(T_1,T_2)$ on $\mathbb{R}^{3d}$ of two currents $T_1,T_2 \in \mathbf{I}_{d-1}(\mathbb{R}^{2d})$ was defined in [12], and that the following projection mappings were defined on $\mathbb{R}^{3d}$:

$$\begin{aligned}
G: &\quad (x,y,n) \mapsto x-y, \\
\pi: &\quad (x,y,n) \mapsto (x,n).
\end{aligned}$$

Whenever both T_1, T_2 have countably $(d-1)$-rectifiable supports, the section $\langle \mathcal{J}(T_1, \rho T_2), G, z\rangle$ is well defined for almost all rotations $\rho \in \mathrm{SO}(d)$ and almost all $z \in \mathbb{R}^d$ (see [12, Section 4]).

Denote

$$\mathcal{J}(T, v, t) := T \llcorner (\mathrm{int}\,(H_{v,t}) \times S^{d-1}) + \pi_{\#}\langle \mathcal{J}(T, N_{H_{v,0}}), G, tv\rangle.$$

In the sequel, ess lim means the approximate limit w.r.t. the Lebesgue measure.

Proposition 5 *Let T be a normal cycle with compact support and denote $X = \pi_0(\operatorname{spt} T)$. The following conditions are equivalent.*

(i) $\mathcal{J}(T, v, t)(\varphi_0) = \chi(X \cap H_{v,t})$ *for $\mathcal{H}^d$-almost all $(v, t) \in S^{d-1} \times \mathbb{R}$.*

(ii) $\iota_T(x, n) = \operatorname{ess\,lim}_{\delta\to 0} \big(\chi(X \cap H_{-n,\delta}(x)) - \chi(X \cap H_{-n,-\delta}(x))\big)$, $x \in \mathbb{R}^d$, *for $\mathcal{H}^{d-1}$-almost all $n \in S^{d-1}$.*

Proof: By (3.3.1.3a) in Fu [8] (as a part of the proof of Fu's uniqueness theorem), we have for T as above and for a.e. $n \in S^{d-1}$ and $a, b \in \mathbb{R}$

$$\mathcal{J}(T, -n, b)(\varphi_0) - \mathcal{J}(T, -n, a)(\varphi_0) = \sum_{x\cdot n\in(a,b)} \iota_T(x, n).$$

(Fu has shown the equality up to a universal sign which can be determined by choosing for T the normal cycle of a ball.) Since the number of summands on the right hand side is finite, this leads to the assertion. $\square$

The Constancy theorem of Federer [7, §4.1.31] leads to a deeper version of Proposition 4 for a general current T as above. Let ζ be the unit $(d-1)$-vector field orienting S^{d-1} so that $\langle \zeta(n) \wedge n, \Omega\rangle = 1$ for any $n \in S^{d-1}$.

Theorem 4 *For any compactly supported, closed Legendrian locally rectifiable $(d-1)$-current T with support in $\mathbb{R}^d \times S^{d-1}$ we have*

(i) $(\pi_1)_{\#}T = T(\varphi_0) \cdot (\mathcal{H}^{d-1} \llcorner S^{d-1}) \wedge \zeta$,

(ii) $T(\varphi_0) = \sum_{x\in\mathbb{R}^d} \iota_T(x, n)$ *for almost all $n \in S^{d-1}$.*

Proof: Since T is a cycle we get $\partial(\pi_1)_{\#}T = (\pi_1)_{\#}\partial T = 0$. The Constancy theorem applied to the manifold S^{d-1} and the integer cycle $(\pi_1)_{\#}T$ yields

$$(\pi_1)_{\#}T = \mathrm{const} \cdot (\mathcal{H}^{d-1} \llcorner S^{d-1}) \wedge \zeta.$$

In view of the arguments in the proof of Proposition 4 we get for the volume form Ω_{d-1} in S^{d-1}

$$(\pi_1)_{\#}\Omega_{d-1} = d\omega_d \varphi_0.$$

Hence,

$$\mathcal{O}_{d-1} T(\varphi_0) = (\pi_1)_{\#}T(\Omega_{d-1}) = \mathrm{const} \cdot d\omega_d,$$

i.e., const $= T(\varphi_0)$, which proves (i).

For (ii), recall that Proposition 4 implies

$$\mathcal{O}_{d-1}T(g\varphi_0) = \int_{S^{d-1}} \sum_{x\in\mathbb{R}^d} g(n)\iota_T(x,n)\mathcal{H}^{d-1}(dn)$$

for any smooth function g on S^{d-1}. On the other hand, the above relationships yield

$$\mathcal{O}_{d-1}T(g\varphi_0) = T(\varphi_0)\int_{S^{d-1}} g(n)\mathcal{H}^{d-1}(dn).$$

Consequently,

$$\int_{S^{d-1}} g(n)\left(T(\varphi_0) - \sum_{x\in\mathbb{R}^d}\iota_T(x,n)\right)\mathcal{H}^{d-1}(dn) = 0$$

for any smooth function g, which proves (ii). □

Corollary 1 *If $X = \pi_0(\operatorname{spt} T)$ is compact and $T(\varphi_0) = \chi(X)$ then the Euler number satisfies the Morse-type index relationship*

$$\chi(X) = \sum_{x\in\mathbb{R}^d}\iota_T(x,n)$$

(with a finite number of summands) for almost all $n \in S^{d-1}$.

We now turn to the case when $T = N_X$ is the normal cycle of a d-dimensional Lipschitz manifold with boundary $X \in \mathcal{MB}_d$, and write $a_X := a_{N_X}$, $i_X := i_{N_X}$, $\iota_X := \iota_{N_X}$. Our aim is to show that for almost all n, the function $\iota_X(x,n)$ can be localized and coincides with a modified version of the index function introduced in Fu [8]. In particular, all Lipschitz manifolds $X \in \mathcal{MB}_d$ are "almost" geometric sets in the sense of [8], i.e., they admit a normal cycle with index function ι_X.

Recall that for $T = N_X$, condition (i) in Proposition 5 is fulfilled (see [12]). Moreover, for any $x \in \mathbb{R}^d$, the halfspaces $H_{-n,\delta}(x)$ and $H_{-n,-\delta}(x)$ do not osculate with X for almost all $n \in S^{d-1}$ and $\delta > 0$, see Proposition 2.

We call a system $\mathcal{V}$ of sets in $\mathbb{R}^d$ a *Vitali system* if for any $x \in \mathbb{R}^d$ and $\delta > 0$ there exists a $V \in \mathcal{V}$ with $\operatorname{diam} V < \delta$ and $x \in \operatorname{int} V$.

Lemma 3 *Let $X \in \mathcal{MB}_d$ be a Lipschitz manifold and let $\mathcal{V} \subseteq \mathcal{MB}_d$ be a countable Vitali system of Lipschitz manifolds non-osculating with X and with the property that $X \cap V \in \mathcal{MB}_d$ whenever $V \in \mathcal{V}$. Then we have*

$$\iota_X(x,n) = \operatorname*{ess\,lim}_{\delta\to 0}\Big(\big(\chi(X\cap V\cap H_{-n,\delta}(x)\big) - \chi\big(X\cap V\cap H_{-n,-\delta}(x)\big)\Big)$$

for all $V \in \mathcal{V}$ with $x \in \operatorname{int} V$, all $x \in \mathbb{R}^d$ and almost all $n \in S^{d-1}$.

Proof: We have by definition

$$N_{X\cap V} = (\mathrm{F})\lim_{\varepsilon\to 0} N_{(X\cap V)_{-\varepsilon}}.$$

This implies

$$N_{X\cap V}\llcorner g_V = (\mathrm{F})\lim_{\varepsilon\to 0} N_{(X\cap V)_{-\varepsilon}}\llcorner g_V$$

for any smooth function g_V with support in $\operatorname{int} V$. Furthermore,

$$N_{(X\cap V)_{-\varepsilon}} \llcorner g_V = N_{X_{-\varepsilon}\cap V_{-\varepsilon}} \llcorner g_V = N_{X_{-\varepsilon}} \llcorner g_V$$

for any $\varepsilon > 0$ with $\operatorname{spt} g_V \subseteq V_{-\varepsilon}$. The last equality follows from the fact that the normal cycle of a set with positive reach is locally determined. Therefore, the above limit is equal to

$$(\mathrm{F}) \lim_{\varepsilon\to 0} N_{X_{-\varepsilon}} \llcorner g_V = N_X \llcorner g_V$$

since $N_X = (\mathrm{F}) \lim_{\varepsilon\to 0} N_{X_{-\varepsilon}}$. Consequently,

$$N_{X\cap V} \llcorner g_V = N_X \llcorner g_V$$

for any g_V and $V \in \mathcal{V}$. From Proposition 4 we conclude

$$\iota_X(x, n) = \iota_{X\cap V}(x, n)$$

for almost all $n \in S^{d-1}$, all $V \in \mathcal{V}$ and all $x \in \mathbb{R}^d$. Then Proposition 5 applied to the Lipschitz manifolds $X \cap V$ leads to the assertion. □

The index function $\iota_X(x, n)$ for almost all n may also be determined by means of local intersections with hyperplanes. (For the case of sets from the convex ring this corresponds to Schneider [13]. In Bernig and Bröcker [3] it is used for another class of singular sets including the subanalytic sets considered in Fu [9].)

Corollary 2 *For any $X \in \mathcal{MB}_d$ and any countable Vitali system $\mathcal{B}$ of balls non-osculating with X and such that $X \cap B \in \mathcal{MB}_d$ for any $B \in \mathcal{B}$ we have*

$$\iota_X(x, n) = 1 - \lim_{\substack{x\in \operatorname{int} B\\ \operatorname{diam} B\to 0, B\in\mathcal{B}}} \operatorname*{ess\,lim}_{\delta\to 0} \chi\big(X \cap B \cap \partial H_{-n,-\delta}(x)\big)$$

for almost all $n \in S^{d-1}$ and all $x \in \partial X$.

Proof: Note that

$$H_{-n,\delta}(x) = H_{-n,-\delta}(x) \cup S$$

for the strip

$$S = S_{n,\delta}(x) := \{y \in \mathbb{R}^d : -\delta \leq (y - x) \cdot n \leq \delta\}.$$

Similar arguments as in the proof of Lemma 3 and the additivity of the Euler characteristic yield for almost all n and δ, and any $B \in \mathcal{B}$

$$\begin{aligned} \chi\big(X \cap B \cap H_{-n,\delta}(x)\big) &= \chi\big(X \cap B \cap H_{-n,-\delta}(x)\big) + \chi(X \cap B \cap S) \\ &\quad -\chi\big(X \cap B \cap \partial H_{-n,-\delta}(x)\big), \end{aligned}$$

since $\partial H_{-n,-\delta}(x) = H_{-n,-\delta}(x) \cap S$. Moreover, $\chi(X \cap B \cap S) = 1$ for $x \in \partial X$, sufficiently small B containing x in its interior, and almost all n and δ. (This follows from the property that for almost all n and δ, $X \cap S$ is again a Lipschitz manifold, hence a local neighbourhood retract.) Substituting this in the above equation and using Lemma 3, we obtain the assertion. □

Remark 4 It follows from Proposition 2 that for any $X \in \mathcal{MB}_d$ there exists a Vitali system of balls meeting the assumptions of Corollary 2.

5 Approximation by outer parallel sets

We shall show that under (1) and (2), outer parallel sets can be used as well as the inner parallel sets for the approximation of the normal cycle. We shall apply again the uniqueness theorem of Fu [9]. As an auxiliary result, the following Morse-type relation for Lipschitz manifolds will be used.

Lemma 4 *If $X \in \mathcal{M}_d$ then*

$$\lim_{\varepsilon \to 0} \chi(X_\varepsilon \cap H) = \chi(X \cap H)$$

holds for almost all halfspaces H in $\mathbb{R}^d$.

Proof: We shall show that the formula is true for any halfspace H which does not osculate with X (this occurs for almost all halfspaces by Proposition 1). We know that $(X \cap H)_\varepsilon$ is homotopy equivalent to $X \cap H$ for sufficiently small ε by [12, Lemma 1 and Theorem 1]. Hence, it is sufficient to show that $(X \cap H)_\varepsilon \sim X_\varepsilon \cap H$. This is easily verified by considering the deformation retraction

$$\theta(z,t) = (1-t)z + t\xi_H(z), \quad z \in (X \cap H)_\varepsilon,\ t \in [0,1],$$

where ξ_H is the projection to H. (It is easy to see that if $z \in (X \cap H)_\varepsilon$ then the whole segment $[z, \xi_H(z)]$ lies in $(X \cap H)_\varepsilon$.) □

Proposition 6 *If $X, \tilde{X} \in \mathcal{MB}_d$ then $X_\varepsilon \in \mathcal{MB}_d$ for sufficiently small ε and*

$$N_X = (F) \lim_{\varepsilon \to 0} N_{X_\varepsilon}.$$

Proof: Clearly $X_\varepsilon \in \mathcal{M}_d$ for sufficiently small ε. Since the properties (1) and (2) are local, we may assume that X is compact. Then the closure of the complement of X_ε has positive reach by [12, Proposition 1] and, hence, $\limsup_{\delta \to 0} \mathbf{M}\big(N_{(X_\varepsilon)_{-\delta}}\big) < \infty$ for sufficiently small ε, by [12, Proposition 3]. Thus X_ε fulfills (1) for sufficiently small ε. Property (2) is fulfilled for X_ε as well since $\widetilde{X_\varepsilon}$ has positive reach and $\mathcal{N} X_\varepsilon$ is the image of $\operatorname{nor} X_\varepsilon$ under the isometry $(x,n) \mapsto (x,-n)$.

In order to show the limit assertion, we shall use the Federer's compactness theorem [7, §4.2.17] as in [12]. Since the closed rectifiable currents N_{X_ε} are uniformly bounded in mass, any subsequence has an (F)-convergent subsequence. Hence, it is sufficient to verify that N_X is the only cumulative point of N_{X_ε} at $\varepsilon = 0$. Let $\varepsilon_i \to 0$ be a sequence such that N_{X_ε} converges in the flat seminorms. We shall show that the current

$$T = (F) \lim_{i \to \infty} N_{X_{\varepsilon_i}}$$

fulfills the assumptions of the Fu's uniqueness theorem (see [9, Theorem 3.2] and [12]). T is closed, Legendrian and compactly supported, since these properties are preserved by flat limits. We have to show that

$$\mathcal{J}(T,v,t)(\varphi_0) = \chi(X \cap H_{v,t})$$

for almost all $(v,t) \in S^{d-1} \times \mathbb{R}$. Let v,t be chosen so that X and $H_{v,t}$ do not osculate. Since

$$\mathcal{J}(N_{X_{\varepsilon_i}}, v, t)(\varphi_0) = \chi(X_{\varepsilon_i} \cap H_{v,t})$$

for sufficiently large i by the Gauss-Bonnet formula and since

$$\mathcal{J}(T,v,t) = (\mathrm{F}) \lim_{i \to \infty} \mathcal{J}(N_{X_{\varepsilon_i}}, v, t)$$

for almost all v,t, the proof is completed by applying Lemma 4. □

Corollary 3 *If $X, \tilde{X} \in \mathcal{MB}_d$ then $N_{\tilde{X}} = -\rho_{\#} N_X$, where $\rho : (x,n) \mapsto (x,-n)$.*

6 Lower-dimensional Lipschitz manifolds

We shall define lower-dimensional Lipschitz manifolds by intersecting d-dimensional Lipschitz manifolds and their boundaries. The non-osculating property will be needed. Generalizing this notion for more than two sets, we say that k d-dimensional Lipschitz manifolds $X_1, \dots, X_k \in \mathcal{M}_d$ *osculate* if there exists a point $z \in \partial X_1 \cap \dots \cap \partial X_k$ and vectors $n_1 \in \mathcal{N}_{X_1}(z), \dots, n_k \in \mathcal{N}_{X_k}(z)$ which are not all equal to zero and with $n_1 + \dots + n_k = 0$.

A *k-dimensional Lipschitz manifold in $\mathbb{R}^d$* is a set which can be locally represented as a bi-Lipschitz image of an open subset of $\mathbb{R}^k$ (see e.g. [15]). A *k-dimensional Lipschitz manifold in $\mathbb{R}^d$ with boundary* is a set locally representable as a bi-Lipschitz image of a relatively open subset of a closed halfspace in $\mathbb{R}^k$.

First of all, consider $X \in \mathcal{MB}_d$. Its boundary, ∂X, is clearly a $(d-1)$-dimensional Lipschitz manifold (without boundary), and we can define its normal cycle by

$$N_{\partial X} = N_X + N_{\tilde{X}}.$$

Lemma 5 *If $X \in \mathcal{MB}_d$ and ∂X is compact then $N_{\partial X}$ fulfills the Gauss-Bonnet formula:*

$$N_{\partial X}(\varphi_0) = \chi(\partial X),$$

where φ_0 is the 0th curvature form.

Proof: Since the Gauss-Bonnet formula holds for both N_X and $N_{\tilde{X}}$ (see [12, Theorem 2]), we have

$$\begin{aligned} N_{\partial X}(\varphi_0) &= N_X(\varphi_0) + N_{\tilde{X}}(\varphi_0) \\ &= \chi(X) + \chi(\tilde{X}) \\ &= \chi(\partial X). \end{aligned}$$

□

To illustrate the following general construction, consider two d-dimensional Lipschitz manifolds with boundaries X, Y such that neither X, Y nor $\tilde{X}, Y$ osculate. Then $Z = \partial X \cap Y$ is a $(d-1)$-dimensional Lipschitz manifold with boundary and if all the sets $X, \tilde{X}, Y, X \cap Y, \tilde{X} \cap Y$ lie in $\mathcal{MB}_d$ we can define

$$N_Z = N_{X \cap Y} + N_{\tilde{X} \cap Y} - N_Y.$$

More generally, given $k = 1, \dots, d-1$, we need a stronger condition than the non-osculation property.

Definition We say that p d-dimensional Lipschitz manifolds $X_1, \dots, X_p \in \mathcal{M}_d$ *intersect transversally* if for any $x_0 \in \partial X_1 \cap \dots \cap \partial X_p$,

$$\bigcap_{i=1}^{p} \operatorname{Lin} \mathcal{N}_{X_i}(x_0) = \{0\}.$$

Denote

$$\mathcal{N}(X_1,\dots,X_p)=\{(x,n):\, x\in X_1\cap\ldots\cap X_p, n\in \operatorname{Lin}\mathcal{N}_{X_1}(x)+\cdots+\operatorname{Lin}\mathcal{N}_{X_p}(x)\}.$$

Further, we define $\mathcal{MB}_k$ to be the family of all subsets of $\mathbb{R}^d$ of the form

$$Z=\partial X_1\cap\cdots\cap\partial X_{d-k}\cap Y, \tag{15}$$

where $X_1,\widetilde{X_1}\ldots,X_{d-k},\widetilde{X_{d-k}},Y\in\mathcal{MB}_d$ are such that

(a) $X_1,\dots,X_{d-k},Y$ intersect transversally,

(b) $\mathcal{N}(X_1,\dots,X_p)$ is locally $(\mathcal{H}^{d-1},d-1)$-rectifiable,

(c) $X_1^{i(1)}\cap\cdots\cap X_{d-k}^{i(d-k)}\cap Y\in\mathcal{MB}_d$ for any $i(1),\dots,i(d-k)\in\{0,1\}$, where $X_j^0:=X_j$ and $X_j^1=\widetilde{X_j}$.

Let further $\mathcal{MB}_k^*$ denote the system of all $Z\in\mathcal{MB}_k$ for which assumption (b) is replaced by the stronger one:

(b*) $\mathcal{N}(X_1,\dots,X_p)$ is countably $(d-1)$-rectifiable,

Note that condition (a) implies that the sets $X_1^{i(1)},\dots,X_{d-k}^{i(d-k)},Y$ do not osculate for any $i(1),\dots,i(d-k)\in\{0,1\}$.

Remark 5 Clearly, any k-dimensional C^1 submanifold of $\mathbb{R}^d$ belongs to $\mathcal{MB}_k$. Also, a boundary of a convex body (or, more generally, of a full-dimensional set with positive reach) lies in $\mathcal{MB}_{d-1}$ (see [12]).

Proposition 7 *Any set from $\mathcal{MB}_k$ is a k-dimensional Lipschitz manifold in $\mathbb{R}^d$ with boundary.*

For the proof we shall need the following auxiliary result.

Lemma 6 *Let $X_1,\dots,X_p\in\mathcal{M}_d$ intersect transversally. Then for any $x_0\in\partial X_1\cap\cdots\cap\partial X_p$ there exists a p-dimensional linear subspace A_p of $\operatorname{Lin}\mathcal{N}_{X_1}(x_0)+\cdots+\operatorname{Lin}\mathcal{N}_{X_p}(x_0)$ such that the orthogonal projection of $\mathcal{N}_{X_1}(x_0)+\cdots+\mathcal{N}_{X_p}(x_0)$ to A_p is nonzero.*

Proof: We shall prove the assertion by induction on p. If $p=1$ then, since $\mathcal{N}_{X_1}(x_0)$ is a proper convex cone, there exists a nonzero vector n_1 such that $n\cdot n_1>0$ whenever $n\in\mathcal{N}_{X_1}(x_0)$, and we can choose the linear hull of n_1 for A_1.

Assume now that $X_1,\dots,X_p\in\mathcal{M}_d$ intersect transversally, $x_0\in\partial X_1\cap\cdots\cap\partial X_p$ and that there exists a $(p-1)$-dimensional subspace A_{p-1} of $E_{p-1}:=\operatorname{Lin}\mathcal{N}_{X_1}(x_0)+\cdots+\operatorname{Lin}\mathcal{N}_{X_{p-1}}(x_0)$ such that the orthogonal projection of $\mathcal{N}_{X_1}(x_0)+\cdots+\mathcal{N}_{X_{p-1}}(x_0)$ to A_{p-1} is nonzero. Take any nonzero vector $n_p\in\mathcal{N}_{X_p}(x_0)$ and set $A_p:=A_{p-1}+\operatorname{Lin}p_{E_{p-1}^{\perp}}a_p$. If $u\in E_{p-1}$ and $v\in\mathcal{N}_{X_p}(x_0)$ then clearly the projection of both $u+v$ and $u-v$ to A_p cannot be a zero vector; hence, A_p fulfills the requirement of the lemma. $\square$

Proof of Proposition 7: Let $Z \in \mathcal{MB}_k$ have the form (15) and let $x_0 \in Z$ be a relatively inner point of Z (i.e., $z \in \partial X_1 \cap \cdots \cap \partial X_{d-k} \cap \operatorname{int} Y$). We can represent each X_i locally at x_0 as a subgraph of a Lipschitz function f_i defined on a hyperplane $n_i^\perp$ $(i = 1, \ldots, d-k)$. Consider the function

$$F = (F_1, \ldots, F_{d-k})$$

from $\mathbb{R}^d$ to $\mathbb{R}^{d-k}$, where the components are given by

$$F_i(x) = f_i(x - (x \cdot n_i)n_i) - x \cdot n_i,$$

and note that the zero set of F coincides with Z locally at x_0. Further, we have

$$\mathcal{N}_{X_i}(x_0) = \partial F_i(x_0),$$

where ∂F_i is the Clarke subgradient of F_i (see [5]). Let A be the subspace of dimension $d - p$ guaranteed by Lemma 6. Then, applying the implicit function theorem for Lipschitz mappings ([5, §7.1]), we get that the zero set of F is locally at x_0 representable as a bi-Lipschitz image of an open subset of $A^\perp$.

If $x_0 \in Z$ is a boundary point ($x_0 \in \partial Y$) we construct as above a subspace A_{d-k} of dimension $d-k$ for the sets $\partial X_1, \cdots, \partial X_{d-k}$, and a subspace A_{d-k+1} of dimension $d-k+1$ for the sets $\partial X_1, \cdots, \partial X_{d-k}, \partial Y$. It is clear from the proof of Lemma 6 that we can guarantee $A \subset A'$. Applying [5, §7.1] as above, we can parametrize $\partial X_1 \cap \cdots \cap \partial X_{d-k} \cap \partial Y$ by an open subset of $A_{d-k+1}^\perp$ and $\partial X_1 \cap \cdots \cap \partial X_{d-k}$ by an open subset of $A_{d-k}^\perp$, locally at x_0. Hence, $\partial X_1 \cap \cdots \cap \partial X_{d-k} \cap Y$ is parametrized at x_0 locally by a halfspace in $A_{d-k}^\perp$ bounded by $A_{d-k+1}^\perp$. $\square$

Definition: Given a set $Z \in \mathcal{MB}_k$ with representation (15), we define its *normal cycle* by induction with respect to $d-k$ as

$$\begin{aligned} N_Z \quad &= \quad N_{\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap X_{d-k} \cap Y} + N_{\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap \widetilde{X_{d-k}} \cap Y} \\ &\quad\; - N_{\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap Y}. \end{aligned}$$

Theorem 5 *For any $Z \in \mathcal{MB}_k$ $(k = 1, 2, \ldots, d-1)$ we have:*

(i) *The normal cycle N_Z is correctly defined and if Z is compact then the Gauss-Bonnet formula holds.*

(ii) *The normal cycle N_Z admits the integral representation* (12) *with index function fulfilling $i_Z(x, n) = (-1)^{\lambda(x,n)} \iota_Z(x, n)$ and*

$$\iota_Z(x, n) = 1 - \lim_{\substack{x \in \operatorname{int} B \\ \operatorname{diam} B \to 0, B \in \mathcal{B}}} \operatorname*{ess\,lim}_{\delta \to 0} \chi\big(Z \cap B \cap \partial H_{-n,-\delta}(x)\big)$$

for almost all $n \in S^{d-1}$ and all $x \in \partial Z$, where $\mathcal{B}$ is an appropriate countable Vitali system of balls.

Proof: let a compact set Z have the form (15) and assume that conditions (a) and (c) hold. We shall show that Z satisfies the Gauss-Bonnet formula, i.e., that $N_Z(\varphi_0) = \chi(Z)$ (note that the current N_Z can be defined without assuming (b)). We can assume without loss of generality that all the sets $X_1, \dots, X_{d-k}, Y$ are compact (otherwise, we can intersect them by a suitable large ball). We shall proceed by induction with respect to $d-k$. For $k = d-1$ we have

$$\begin{aligned} N_Z(\varphi_0) &= N_{X_1\cap Y}(\varphi_0) + N_{\widetilde{X_1}\cap Y}(\varphi_0) - N_Y(\varphi_0) \\ &= \chi(X_1 \cap Y) + \chi(\widetilde{X_1} \cap Y) - \chi(Y) \\ &= \chi((X_1 \cap Y) \cap (\widetilde{X_1} \cap Y)) \\ &= \chi(Z). \end{aligned}$$

Assume now that the formula is true for $k = d-1, d-2, \dots, d-k-1$, and let Z be as above. Then

$$\begin{aligned} N_Z(\varphi_0) &= N_{\partial X_1\cap\cdots\cap\partial X_{d-k-1}\cap X_{d-k}\cap Y}(\varphi_0) + N_{\partial X_1\cap\cdots\cap\partial X_{d-k-1}\cap\widetilde{X_{d-k}}\cap Y}(\varphi_0) \\ &\quad - N_{\partial X_1\cap\cdots\cap\partial X_{d-k-1}\cap Y}(\varphi_0) \\ &= \chi(\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap X_{d-k} \cap Y) \\ &\quad + \chi(\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap \widetilde{X_{d-k}} \cap Y) \\ &\quad - \chi(\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap Y) \\ &= \chi(Z); \end{aligned}$$

we have used the induction assumption for the sets $\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap X_{d-k} \cap Y$, $\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap \widetilde{X_{d-k}} \cap Y$, $\partial X_1 \cap \cdots \cap \partial X_{d-k-1} \cap Y$. It remains to show the correctness of the definition, i.e., that it does not depend on the particular representation (15). We shall proceed again by induction. Let now $Z \in \mathcal{MB}_k$. Since the normal cycles are defined locally, it is sufficient to consider a compact set Z with representation (15). Using assumption (b), it follows as in the proof of Proposition 1 that almost all hyperplanes H have the property that there is no $(x, n) \in \mathcal{N}(X_1, \dots, X_{d-k}, Y)$ with $x \in H$ and $n \perp H$. For such hyperplanes H, the sets $X_1, \dots, X_{d-k}, Y \cap H$ intersect transversally. It follows further from the induction assumption and from Proposition 2 that $X_1^{i(1)} \cap \cdots \cap X_{d-k}^{i(d-k)} \cap Y \cap H \in \mathcal{MB}_d$ for any $i(1), \dots, i(d-k) \in \{0, 1\}$, where $X_j^{j(i)}$ are as above. Thus $Z \cap H$ can be written in the form (15) (with $Y \cap H$ instead of Y) and the representation fulfills conditions (a) and (c). Hence, $Z \cap H$ satisfies the Gauss-Bonnet formula. It follows then by the Fu's uniqueness theorem that N_Z is unique, independent of the representation (15).

To verify (ii), let $\mathcal{B}$ be a countable Vitali system of ball such that any $B \in \mathcal{B}$ does not osculate with $X_1^{i(1)} \cap \cdots \cap X_{d-k}^{i(d-k)} \cap Y$ and $B \cap X_1^{i(1)} \cap \cdots \cap X_{d-k}^{i(d-k)} \cap Y \in \mathcal{MB}_d$ for any $i(1), \dots, i(d-k) \in \{0, 1\}$ (such a system exists by Proposition 2). The formula for ι_Z follows then from Corollary 2 and from the additivity of the Euler number. □

An important property of lower-dimensional Lipschitz manifolds is that their curvature measures of higher order vanish.

Lemma 7 *If $0 \le k < l \le d$ and $Z \in \mathcal{MB}_k$ then $C_l(Z; \cdot) = 0$.*

Proof: If $l = d$ then it is enough to realise that Z is obtained in a $(d-1)$-dimensional manifold which has zero Lebesgue measure. Assume thus that $0 \le k < l \le d-1$, let Z be given in the form (15) and let $z \in Z$. Let $n_i \in \mathcal{N}_{X_i}(z)$, $i = 1, \dots, d-k$. The linear hull $L = \mathrm{Lin}\,\{n_1, \dots, n_{d-k}\}$ has dimension $d-k$ by the transversality assumption and we have

$$\{x\} \times L \subseteq \mathcal{N}(X_1, \dots, X_{d-k}).$$

Consequently, if $(x,n) \in \mathcal{N}(X_1, \dots, X_{d-k})$ then

$$\{0\} \times V \subseteq \mathrm{Tan}^{d-1}\Big(\mathcal{N}(X_1, \dots, X_{d-k}), (x,n)\Big)$$

for some subspace V of dimension $d-k-1$ (V is the orthogonal complement of n in L). By assumption (b), $\mathcal{N}(X_1, \dots, X_{d-k})$ is locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable. Let $W(N_Z)$ be the locally $(\mathcal{H}^{d-1}, d-1)$-rectifiable set corresponding to the current N_Z as described in Section 3. Since clearly $W(N_Z) \subseteq \mathcal{N}(X_1, \dots, X_{d-k})$, we have

$$\mathrm{Tan}^{d-1}(W(N_Z), (x,n)) = \mathrm{Tan}^{d-1}\Big(\mathcal{N}(X_1, \dots, X_{d-k}), (x,n)\Big)$$

for $\mathcal{H}^{d-1}$-almost all $(x,n) \in W(N_Z)$, see [7, §3.2.19]. Consequently, $\mathrm{Tan}^{d-1}(W(N_Z), (x,n))$ contains a subspace $\{0\} \times V$ with a $(d-k-1)$-dimensional subspace V of $\mathbb{R}^d$ for almost all $(x,n) \in W(N_Z)$. Comparing this fact with Proposition 3, we see that at least $d-k-1$ of the principal curvatures $\kappa_1(x,n), \dots, \kappa_{d-1}(x,n)$ are infinite. Applying (13), we thus get $T(\varphi_l) = 0$ for any $l > k$. □

Finally, we state the Principal kinematic formula for lower-dimension Lipschitz manifolds.

Theorem 6 *Let $X \in \mathcal{MB}_p^*$ and $Y \in \mathcal{MB}_q^*$ with $p + q > d$. Then $X \cap gY \in \mathcal{MB}_{p+q-d}$ for almost all euclidean motions g and for any $0 \le k \le p+q-d$ and bounded Borel subsets A, B of $\mathbb{R}^d$ we have,*

$$\int_{\mathcal{G}_d} \bar{C}_k\big(X \cap gY; A \cap gB\big)\mu_d(dg) = \sum_{\substack{1 \le r \le k, 1 \le s \le l \\ r+s = d+k}} c(d,r,s)\bar{C}_r(X;A)\bar{C}_s(Y;B),$$

with the constant $c(d,r,s)$ as in Theorem 2.

Proof: Let X, Y have representations (15) with

$$X = \partial U_1 \cap \dots \cap \partial U_{d-p} \cap U_0$$

and

$$Y = \partial V_1 \cap \dots \cap \partial V_{d-q} \cap V_0.$$

Applying Lemma 1 to the product

$$\mathcal{N}(U_1, \dots, U_{d-p}, U_0) \times \mathcal{N}(V_1, \dots, V_{d-q}, V_0),$$

we obtain that for almost all motions g there is no $(x,n) \in \mathcal{N}(U_1, \dots, U_{d-p}, U_0)$ with $(x,-n) \in \mathcal{N}(V_1, \dots, V_{d-q}, V_0)$. It follows that the sets $U_1, \dots, U_{d-p}, gV_1, \dots, gV_{d-q}, U_0 \cap gV_0$ intersect transversally for almost all motions g. Together with Proposition 2, this implies that

$$X \cap gY = \partial U_1 \cap \dots \cap \partial U_{d-p} \cap \partial gV_1 \cap \dots \cap \partial gV_{d-q} \cap U_0 \cap gV_0 \in \mathcal{MB}_{p+q-d}.$$

The principal kinematic formula can be shown by induction on $d-p$ and $d-q$. For $p=q=d$ the formula was proved in [12], see Theorem 2. Suppose that the formula holds for $p'=p+1$ and $q'=q$, and let X,Y be as above. The we can write by definition

$$N_X = N_Z + N_{Z'} - N_W$$

with

$$\begin{aligned} Z &= \partial U_1 \cap \cdots \cap \partial U_{d-p-1} \cap U_{d-p} \cap U_0, \\ Z' &= \partial U_1 \cap \cdots \cap \partial U_{d-p-1} \cap \widetilde{U_{d-p}} \cap U_0, \\ W &= \partial U_1 \cap \cdots \cap \partial U_{d-p-1} \cap U_0, \end{aligned}$$

with $Z, Z', W \in \mathcal{MB}_{p+1}$. Analogously, we can write

$$N_{X\cap gY} = N_{Z\cap gY} + N_{Z'\cap gY} - N_{W\cap gY}$$

for almost all motions g. Applying now the additivity decompositions on both sides of the principal kinematic formula and using its validity for the pairs Z, Y, Z', Y and W, Y by induction hypothesis, the proof is completed. $\square$

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