Commun. Math. Phys. 84, 239–250 (1982)

Communications in Mathematical Physics

General Quantum Measurements: Local Transition Maps

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Abstract. On the basis of four physically motivated assumptions, it is shown that a general quantum measurement of commuting observables can be represented by a "local transition map," a special type of positive linear map on a von Neumann algebra. In the case that the algebra is the bounded operators on a Hilbert space, these local transition maps share two properties of von Neumann-type measurements: they decrease "matrix elements" of states and increase their entropy. It is also shown that local transition maps have all the properties of a conditional expectation of a von Neumann algebra onto a subalgebra except that their range is not restricted to the subalgebra. The notion of locality arises from requiring that a quantum measurement may be treated classically when restricted to the commutative algebra generated by the measured observables. The formalism established applies to observables with arbitrary spectrum. In the case that the spectrum is continuous we have only "incomplete" measurements.

1. Introduction

A long-standing problem in mathematical physics is the description of the quantum mechanical measurement of an observable with continuous spectrum. The lack of such a description is an obstacle to the development of an adequate theory of quantum stochastic processes, and to a complete understanding of non-relativistic quantum mechanics. This paper attempts to clear up one aspect of the problem, the effect of a measurement on the state of the system.

The following conventions and notation will be used in this paper. Hilbert spaces will be complex and separable with inner products conjugate linear in the first entry. $\mathcal{B}(\mathcal{H})$ will denote the bounded linear operators on a Hilbert space $\mathcal{H}$. $\mathcal{T}(\mathcal{H})$, $\mathcal{T}^2(\mathcal{H})$, and $\Sigma(\mathcal{H})$ will denote respectively the trace-class operators, Hilbert–Schmidt operators, and normal states on $\mathcal{B}(\mathcal{H})$. A state $\rho \in \Sigma(\mathcal{H})$ will be thought of interchangeably as a linear functional on $\mathcal{B}(\mathcal{H})$ or a trace-class operator in $\mathcal{B}(\mathcal{H})$. A subset of a topological vector space is "total" if its linear span is dense.

The following description, introduced by von Neumann [1], of the complete measurement of an observable with simple discrete spectrum is widely accepted.

0010-3616/82/0084/0239/$02.40

Let $A \in \mathscr{B}(\mathscr{H})$ be such an observable with distinct eigenvalues λ_i and one-dimensional eigenprojections P_i, so that $A = \sum_i \lambda_i P_i$. If the system is in the state $\rho \in \Sigma(\mathscr{H})$ prior to the measurement, then the probability of measuring A to have value λ_i is $\mathrm{tr}(\rho P_i)$, and afterwards the system will be in the pure state $\rho' = P_i$. Thus the statistical average for the final state is $\sum_i \mathrm{tr}(\rho P_i) P_i = \sum_i P_i \rho P_i$. Taking the statistical point of view we say that the measurement of A changes the state from ρ to $\rho' = \sum_i P_i \rho P_i$.

In preparation for generalizing this prescription we should note the characteristic properties which we wish to retain:

1. The map $\rho \to \rho'$ is affine.
2. Complete positivity (see Proposition 2.2).
3. If ρ expressed as a matrix with respect to an orthonormal basis of eigenvectors of A, then off-diagonal matrix elements are annihilated.
4. The entropy of ρ' is greater than or equal to the entropy of ρ. (See e.g. [1–3]).
5. If $\mathscr{C}_0$ is the commutative algebra generated by the P_i, then $\rho' \in \mathscr{C}_0$, and $\rho'(x) = \rho(x)$ for $x \in \mathscr{C}_0$.

All of these properties, or generalized versions, will be retained by the formalism presented in this paper.

Note that $\rho' = \sum_i P_i \rho P_i$ represents a *complete* measurement, in the sense that the measurement is performed with sufficient precision that a result of λ_i leaves the system in the pure state P_i. A less precise measurement which cannot completely resolve the eigenvalues of A may well leave the system in a state still retaining non-zero off-diagonal terms. The statistical argument given above for $\rho' = \sum_i P_i \rho P_i$ would then not apply to such an "incomplete" measurement. Off-diagonal terms will not be eliminated, but we might hope that they are reduced in magnitude. In Sect. 4 we will see that they are.

In the case of a continuous spectrum, it should be clear that we cannot expect a complete measurement, one that resolves the spectrum of A with absolute precision. Even if the measurement can be made with arbitrarily high precision, any individual measurement must have some imprecision and hence be incomplete.

There has been interest for some time in the use of conditional expectations on von Neumann algebras to describe quantum measurements. (See e.g., [4–8]). A conditional expectation of a von Neumann algebra $\mathscr{A}$ onto a von Neumann subalgebra $\mathscr{A}_0$ may be defined as a normal linear map $\Phi : \mathscr{A} \to \mathscr{A}_0$ satisfying the following properties: ([9–12])

1. $\Phi(1) = 1$.
2. $\|\Phi(x)\| \leqq \|x\|$, $x \in \mathscr{A}$.
3. $\Phi(x) \geqq 0$ whenever $x \geqq 0$, $x \in \mathscr{A}$.
4. $\Phi(axb) = a\Phi(x)b$, $a, b \in \mathscr{A}_0$, $x \in \mathscr{A}$.
5. $\Phi(x)^* \Phi(x) = \Phi(x^* x)$, $x \in \mathscr{A}$.

In the above example $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{C}_0$ given by $\Phi x = \sum_i P_i x P_i$ is a conditional

expectation [4]. However, attempts to generalize this formulation have been discouraged by the fact that conditional expectations do not exist in general unless the observable A has a discrete spectrum. (For some explicit theorems, see [6, p. 60], [13].) In his recent contribution Srinivas [8] proposed that the condition of normality be relaxed. In this paper it is proposed instead that the condition that Φ map into $\mathscr{C}_0$ be relaxed.

In Sect. 2 transition maps are defined. The results here are not new, but an attempt has been made to list the explicit minimal conditions under which transition maps may be used to describe quantum processes. The notion of locality is introduced in Sect. 3; the results are presented in somewhat more generality than will be used in this paper. Section 4 contains the main results of this paper. A measurement is defined as a local transition map, and it is shown that all such measurements decrease matrix elements (Theorem 4.5) and increase entropy (Theorem 4.8), thus providing justification for this definition.

2. Transition Maps

Let $\mathscr{A}$, $\mathscr{B}$ be von Neumann algebras. A map $\Phi: \mathscr{A} \to \mathscr{B}$ will be called a *transition map* if Φ is a completely positive normal linear map with $\Phi(1) = 1$. (Here "normal" means "continuous with respect to the σ-weak topologies of $\mathscr{A}$ and $\mathscr{B}$." For a discussion of complete positivity see e.g. [14, p. 192].) If $0 < \Phi(1) \leqq 1$ but the other conditions are satisfied we call Φ a *subtransition* map.

We claim that the effect of a quantum measurement on the state of a system can be represented by a transition map. In fact, this is true not only for measurements but for a large class of quantum processes. Following are the explicit assumptions necessary to justify this claim. Let $\mathscr{A}$ be the von Neumann algebra representing the observables of the system, with $\mathscr{A}_*$ its predual and $\Sigma(\mathscr{A})$ the set of normal states on $\mathscr{A}$. (The most common situation is $\mathscr{A} = \mathscr{B}(\mathscr{H})$, $\mathscr{A}_* = \mathscr{T}(\mathscr{H})$, $\Sigma(\mathscr{A}) = \Sigma(\mathscr{H})$.)

(A) *Well Definedness*: There is a map $\Phi_*: \Sigma(\mathscr{A}) \to \Sigma(\mathscr{A})$ which represents the measurement (or other process) in the sense that if ρ is the state of the system immediately preceding the measurement and ρ' the state immediately following, then $\rho' = \Phi_* \rho$.

(B) *Superposition*: The map Φ_* is affine.

(C) *Extension*: Let $\mathscr{B}$ be the von Neuman algebra of a second quantum system. There is a map $\bar{\Phi}_*: \Sigma(\mathscr{A} \otimes \mathscr{B}) \to \Sigma(\mathscr{A} \otimes \mathscr{B})$ satisfying (A) and (B) such that for $\rho \in \Sigma(\mathscr{A})$, $\omega \in \Sigma(\mathscr{B})$, $\bar{\Phi}_*(\rho \otimes \omega) = \Phi_* \rho \otimes \omega$.

Condition (A) in essence claims that we know everything about the measurement except the state of the system to be measured. In particular we must know the initial state of the measurement apparatus and the details of the possible interaction(s) between the apparatus and the system.

Condition (C) merely states that a measurement process may be extended to include a non-interacting second system with trivial evolution (i.e., nothing happens). That a quantum system with trivial evolution may never occur in reality

is irrelevant here; the point is that there should be no obstacle to such an extension as no physical process is involved, only a redefinition of "system" and "external universe."

Proposition 2.1. *Φ_* satisfies conditions* (A), (B), (C) *if and only if there is a transition map $\Phi : \mathscr{A} \to \mathscr{A}$ such that $\Phi_* \rho = \rho \circ \Phi$ for all $\rho \in \Sigma(\mathscr{A})$.*

Proof Sketch. It is well known that an affine map on $\Sigma(\mathscr{A})$ has a unique extension to a positive linear map on $\mathscr{A}_*$, also denoted by Φ_*, which is then automatically norm continuous [6, p. 17]. It then follows from [21, p. 205, 21.5] that the adjoint map Φ on $\mathscr{A}$ is σ-weakly continuous, and $\Phi_* \rho = \rho \circ \Phi$ for $\rho \in \mathscr{A}_*$ follows from the definition of adjoint map. Since Φ_* maps $\Sigma(\mathscr{A})$ into $\Sigma(\mathscr{A})$, Φ must be positive with $\Phi(1) = 1$. Conversely, any positive normal linear map on $\mathscr{A}$ with $\Phi(1) = 1$ yields an affine map on $\Sigma(\mathscr{A})$ via $\Phi_* \rho = \rho \circ \Phi$.

For an argument that (C) is equivalent to complete positivity, see [15, p.121]. □

If we relax condition (A) somewhat and require only that $\Phi_* : \Sigma(\mathscr{A}) \to \mathscr{A}_*^+$ with $\Phi_* \rho(1) \leqq \rho(1) = 1$ for all $\rho \in \Sigma(\mathscr{A})$, then we may conclude the existence of a subtransition map Φ such that $\Phi_* \rho = \rho \circ \Phi$. This is often useful when only a part of a quantum process is being considered, as in the positive-map-valued measures of Davies [6]. Our primary interest in this paper is transition maps, but when results can be cheaply extended to subtransition maps we will do so.

The following proposition is very useful in dealing with subtransition maps; it is a consequence of the Stinespring Theorem on completely positive maps [16].

Proposition 2.2. *Let $\mathscr{H}$, $\mathscr{K}$ be Hilbert spaces. Let $\mathscr{A}$ be a von Neumann algebra in $\mathscr{B}(\mathscr{H})$ and let Φ be a completely positive normal linear map from $\mathscr{A}$ into $\mathscr{B}(\mathscr{K})$. Then there exist $A_i \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ such that for all $x \in \mathscr{A}$, $\Phi(x) = \sum_i A_i^* x A_i$, where the sum converges σ-weakly. Conversely, any map of this form is a completely positive normal linear map.* [22, *p.* 34]

3. Locality

In the last section we saw that a large class of quantum processes, including measurements, can be represented by transition maps. This raises the question of what additional conditions are necessary to qualify a transition map as a quantum measurement. The theme of this paper is that a single extra condition will suffice—the map in question must be local with respect to some subalgebra of $\mathscr{B}(\mathscr{H})$. Before defining locality we offer some motivation.

It is generally recognized that an important difference quantum and classical probability is that in quantum mechanics the algebra of observables is non-commutative. However, even in quantum systems if we restrict our attention to a commutative algebra of observables we may deal with the system using classical probability, since a commutative algebra may be represented as an algebra of functions on a measurable space. This is precisely what is done when measuring observables in quantum mechanics.

Consider the case of a single observable $A \in \mathscr{B}(\mathscr{H})$. Let $\mathscr{C}_0$ be the commutative

von Neumann algebra in $\mathscr{B}(\mathscr{H})$ generated by A (and the identity). By the spectral theorem $\mathscr{C}_0$ is isomorphic to the algebra of bounded Borel functions on the spectrum of A [17, p. 225]. Normal states on $\mathscr{B}(\mathscr{H})$, when restricted to $\mathscr{C}_0$, may therefore be represented by Borel measures on the spectrum of A. We are then placed in the formalism of classical probability as claimed.

In the classical theory of stochastic processes it is implicitly assumed that in principle measurements can be made with complete accuracy and without disturbing the state of the system. We make an analogous assumption in the case of a quantum measurement, but restricted to that aspect of the measurement which can be treated classically.

Measurement Principle. *In a quantum measurement, expectation values of the measured observables are not disturbed, nor are expectation values of observables in the commutative von Neumann algebra generated by the measured observables and the identity.*

If this commutative algebra is denoted by $\mathscr{C}_0$ and if ρ, ρ' are the states immediately preceding and following the measurement, we have $\rho'(x) = \rho(x)$ for all $x \in \mathscr{C}_0$. If the measurement is effected by a transition map Φ, then

$$(*)\rho(\Phi x) = \rho(x) \quad \text{for all } \rho \in \Sigma(\mathscr{H}), x \in \mathscr{C}_0.$$

Note that $(*)$ is not a repeatability condition, as there is no guarantee that a repeated measurement will give a result even close to the first. It is perhaps best thought of as a "consistency" condition.

As a consequence of $(*)$ we have $\Phi x = x$ for all $x \in \mathscr{C}_0$. A weakening of this condition which costs nothing for transition maps but also applies to subtransition maps is to require $\Phi p \leqq p$ for projections $p \in \mathscr{C}_0$.

Let $\mathscr{A}$, $\mathscr{A}_0$ be von Neumann algebras in $\mathscr{B}(\mathscr{H})$ with $\mathscr{A}_0 \subset \mathscr{A}$. A subtransition map $\Phi : \mathscr{A} \to \mathscr{A}$ is said to be $\mathscr{A}_0$-*local* if $\Phi(p) \leqq p$ for all projections $p \in \mathscr{A}_0$.

Note that von Neumann algebras are the natural setting for discussing locality since each element is approximated by finite linear combinations of projections [11, 1.7.5, 1.3.1], and the spectral projections of any self-adjoint element are in the algebra [11, 1.11.3].

The following proposition characterizes local subtransition maps.

Proposition 3.1 *Let $\Phi : \mathscr{A} \to \mathscr{A}$ be a subtransition map with $\Phi x = \sum_i A_i^* x A_i$ for $x \in \mathscr{A}$, with each $A_i \in \mathscr{B}(\mathscr{H})$. Then Φ is $\mathscr{A}_0$-local if and only if $A_i \in \mathscr{A}_0'$ for each i.*

Proof. If $A_i \in \mathscr{A}_0'$ for each i, then also $A_i^* \in \mathscr{A}_0'$ and hence for $p \in \mathscr{A}_0$ a projection we have $p\Phi(p)p = p\left(\sum_i A_i^* p A_i\right)p = \sum_i pA_i^* pA_i p = \sum_i A_i^* pA_i = \Phi(p)$. Therefore $\Phi(p) = p\Phi(p)p \leqq p1p = p$ and hence Φ is $\mathscr{A}_0$-local.

Now assume that Φ is $\mathscr{A}_0$-local. Fixing a projection $p \in \mathscr{A}_0$, we have $\Phi(p) = \sum_i A_i^* pA_i \leqq p$. We must have $0 \leqq A_i^* pA_i \leqq p$ for each i. It follows that $0 = (1-p)A_i^* pA_i(1-p) = (pA_i(1-p))^*(pA_i(1-p))$, and hence $pA_i(1-p) = 0$, or $pA_i = pA_ip$. Since this holds for all projections on $\mathscr{A}_0$ we may replace p by $1-p$ to obtain $(1-p)A_i = (1-p)A_i(1-p)$, or $A_ip = pA_ip$. Therefore, $pA_i = pA_ip. = A_ip$ for all

projections p in $\mathscr{A}_0$. Since the projections are total in $\mathscr{A}_0$ we may conclude that $xA_i = A_i x$ for all $x \in \mathscr{A}_0$, that is $A_i \in \mathscr{A}_0'$. □

Corollary 3.2. *If $\Phi : \mathscr{A} \to \mathscr{A}$ is an $\mathscr{A}_0$-local subtransition map, then $\Phi(axb) = a\Phi(x)b$ for $x \in \mathscr{A}$, $a, b \in \mathscr{A}_0$.*

Proof. Let $\Phi x = \sum_i A_i^* x A_i$ with each $A_i \in \mathscr{A}_0'$ by Proposition 3.1. Then $\Phi(axb) =$

$$\sum_i A_i^* axb A_i = \sum_i aA_i^* xA_i b = a\left(\sum_i A_i^* xA_i\right)b = a\Phi(x)b. \quad \square$$

It is any easy consequence of Corollary 3.2 with $x = b = 1$ that if Φ is an $\mathscr{A}_0$-local transition map on $\mathscr{A}$, then $\Phi(a) = a$ for all $a \in \mathscr{A}_0$. Also, it follows from [14, p. 199] that any completely positive linear map of C^*-algebras with $\Phi(1) \leqq 1$ satisfies $\Phi(x)^* \Phi(x) \leqq \Phi(x^* x)$.

Thus we see that an $\mathscr{A}_0$-local transition map satisfies all the properties of a conditional expectation onto $\mathscr{A}_0$, except that it need not map into $\mathscr{A}_0$.

We close this section with a comment on the interpretation of the locality condition: $\Phi p \leqq p$ for projections $p \in \mathscr{A}_0$. If $\mathscr{A}_0$ is the (commutative) algebra generated by the position observable, then this condition requires that if the system was contained in a region of space prior to the measurement, it will still be contained in that region after the measurement. This is the reason for using the term "local". Our notion of locality applies to any observable, discrete or continuous.

4. Measurements

Based on the discussions in Sect. 1, 2, and 3, we define a *measurement* on $\mathscr{B}(\mathscr{H})$ to be a transition map $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ together with a von Neumann algebra $\mathscr{A}_0$ in $\mathscr{B}(\mathscr{H})$, such that Φ is $\mathscr{A}_0$-local. The idea is that if observables $x_1, \ldots, x_N \in \mathscr{B}(\mathscr{H})$ are to be measured simultaneously, we let $\mathscr{A}_0$ be the von Neumann algebra in $\mathscr{B}(\mathscr{H})$ generated by $x_1, \ldots, x_N, 1$ (the identity), and require that Φ be $\mathscr{A}_0$-local.

For the remainder of this paper we will assume that $\mathscr{A}_0 = \mathscr{C}_0$ is commutative, as in quantum mechanics one usually requires that observables commute in order for simultaneous measurement to be possible.

Consider the direct integral decomposition $\mathscr{H} = \int_\Gamma^\oplus \mathscr{H}(t) d\mu(t)$ corresponding to $\mathscr{C}_0$ ([14, p. 275]), for which $\mathscr{C}_0$ is the algebra of multiplication operators. If $\mathscr{C}_0$ is maximal commutative, then $\mathscr{C}_0' = \mathscr{C}_0$ and each $\mathscr{H}(t)$ is one-dimensional. In this case the measured observables generating $\mathscr{C}_0$ are said to be a "complete set of commuting observables." If on the other hand $\mathscr{C}_0$ is not maximal commutative the $\mathscr{H}(t)$ will be multidimensional. Observables in $\mathscr{C}_0' \backslash \mathscr{C}_0$ represent observables not measured, but compatible with those that are. It seems reasonable that a measurement could be performed without disturbing the expectation values of these additional observables.

We therefore define a *strong* measurement to be a transition map which is $\mathscr{C}_0'$-local. When a distinction is necessary, a transition map local only with respect to $\mathscr{C}_0$ will be called a weak measurement.

The remainder of this section is devoted to showing that strong measurements "decrease matrix elements" and "increase entropy" (Theorems 4.5 and 4.8).

Lemma 4.1. *Let $\mathscr{K}_1$, $\mathscr{K}_2$ be Hilbert spaces. Let $\{A_i\}$, $\{B_i\}$ be sequences in $\mathscr{B}(\mathscr{K}_1)$ and $\mathscr{B}(\mathscr{K}_2)$ respectively. Assume that*

$$\sum_i A_i^* A_i \leqq 1_{\mathscr{K}_1} \text{ and } \sum_i B_i^* B_i \leqq 1_{\mathscr{K}_2}.$$

Then for any $C \in \mathscr{B}(\mathscr{K}_2, \mathscr{K}_1)$, $\| \sum_i A_i^* C B_i \| \leqq \| C \|$.

Proof. For $f \in \mathscr{K}_1$, $g \in \mathscr{K}_2$,

$$\begin{aligned}\left| \left\langle f, \sum A_i^* C B_i g \right\rangle \right| &\leqq \sum_i |\langle A_i f, C B_i g \rangle| \leqq \sum_i \| A_i f \| \, \| C \| \, \| B_i g \| \\ &\leqq \| C \| \left(\sum_i \| A_i f \|^2 \right)^{1/2} \left(\sum_i \| B_i g \|^2 \right)^{1/2} \\ &= \left(\left\langle f, \sum_i A_i^* A_i f \right\rangle \right)^{1/2} \left(\left\langle g, \sum_i B_i^* B_i g \right\rangle \right)^{1/2} \| C \| \\ &\leqq \| C \| \, \| f \| \, \| g \|.\end{aligned}$$

Therefore $\left\| \sum A_i^* C B_i \right\| = \sup_{f,g} (\| f \| \, \| g \|)^{-1} \left| \left\langle f, \sum_i A_i^* C B_i g \right\rangle \right| \leqq \| C \|$. □

Lemma 4.2. *If $\mathscr{H} = \int_\Gamma^\oplus \mathscr{H}(t) d\mu(t)$, let $\{A_i\}$ be a sequence in $\mathscr{B}(\mathscr{H})$ with $\sum_i A_i^* A_i \leqq 1$. If each $A_i \in \mathscr{C}_0'$ with $A_i = \int_\Gamma^\oplus A_i(t) d\mu(t)$, then $\sum_i A_i(t)^* A_i(t) \leqq 1$ a.e. (μ). If $\sum_i A_i^* A_i = 1$ then $\sum_i A_i(t)^* A_i(t) = 1$ a.e. (μ)*

Proof. For each $f, g \in \mathscr{H}$ we must have

$$\begin{aligned}\int_\Gamma \langle f(t), g(t) \rangle d\mu(t) = \langle f, g \rangle &\geqq \left\langle f, \sum_i A_i^* A_i g \right\rangle = \sum_i \langle A_i f, A_i g \rangle \\ &= \sum_i \left\langle \int_\Gamma^\oplus A_i(s) f(s) d\mu(s), \int_\Gamma^\oplus A_i(t) g(t) d\mu(t) \right\rangle \\ &= \sum_i \int_\Gamma \langle A_i(t) f(t), A_i(t) g(t) \rangle d\mu(t) \\ &= \int_\Gamma \left\langle f(t), \sum_i A_i^*(t) A_i(t) g(t) \right\rangle d\mu(t),\end{aligned}$$

from which the result follows. □

The next two theorems deal with operator-valued kernels on direct integrals, a generalization of bounded operators on $\mathscr{L}^2$-spaces as in [19]. We present here only needed definitions and results—for details see [18].

Let $\mathscr{H} = \int_\Gamma^\oplus \mathscr{H}(t) d\mu(t)$ be a direct integral of Hilbert spaces. We define an *absolutely bounded kernel* on $\mathscr{H}$ to be a measurable field of operators ([14, p. 272])

$\{x(s,t): s,t\in\Gamma\}$ with $x(s,t)\in\mathscr{B}(\mathscr{H}(t),\mathscr{H}(s))$, such that $\{\|x(s,t)\|: s,t\in\Gamma\}$ is a bounded kernel on $\mathscr{L}^2(\Gamma,\mu)$.

Proposition 4.3. (i) *If $x(s,t)$ is an absolutely bounded kernel there exists $A\in\mathscr{B}(\mathscr{H})$ such that for $f,g\in\mathscr{H}$, $\langle f,Ag\rangle=\int_{\Gamma\times\Gamma}\langle f(s),x(s,t)g(t)\rangle d\mu(s)d\mu(t)$.* (ii) *If $x(s,t)$ is an absolutely bounded kernel and $y(s,t)$ is a measurable field of operators such that $\|y(s,t)\|\leqq\|x(s,t)\|$ for all s,t, then $y(s,t)$ is an absolutely bounded kernel.* (iii) *If $x(s,t)$ is an absolutely bounded kernel which induces the zero operator, then $x(s,t)=0$ a.e. $(\mu\times\mu)$.* (iv) *If $A\in\mathscr{T}^2(\mathscr{H})$, then there exists an absolutely bounded kernel which induces A as in* (i).

Theorem 4.4 *Let Φ be a subtransition map on $\mathscr{B}(\mathscr{H})$. The following is a necessary and sufficient condition for Φ to be $\mathscr{C}_0$-local:*

If $x\in\mathscr{B}(\mathscr{H})$ has an absolutely bounded kernel $x(s,t)$ with respect to $\mathscr{C}_0$, then $x'=\Phi x$ also has an absolutely bounded kernel $x'(s,t)$ with respect to $\mathscr{C}_0$ and

$$\|x'(s,t)\|\leqq\|x(s,t)\| \quad a.e.(\mu\times\mu).$$

Proof. Necessity: Let $\Phi=\sum_i A_i^*\cdot A_i$ with each $A_i\in\mathscr{B}(\mathscr{H})$. Since Φ is $\mathscr{C}_0$-local each A_i and A_i^* are in $\mathscr{C}_0'$ and hence decomposable. Let $f,g\in\mathscr{H}$. Then

$$\langle g,\Phi x f\rangle=\left\langle g,\sum_i A_i x A_i f\right\rangle=\int_{\Gamma\times\Gamma}\left\langle g(s),\sum_i A_i(s)^* x(s,t)A_i(t)f(t)\right\rangle d\mu(t)d\mu(s).$$

Since $\sum_i A_i(s)^*A_i(s)\leqq 1_{\mathscr{H}(s)}$ and $\sum_i A_i(t)^*A_i(t)\leqq 1_{\mathscr{H}(t)}$ (each μ-almost everywhere) by Lemma 4.2, it follows from Lemma 4.1 that $\left\|\sum_i A_i(s)^*x(s,t)A_i(t)\right\|\leqq\|x(s,t)\|$. By Proposition 4.3(ii) $\sum_i A_i(s)^*x(s,t)A_i(t)$ is an absolutely bounded kernel, and by the above calculation and the uniqueness of kernels (Proposition 4.3(iii)) $x'(s,t)=\sum_i A_i(s)^*x(s,t)A_i(t)$.

Sufficiency: Assume that Φ is not $\mathscr{C}_0$-local. Then by Proposition 3.1 one of the A_i, say A_1, is not in $\mathscr{C}_0'$. We will produce an $x\in\mathscr{B}(\mathscr{H})$ such that the condition of the theorem is false.

A_1^* is also not in $\mathscr{C}_0'$, so there is a projection $p\in\mathscr{C}_0$ and a vector $f\in p\mathscr{H}$ such that $A_1^*f\notin p\mathscr{H}$. Since p is a multiplication operator and a projection, we must have $p=P(E)=\int_E^{\oplus}1(t)d\mu(t)$ for some measurable $E\subset\Gamma$, where $1(t)$ is the identity on $\mathscr{H}(t)$. There is then a set $E'\subset\Gamma$ with $\mu(E')>0$ such that $E\cap E'=\varnothing$, and for $t\in E'$, $f(t)=0$ but $A_1^*f(t)\neq 0$.

Let $x=\langle f,\cdot\rangle f$. Since $x\in\mathscr{T}^2(\mathscr{H})$, x has an absolutely bounded kernel (Proposition 4.3(iv)) given by $x(s,t)=\langle f(t),\cdot\rangle f(s)$. If $s,t\in E'$ then $f(s)=f(t)=0$, so $x(s,t)=0$. On the other hand, $x'=\Phi x=\sum_i A_i^*(\langle f,\cdot\rangle f)A_i=\sum_i\langle A_i^*f,\cdot\rangle A_i^*f$. If $P(E')=\int_{E'}^{\oplus}1(t)d\mu(t)$, then $P(E')x'P(E')$ is a positive operator, and is nonzero since

$P(E')\langle A_1^* f, \cdot \rangle A_1^* f P(E') > 0$ and $\sum_{i=2}^{\infty} \langle A_i^* f, \cdot \rangle A_i^* f \geqq 0$. Therefore if x' has an absolutely bounded kernel $x'(s,t)$, then $x'(s,t) \neq 0$ on some set $E'' \subset E' \times E'$ of positive measure, as only the zero operator has zero kernel (Proposition 4.3(iii)). For $(s,t) \in E''$, $\|x'(s,t)\| > \|x(s,t)\| = 0$. □

Our objective is to prove a similar result for states. The predual map is given by $\Phi_* \tau = \sum_i A_i \tau A_i^*$ for $\tau \in \mathscr{T}(\mathscr{H})$. By Proposition 4.3(iv) each state ρ on $\mathscr{H}$ has a kernel $\rho(s,t)$ with respect to $\mathscr{C}_0$, as $\rho \in \mathscr{T}(\mathscr{H}) \subset \mathscr{T}^2(\mathscr{H})$. However, Theorem 4.4 cannot be applied immediately because in general $\sum_i A_i A_i^*$ is not $\leqq 1$.

In any situation where $\sum_i A_i A_i^* \leqq 1$, the roles of A_i and A_i^* may be exchanged and Theorem 4.4 will apply. It would suffice for each A_i to be normal. There is no obvious physical reason to assume this directly, but it is a consequence of assuming Φ to be a strong measurement.

Theorem 4.5 *Let Φ be a $\mathscr{C}_0'$-local subtransition map on $\mathscr{B}(\mathscr{H})$. Let $\rho \in \Sigma(\mathscr{H})$ and let $\rho' = \rho \circ \Phi$. If ρ, ρ' have kernels $\rho(s,t)$, $\rho'(s,t)$ with respect to $\mathscr{C}_0$, then*

$$\|\rho'(s,t)\| \leqq \|\rho(s,t)\| \quad a.e.\ (\mu \times \mu).$$

Proof. Suppose $\Phi = \sum A_i^* \cdot A_i$. Then $\sum_i A_i^* A_i \leqq 1$, and since $A_i, A_i^* \in (\mathscr{C}_0')' = \mathscr{C}_0$ (Proposition 3.1), A_i commutes with A_i^* for each i. Hence $\sum_i A_i A_i^* \leqq 1$ also. $\tilde{\Phi} = \sum_i A_i \cdot A_i^*$ is then also a subtransition map on $\mathscr{B}(\mathscr{H})$ and we may apply Theorem 5.4 to $\tilde{\Phi}\rho = \rho \circ \Phi$. □

We next consider the effect of a measurement on the entropy of a state. An excellent reference on quantum (and classical) entropy is Wehrl's review article [2]. Further references are to be found there.

There are several reasonable definitions for the entropy of a quantum state. The following is the most widely accepted:

Let ρ be a state on $\mathscr{H}$ with $\rho = \sum_k \lambda_k p_k$, where $\lambda_k > 0$, $\sum_k \lambda_k = 1$, and $\{p_k\}$ is an orthogonal set of one-dimensional projections. The entropy of ρ, denoted by $S(\rho)$, is given by $S(\rho) = -\sum_k \lambda_k \ln \lambda_k$ if this series converges, ∞ otherwise.

We may also write $S(\rho)$ in the form $S(\rho) = \mathrm{tr}(f(\rho))$ where $f(x) = -x \ln x$. The results in this section will apply to any entropy functional S of this form when f is a concave function on $[0,1]$. We will use a result of Wehrl [2, p. 239]:

Proposition 4.6 *Let $\rho, \rho' \in \Sigma(\mathscr{H})$ and let $\{a_k\}$, $\{a_k'\}$ be the ordered sequence of eigenvalues (with multiplicity) for ρ and ρ' respectively. If $\sum_{k=1}^{N} a_k' \leqq \sum_{k=1}^{N} a_k$ for each positive integer N, then $S(\rho') \geqq S(\rho)$ whenever the functional S is given by $S(\rho) = \mathrm{tr}(f(\rho))$ for some function f concave on $[0,1]$.*

Proposition 4.7. *Let* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *be a transition map given by* $\Phi x = \sum_{i=1} A_i^* x A_i$ *with* $A_i \in \mathcal{B}(\mathcal{H})$.

Suppose that $\sum_{i=1}^{\infty} A_i A_i^* = \sum_{i=1}^{\infty} A_i^* A_i = 1$. *Then* Φ *is entropy-increasing in the sense that for any state* ρ *on* $\mathcal{B}(\mathcal{H})$, *if* $\rho' = \rho \circ \Phi$, *then* $S(\rho') \geqq S(\rho)$.

Proof. Let $\{\phi_k\}$ ($\{\phi_k'\}$) be a complete orthonormal set of eigenvectors for $\rho(\rho')$ and $\{a_k\}$ ($\{a_k'\}$) the corresponding eigenvalues. We assume the eigenvalues are listed in decreasing order.

We will show that for each positive integer N, $\sum_{k=1}^{N} a_k' \leqq \sum_{k=1}^{N} a_k$. It will then follow that $S(\rho') \geqq S(\rho)$ by Proposition 4.6.

Note that $\rho' = \sum_{i=1}^{\infty} A_i \rho A_i^*$, where $\rho = \sum_{l=1}^{\infty} a_l \langle \phi_l, \cdot \rangle \phi_l$. Then

$$\sum_{k=1}^{N} a_k' = \sum_{k=1}^{N} \langle \phi_k', \rho' \phi_k' \rangle = \sum_{k=1}^{N} \sum_{i=1}^{\infty} \langle A_i^* \phi_k', \rho A_i^* \phi_k' \rangle$$

$$= \sum_{k=1}^{N} \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} a_l |\langle A_i^* \phi_k', \phi_l \rangle|^2 = \sum_{l=1}^{\infty} \gamma_l a_l,$$

where $\gamma_l = \sum_{k=1}^{N} \sum_{i=1}^{\infty} |\langle A_i^* \phi_k', \phi_l \rangle|^2$. We have

$$\gamma_l = \sum_{i=1}^{\infty} \sum_{k=1}^{N} |\langle A_i \phi_l, \phi_k' \rangle|^2 \leqq \sum_{i=1}^{\infty} \langle A_i \phi_l, A_i \phi_l \rangle$$

$$= \sum_{i=1}^{\infty} \langle \phi_l, A_i^* A_i \phi_l \rangle = 1.$$

Also

$$\sum_{l=1}^{\infty} \gamma_l = \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \sum_{i=1}^{N} |\langle A_i^* \phi_k', \phi_l \rangle|^2$$

$$= \sum_{k=1}^{N} \sum_{i=1}^{\infty} \langle A_i^* \phi_k', A_i^* \phi_k' \rangle = \sum_{k=1}^{N} \sum_{i=1}^{\infty} \langle \phi_k', A_i A_i^* \phi_k' \rangle$$

$$= \sum_{k=1}^{N} \langle \phi_k', \phi_k' \rangle = N.$$

Since $0 \leqq \gamma_l \leqq 1$ for each l and $\sum_{l=1}^{\infty} \gamma_l = N$, $\{\gamma_l\}$ may be thought of as a measure of mass N on the positive integers with each integer having mass at most 1. Since $\{a_l\}$ is decreasing, the maximum value of $\sum_{l=1}^{\infty} \gamma_l a_l$ would occur if $\gamma_l = 1$ for $1 \leqq l \leqq N$. Therefore $\sum_{k=1}^{N} a_k' = \sum_{l=1}^{\infty} \gamma_l a_l \leqq \sum_{l=1}^{N} a_l$, from which the result follows. □

Theorem 4.8. *Let $\mathscr{C}_0$ be a commutative von Neumann algebra in $\mathscr{B}(\mathscr{H})$. If Φ is a $\mathscr{C}_0'$-local transition map on $\mathscr{B}(\mathscr{H})$, then Φ is entropy-increasing in the sense of Proposition 4.7.*

Proof. From Corollary 3.5 we see that if $\Phi = \sum_i A_i^* \cdot A_i$, then A_i, A_i^* are in $\mathscr{C}_0'' = \mathscr{C}_0$, so that A_i and A_i^* commute and Proposition 4.7 applies. □

Lindblad has proven a similar result in [20]. Our result, however, depends not on the exact form of the entropy functional, but only on the fact that $S(\rho) = \mathrm{tr}(f(\rho))$ for some concave function f. In addition the hypotheses on Φ are more general than Lindblad's.

5. Discussion

We have proven what we set out to prove in the case of a strong measurement. Analogous results hold in the case of a weak measurement for which $\mathscr{C}_0$ is not maximal commutative, but these results require development of considerably more background material and will be presented in a separate article.

Ever since the paper of Davies and Lewis [5], the concept of a "positive-map-valued measure" has been the dominant idea in quantum measurement theory. However, Srinivas [8] has shown that under certain assumptions the Davies–Lewis formalism forces the use of non-normal conditional expectations, i.e. conditional expectations satisfying the conditions in Sect. 1 except that of normality. The most serious problem with non-normal conditional expectations is not the loss of σ-additivity, but the fact that a non-normal conditional expectation will necessarily map some normal states to non-normal states.

As the usual formulation of algebraic quantum mechanics allows a finite system to have only normal states, it is desirable to discuss measurement of continuous observables in a conditional expectation setting without using non-normal states. This is what has been done in this paper. We have made no attempt to discuss a "collapse postulate" as in [8]; the results therein indicate that the Davies–Lewis formalism may not be the best approach. One promising possibility is the use of "positive-map-valued distributions" (as opposed to measures), where the positive maps used would be the subtransition maps defined in this paper.

Finally, the concept of local transition map provides an alternative to the "coarse-graining" and "discrete spectrum approximation" formalisms of quantum measurement ([1], [7]).

Acknowledgements. The author would like to thank the referee for suggestions on shortening some proofs, Dr. D. Ragozin for pointing out errors in an earlier version of this work, and Dr. M. J. Westwater for many conversations on quantum mechanics.

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Communicated by R. Haag

Received April 1, 1981; in revised form October 6, 1981