GENERAL RELATIVITY AND MACH'S PRINCIPLE

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Summary

Sciama, among others, has taken the view that general relativity has failed to account satisfactorily for the inertial properties of matter. This paper shows that general relativity is entirely consistent in principle with Sciama's ideas of inertia as an inductive effect predominantly of distant matter, and that therefore his remarks concerning general relativity are not justified. It is shown that general relativity provides a superior presentation of his idea of Mach's principle and appears to be the general tensor theory he was looking for.

Arguments are put forward to show that general relativity may fully incorporate Mach's principle contrary to Einstein's own belief. This paper emphasizes the fitness of the steady state theory as a cosmological solution which permits this possibility.

1. *Introduction.*—A tentative theory has been presented by Sciama (1), with Maxwell type equations, which is designed to provide a combination of Newton's laws of motion and of gravitation with the inertial frames determined by Mach's principle. In the introduction to his paper Sciama states that general relativity has failed to provide an adequate theory of inertia. He claims that his theory differs from general relativity principally in the following respects:

(i) it enables the amount of matter in the universe to be estimated from a knowledge of the gravitational constant;

(ii) the principle of equivalence is a consequence of his theory, not an initial axiom; and

(iii) it implies that gravitation must be attractive.

The chief characteristic of Sciama's theory is that "in the rest frame of any body the gravitational field of the universe as a whole cancels the gravitational field of local matter so that in this frame the body is 'free'. Thus in this theory inertial effects arise from the gravitational field of a moving universe." For this purpose Sciama employs a scalar potential Φ and a vector potential $\mathbf{A}$ to calculate gravitational effects, using Maxwell type field equations in flat space-time.

It is the purpose of this paper to show that general relativity is fully consistent with this interpretation of Mach's principle by Sciama, and to indicate that general relativity may fully incorporate Mach's principle.

2. *Free particle in general relativity.*—The motion of a free particle in general relativity, when the gravitational field is weak and when the reference frame is such that the spatial velocity of the particle is small compared with the velocity of light, can be described by a Maxwell-type pondermotive equation. This idea

is not new and has in fact, with limited application, been presented by Einstein (2). But since Einstein's derivation of the result appears to contain errors of detail we give our own derivation here, before investigating its significance for Mach's principle.

Let Latin letters indicate space coordinates running over indices 1, 2, 3 while Greek letters cover the space-time indices 1, 2, 3, 4. The world line of a free particle in general relativity is a geodesic in the field having equations, in a standard form,

$$\frac{d}{ds}\left(g_{\mu\alpha}\frac{dx^\alpha}{ds}\right)=\tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{1}$$

where

$$ds^2=g_{44}(dx^4)^2+2g_{4p}dx^4dx^p+g_{pq}dx^pdx^q. \tag{2}$$

For $\mu=i$ equations (1) may be written

$$\left(\frac{ds}{dx^4}\right)^2\frac{d}{ds}\left(g_{ip}\frac{dx^p}{ds}\right)=\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i}-\left(\frac{ds}{dx^4}\right)^2\frac{d}{ds}\left(g_{i4}\frac{dx^4}{ds}\right)+\frac{\partial g_{p4}}{\partial x^i}\frac{dx^p}{dx^4}+\tfrac{1}{2}\frac{\partial g_{pq}}{\partial x^i}\frac{dx^p}{dx^4}\frac{dx^q}{dx^4}.$$

Write now $x^4\equiv t$, $dx^p/dt\equiv v^p$, and neglect squares and products of the spatial coordinate velocities v^p, getting

$$\frac{ds}{dt}\frac{d}{dt}\left(g_{ip}\frac{dt}{ds}v^p\right)=\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i}-\frac{ds}{dt}\frac{d}{dt}\left(g_{i4}\frac{dt}{ds}\right)+\frac{\partial g_{p4}}{\partial x^i}v^p. \tag{3}$$

We shall now represent the metric (2) as that of a weak field in the form

$$ds^2=(1+\gamma_{44})dt^2+2\gamma_{4p}dx^pdt-(1-\gamma_{11})(dx^1)^2-(1-\gamma_{22})(dx^2)^2-(1-\gamma_{33})(dx^3)^2+\gamma_{pq}dx^pdx^q \qquad (p\neq q) \tag{4}$$

Here we take the velocity of light, c, as unity. We see that the $\gamma_{\mu\nu}$ are the deviations of the $g_{\mu\nu}$ from the Galilean values in the so-called inertial frames. They are the $\gamma_{\mu\nu}$ of Einstein's treatment of the problem except for sign due to his employment of imaginary x^4.

We make the assumption that the squares and products of the $\gamma_{\mu\nu}$ and those of their derivatives can be neglected. In solving the field equations to this approximation Einstein showed (3) that the $\gamma_{\mu\nu}$ were the solutions of the equations

$$\left\{\left(\frac{\partial}{\partial x^1}\right)^2+\left(\frac{\partial}{\partial x^2}\right)^2+\left(\frac{\partial}{\partial x^3}\right)^2-\left(\frac{\partial}{\partial t}\right)^2\right\}\gamma_{\mu\nu}=2\kappa(T_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}T) \tag{5}$$

provided

$$\frac{\partial^2}{\partial x^\nu\partial x^\alpha}(\gamma^\alpha_\mu-\tfrac{1}{2}\delta^\alpha_\mu\gamma^\beta_\beta)+\frac{\partial^2}{\partial x^\mu\partial x^\alpha}(\gamma^\alpha_\nu-\tfrac{1}{2}\delta^\alpha_\nu\gamma^\beta_\beta)=0 \tag{6}$$

to the order of the approximation. Here $\kappa=8\pi G/c^2$ where G is the Newtonian constant of gravitation, and $\gamma^\alpha_\mu=\delta^{\alpha\beta}\gamma_{\mu\beta}$ where $\delta^{\alpha\beta}$ are the Galilean values of the $g^{\alpha\beta}$. Assuming the contribution of stress to the energy momentum tensor to be vanishingly small compared with the density and momentum components for the case considered by Einstein, equation (5) yields Einstein's solution:

$$\left.\begin{array}{ll}\gamma_{11}=\gamma_{22}=\gamma_{33}=\gamma_{44}=-\dfrac{\kappa}{4\pi}\displaystyle\int\frac{[\rho]}{r}dV, & \gamma_{4p}=\dfrac{\kappa}{2\pi}\displaystyle\int\frac{[\rho u^p]}{r}dV \\ \text{and} \qquad \gamma_{pq}=0, & p\neq q\end{array}\right\}. \tag{7}$$

In this solution ρ is the mass density, u^p the space velocity, of the element of mass in volume dV at distance r from the point where the $\gamma_{\mu\nu}$ are evaluated, all quantities being measured by observers at rest in the reference frame. Square brackets indicate retarded values corresponding to the propagation of the field with the unit velocity.

The integrals, supposed convergent, are over all matter producing the field. Such a solution of the wave equation of Lorentz (equation (5)) is well known to be valid only if the quantities solved for (the $\gamma_{\mu\nu}$) tend to zero in a suitable way. Thus it is clear that the case considered by Einstein involves mass concentrations only in the neighbourhood of the space origin and a field metric which is Galilean at "infinity". The integrals in (7) evaluated over such mass concentrations are therefore evidently convergent. The solution has to be consistent with conditions (6) which will be satisfied if the expressions

$$\frac{\partial}{\partial x^\alpha}(\gamma^\alpha_\mu - \tfrac{1}{2}\delta^\alpha_\mu \gamma^\beta_\beta)$$

vanish, for all μ, to the first order in the $\gamma_{\mu\nu}$. Using (7) and the fundamental equations $T,^{\mu\nu}_{\nu}=0$, it is easily seen that for integrals over a finite region of mass these expressions are indeed second order quantities.

To this approximation therefore, retaining only first order terms, we can reduce equation (3) to

$$\frac{d}{dt}\left(\frac{g_{ii}v^i}{\sqrt{g_{44}}}\right) = \tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{d}{dt}(g_{i4}) + \frac{\partial g_{p4}}{\partial x^i}v^p,$$

on using (2) to find the appropriate approximation for ds/dt in each term. There is of course no summation over i on the left. Rearranging we can write:

$$\frac{d}{dt}\left(-\frac{g_{ii}v^i}{\sqrt{g_{44}}}\right) = -\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-g_{4i}) + \left\{\frac{\partial}{\partial x^p}(g_{4i}) - \frac{\partial}{\partial x^i}(g_{4p})\right\}v^p \tag{8}$$

so that

$$\frac{d}{dt}\{(1-\gamma_{ii}-\tfrac{1}{2}\gamma_{44})v^i\} = -\tfrac{1}{2}\frac{\partial \gamma_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-\gamma_{4i}) + \left\{\frac{\partial}{\partial x^p}(\gamma_{4i}) - \frac{\partial}{\partial x^i}(\gamma_{4p})\right\}v^p. \tag{9}$$

Write now

$$\phi = -G\int\frac{[\rho]dV}{r}, \quad A^p = -4G\int\frac{[\rho u^p]}{r}dV \tag{10}$$

so that by (7),

$$\gamma_{11}=\gamma_{22}=\gamma_{33}=\gamma_{44}=2\phi, \quad \gamma_{4p}=-A^p. \tag{11}$$

Equation (9) can therefore be written in vector form covering $i=1, 2, 3$

$$\frac{d}{dt}\{(1-3\phi)\mathbf{v}\} = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\wedge\operatorname{curl}\mathbf{A}. \tag{12}$$

Equation (12) is the required Maxwell-type pondermotive equation of the field. The assumptions made during its derivation are:

(i) The particle velocity $\mathbf{v}$ in the reference frame is assumed small such that v^2/c^2 is negligible compared with v/c.

(ii) The deviations of the $g_{\mu\nu}$ from the Galilean values are small such that their squares and products and those of their derivatives can be neglected.

(iii) The deviations $\gamma_{\mu\nu}$ vanish at "infinity" so that the quantities $\mathbf{A}$, ϕ are defined in terms of convergent integrals. If *in addition* we now further assume that

(iv) The source velocities of the field are also small in the reference frame so that the same remark as in (i) applies for them, then equation (12) reduces to

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\wedge\operatorname{curl}\mathbf{A}, \tag{13}$$

The equation obtained by Einstein was (our notation)

$$\frac{d}{dt}[(1-\phi)\mathbf{v}] = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \wedge \operatorname{curl}\mathbf{A}.$$

Since he assumed condition (iv) as well as (i), (ii), (iii), his result is incorrect to the order he was considering, and misleading. In obtaining this result he put

$$[p4, i] = \tfrac{1}{2}\left(\frac{\partial g_{4i}}{\partial x^p} - \frac{\partial g_{4p}}{\partial x^i}\right)$$

thereby neglecting the term $\partial g_{ip}/\partial x^4$ which, when $p=i$, contributes to our result in equation (9) as the term $d(-\gamma_{ii})/dt$ in the coefficient of v^i in the left hand side. The neglect of this term is of course consistent with condition (iv), but on the other hand the retention of the term $d(-\phi)/dt$ in the coefficient of $\mathbf{v}$, arising in our approximation from the term $d(-\frac{1}{2}\gamma_{44})/dt$ in the coefficient, is not consistent with Einstein's assumptions.

3. *Interpretation of the pondermotive equation.*—As Einstein pointed out, equation (12) indicates that general relativity goes far towards incorporating Mach's principle. It may be compared with the Newtonian equation, viz.,

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\phi.$$

The additional terms are small in the quasi-Galilean frame considered by Einstein, and, as he said, beyond physical measurement. Nevertheless they show in the sense of Mach's principle how concentrated matter affects the inertial mass of a freely moving particle, and the acceleration of its locally inertial rest frame relative to the given frame, in the following respects:

(i) The inertial mass is apparently proportional to $1-3\phi$.

(ii) The locally inertial rest frame of the particle is accelerated by means of:

(*a*) gravitational attraction towards the local mass concentrations indicated by the term $-\operatorname{grad}\phi$;

(*b*) an inductive effect of local accelerating matter in the same sense as the acceleration, indicated by the term $-\partial\mathbf{A}/\partial t$;

(*c*) an inductive effect of matter which is rotating relative to the compass of inertia (to use Gödel's phrase) at "infinity", in the sense of the rotation, as indicated by the term $\mathbf{v} \wedge \operatorname{curl}\mathbf{A}$. This is of the same type as the "fictitious" Coriolis force familiar in Newtonian dynamics, when a reference frame is used which is rotating relative to the compass of inertia. Centrifugal force also arises in this case as a fictitious gravitational force.

It is clear therefore that general relativity certainly incorporates in detailed manner the aspects of Mach's principle indicated above. For a satisfactory theory of Mach's principle, however, Einstein realised the necessity of showing how inertia depended on the entire cosmic distribution of matter. Because he assumed that the metric was Galilean at "infinity" and therefore excluded any contribution to $\mathbf{A}$ and ϕ other than that of the matter concentrated near the space origin, he was unable to examine the cosmic influence on inertia.

We shall here put forward an analysis to show that general relativity actually permits the same interpretation of inertia which has been presented by Sciama as the inductive effect of the whole universe.

4. *Inductive effect of the universe in general relativity.*—We shall investigate the extent to which we may generalize the circumstances when the motion of a free particle may be described by a Maxwell-type pondermotive equation. For this purpose we make the assumptions less restrictive than in the quasi-Galilean case as follows:

(i) The particle velocity $\mathbf{v}$ in the reference frame is assumed small such that v^2/c^2 is negligible compared with v/c.

(ii) The velocities of the sources of the field, in the region of space-time coordinates with which we shall be concerned, are also small of the same order so that the same remark applies.

(iii) The deviations of the $g_{\mu\nu}$ from the Galilean values are small in the above quoted range of space-time coordinates, such that their squares and products and those of their derivatives can be neglected. We do not however assume that these deviations vanish at "infinity", nor that they even remain small outside the specified range.

It is clear from equation (8) that the equation of motion of a free particle can in these circumstances be written

$$\frac{dv^i}{dt} = -\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-g_{4i}) + \left\{\frac{\partial}{\partial x^p}(g_{4i}) - \frac{\partial}{\partial x^i}(g_{4p})\right\} v^p \tag{14}$$

for $i = 1, 2, 3$.
This equation is generally covariant, in the sense that, in all reference frames and regions of space-time which do not violate the assumptions above, it describes the space motion of the free particle in terms of the derivatives of the $g_{\mu\nu}$ involved. We now generalize the quantities $\mathbf{A}$, ϕ occurring in the quasi-Galilean analysis by defining

$$(\mathbf{A}, \Phi) \equiv (-g_{4i}, \tfrac{1}{2}g_{44}). \tag{15}$$

The three equations in (14) may then be written concisely

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\wedge\operatorname{curl}\mathbf{A}. \tag{16}$$

The vector notation implies the vector character of the terms for purely spatial transformations. For space-time transformations however the quantities $(\mathbf{A}, \Phi)$ do not transform as a 4-vector but as components of the tensor $g_{\mu\nu}$. This is because, unlike the corresponding electromagnetic pondermotive equation, the permitted transformations are not necessarily between inertial frames and therefore not in general linear.

It is to be noted that here we have not as in the quasi-Galilean case identified $\mathbf{A}$, Φ with the deviations of the $g_{\mu\nu}$ involved, from their Galilean values but, consistent with our endeavour to account for the whole of inertia according to Mach's principle, in terms of the total $g_{\mu\nu}$. The covariance of (16) is secured by the tensor character of the total $g_{\mu\nu}$ involved; the deviations do not transform as tensors for general transformations. Indeed according to the field equations it is the total $g_{\mu\nu}$ field that is related inseparably to the distribution of matter in the whole universe.

Bearing in mind therefore the physical interpretation of the quantities $\mathbf{A}$, ϕ in the quasi-Galilean case we should expect analogous interpretation of $\mathbf{A}$, Φ in (16) which would, if Mach's principle is to be satisfied, take account of the distribution and motion of matter in the whole universe, relative to the particular reference frame being used. It would be an immediate consequence of such an interpretation of the terms in (16) that the "fictitious" forces of Newtonian mechanics in accelerating or rotating reference frames would become directly attributable to the inductive effect of a moving universe.

In particular, in a reference frame in which a freely moving particle was permanently at rest, equation (16) would reduce to

$$-\operatorname{grad}\Phi - \frac{\partial \mathbf{A}}{\partial t} = 0, \tag{17}$$

holding at the particle. This is the equation *postulated* by Sciama. To use Sciama's expression the "gravoelectric" field of the whole universe would be zero at the particle and it would be gravitationally "free" in its own rest frame.

For a reference frame at rest relative to the averaged motions of the rest of the matter in the universe (the "smoothed-out" universe) we should expect by Mach's principle that, in the neighbourhood of the space origin, the derivatives of **A**, Φ on the right of (16) would vanish and therefore that the left hand side must vanish. The real existence of such frames which are locally inertial is the basis of Newtonian mechanics. This aspect of Mach's principle is built into general relativity theory since the field equations predict that such a reference frame will be Galilean near the space origin, because of the spherical symmetry about it. Thus in this neighbourhood the metric will approximate to

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{18}$$

It is emphasized that in this paper we attach importance for Mach's principle to the total $g_{\mu\nu}$ involved in equation (16) and not just their derivatives. General reasons for this have already been given and further justification provided in Sections 5, 6. Accordingly it is important to obtain the total value of Φ. It follows from equation (18) that the static potential Φ_0, at the origin of such a frame, of the whole universe would be

$$\Phi_0 = \tfrac{1}{2}g_{44}(0) = \tfrac{1}{2} \text{ (or } \tfrac{1}{2}c^2 \text{ in general units).} \tag{19}$$

The dimensions of Φ and the significance we are trying to associate with it would require Φ_0 to be of order $-GM/R$ where M is the effective gravitational mass of the universe and R its effective radius. Sciama's approach is to *define* Φ_0 as $-\int_{r=0}^{r=R} \frac{\sigma dV}{r}$ where σ is the gravitational mass density, and he gets

$$G\Phi_0 = -c^2.$$

Both results are numerically of the same order. The discrepancy in sign will occupy us later. Before investigating to what extent general relativity theory justifies this tentative physical interpretation of **A**, Φ, we give some applications of our theory.

5. *Applications of the inductive theory in general relativity.*—(i) Sciama considers the case of a free particle in rectilinear motion in the gravitational field of a mass M which is at rest relative to the smoothed-out universe. If we choose a reference frame at rest relative to the smoothed out universe with this mass M at the space origin, we can neglect in that neighbourhood the deviations from the Galilean values of the $g_{\mu\nu}$ as far as they arise from the universe as a whole, and

include only the deviations due to the mass M. Thus to this approximation the metric will be

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 + \frac{2GM}{r}\right)(dx^2 + dy^2 + dz^2). \tag{20}$$

It is to be noted that, according to the ideas presented in this paper, the contribution to the $g_{\mu\nu}$ potentials from the universe as a whole is present in the Galilean terms of the $g_{\mu\nu}$.

Since the universe is at rest in this frame we have

$$\left.\begin{aligned} \mathbf{A} &= 0 \\ \Phi &= \tfrac{1}{2}\left(1 - \frac{2GM}{r}\right) \end{aligned}\right\}. \tag{21}$$

while

Suppose the particle is moving freely towards the mass M along the x axis. If its space coordinates are $(x_1, 0, 0)$ at coordinate time t then its coordinate speed is $dx_1/dt = -v$, where $v > 0$. Make now the transformation to a suitable rest frame for the particle, by means of the relations

$$x = X + x_1, y = Y, z = Z, t = T \tag{22}$$

yielding $dx = dX - vdT$, $dy = dY$, $dz = dZ$, $dt = dT$. We get therefore to sufficient order for the covariance of (16)

$$ds^2 = \left(1 - v^2 - \frac{2GM}{r}\right) dT^2 + 2v dX dT - \left(1 + \frac{2GM}{r}\right)(dX^2 + dY^2 + dZ^2). \tag{23}$$

Thus in the particle's rest frame

$$\left.\begin{aligned} \mathbf{A} &= (-v, 0, 0) \\ \Phi &= \tfrac{1}{2}\left(1 - v^2 - \frac{2GM}{r}\right) \end{aligned}\right\}. \tag{24}$$

Apply now equation (17) in the particle's rest frame, yielding

$$-\frac{\partial}{\partial X}\left\{\tfrac{1}{2}\left(1 - v^2 - \frac{2GM}{r}\right)\right\} - \frac{\partial}{\partial T}(-v) = 0 \tag{25}$$

leading to

$$-\frac{GM}{r^2} + \frac{dv}{dt} = 0 \tag{26}$$

on substituting the original coordinates. This is the Newtonian equation of motion of the particle and is also the equation which would follow from the general pondermotive equation (16), applied in the original frame, using (21).

On examining (24) and (25) we see that the origin of the inertial term

$$-\frac{\partial}{\partial T}(-v)$$

in (25) lies in the relative motion of the universe, yielding $\mathbf{A} = (-v, 0, 0)$ in the particle's rest frame, and thus creating an inductive field at the particle which balances the local gravitational attraction due to the mass M, thus connecting with Sciama's ideas.

We note also that the $\mathbf{A}$, Φ in (24) arise by transformation of the *whole* $g_{\mu\nu}$ and not just their deviations from the Galilean values, in accordance with our tentative interpretation of the Galilean values as the static potentials of the whole universe.

(ii) The other case considered by Sciama is that of a particle moving with uniform motion in a circle under the attraction of a mass M at the centre, this mass being again at rest relative to the smoothed-out universe.

Transform therefore from the metric (20) to a suitable rest frame for the particle according to the relations

$$\left.\begin{aligned} x &= X\cos\omega T - Y\sin\omega T \\ y &= Y\cos\omega T + X\sin\omega T \\ z &= Z \\ t &= T \end{aligned}\right\} \tag{27}$$

so that to sufficient order

$$\begin{aligned} ds^2 = (1 - 2GM/R - \omega^2R^2)\, dT^2 - 2\omega\,(-YdXdT + XdYdT) \\ -(1 + 2GM/R)\,(dX^2 + dY^2 + dZ^2) \end{aligned} \tag{28}$$

with $R^2 = X^2 + Y^2$.

Thus in this frame

$$\left.\begin{aligned} \mathbf{A} &= (-\omega Y, \omega X, 0) \\ \Phi &= \tfrac{1}{2}\left(1 - \frac{2GM}{R} - \omega^2R^2\right). \end{aligned}\right\} \tag{29}$$

The equation (17) then yields

$$-\frac{GM}{R^2} + \omega^2 R = 0 \tag{30}$$

which is the Newtonian equation of motion, and also, putting $R = r$, what would be given by (16) in the original frame.

Connecting with Sciama's ideas we say that the gravitational attraction by M is balanced by the gravitational field induced by a rotating universe, whose rotational momentum is indicated by **A** in (29).

(iii) As a final example we shall show how, by means of the covariance of (16), the Newtonian "fictitious" forces may be attributed to the inductive effect of a moving universe in the most general Newtonian motion of the reference frame relative to a locally inertial frame.

Consider a free particle at rest in a reference frame which is locally inertial, so that the metric is approximately as given by (18) in that region. Let **r** be the position vector of the particle in that frame. Then by (15), (16) we have

$$\mathbf{r} = \text{constant}. \tag{31}$$

Transform to a second frame whose space origin has variable velocity **V** and which has variable spin **ω** relative to the first frame. If the position vector of the particle in this frame is **R**, then a well-known kinematic result of Newtonian motion gives

$$\dot{\mathbf{r}} = \mathbf{V} + \dot{\mathbf{R}} + \boldsymbol{\omega} \wedge \mathbf{R} \tag{32}$$

$$\ddot{\mathbf{r}} = \dot{\mathbf{V}} + \boldsymbol{\omega} \wedge \mathbf{V} + 2\boldsymbol{\omega} \wedge \dot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \wedge \mathbf{R} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R}) + \ddot{\mathbf{R}} \tag{33}$$

differentiation being with respect to the common Newtonian time of either frame. Thus for the particle in the second frame

$$\ddot{\mathbf{R}} = -[\dot{\mathbf{V}} + \boldsymbol{\omega} \wedge \mathbf{V} + 2\boldsymbol{\omega} \wedge \dot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \wedge \mathbf{R} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R})]. \tag{34}$$

The transformation connecting the two frames is, by (32), in differential form

$$\left.\begin{aligned} d\mathbf{r} &= (\mathbf{V}+\boldsymbol{\omega}\wedge\mathbf{R})dT + d\mathbf{R} \\ dt &= dT \end{aligned}\right\}. \tag{35}$$

Hence $ds^2 = dt^2 - d\mathbf{r}^2$

$$= [1-\mathbf{V}^2 - 2\mathbf{V}.(\boldsymbol{\omega}\wedge\mathbf{R}) - (\boldsymbol{\omega}\wedge\mathbf{R})^2]dT^2 - 2(\mathbf{V}+\boldsymbol{\omega}\wedge\mathbf{R}).d\mathbf{R}dT - d\mathbf{R}^2 \tag{36}$$

so that in the second frame

$$\left.\begin{aligned} \mathbf{A} &= \mathbf{V}+\boldsymbol{\omega}\wedge\mathbf{R} \\ \Phi &= \tfrac{1}{2}[1-\mathbf{V}^2-2\mathbf{V}.(\boldsymbol{\omega}\wedge\mathbf{R})-(\boldsymbol{\omega}\wedge\mathbf{R})^2] \end{aligned}\right\}. \tag{37}$$

Now

$$\operatorname{grad}\Phi = \boldsymbol{\omega}\wedge\mathbf{V} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R})$$

$$\frac{\partial\mathbf{A}}{\partial T} = \frac{\partial\mathbf{V}}{\partial T} + \frac{\partial\boldsymbol{\omega}}{\partial T}\wedge\mathbf{R}$$

$$= \dot{\mathbf{V}} + \dot{\boldsymbol{\omega}}\wedge\mathbf{R}$$

while curl $\mathbf{A} = 2\boldsymbol{\omega}$.

Hence by (16) $\ddot{\mathbf{R}} = -[\dot{\mathbf{V}} + \boldsymbol{\omega}\wedge\mathbf{V} + 2\boldsymbol{\omega}\wedge\dot{\mathbf{R}} + \dot{\boldsymbol{\omega}}\wedge\mathbf{R} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R})]$

giving complete agreement with (34).

Thus our theory gives an exact treatment of the fictitious forces as the inductive effect of a moving universe.

6. *Physical interpretation of* **A,** Φ *in general relativity.*—The analysis in this section is intended to be of a tentative nature, since complete rigour cannot be claimed for it.

In Section 4, equation (19), we obtained the result

$$\Phi_0 = \tfrac{1}{2}c^2 = \tfrac{1}{2}g_{44}(0)$$

for that value of the gravitational potential Φ of the whole universe which enters into the pondermotive equation (16), when evaluated at the space origin of a reference frame locally inertial there. In order to interpret this result in terms of Mach's principle we recall the expressions for ϕ in the quasi-Galilean case given by (7) and (11). Since the field equations are relations for the whole $g_{\mu\nu}$ in terms of the matter in the whole universe, we make the tentative inference that in some way the Galilean terms themselves are related to world gravitation, so that inertia would arise in accordance with Mach's principle. To what extent does general relativity provide justification of this inference?

In all cosmological models of general relativity in which the average inertial density ρ does not vanish there is an effective radius R of the model which is the distance, measured in a suitable way, to the horizon of the model where the velocity of the matter relative to the space origin equals the velocity of light. For an observer at the origin matter which goes beyond this distance virtually ceases to exist because of the Doppler effect on its light and presumably on its gravitation. This is the case whether the model be of the homogeneous rotating type (Gödel's models) or the isotropic expanding or contracting types. We shall discuss the latter as an example. These have the general metric

$$ds^2 = c^2dt^2 - e^{g(t)}\frac{(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)}{(1+r^2/4R_0^2)^2} \tag{38}$$

where R_0^2 may be positive, negative, or infinite, and the fundamental particles have constant r, θ, ϕ.

The distance to the particle at (r, θ, ϕ), measured in the simultaneity of the fundamental observers at cosmological time t, from $r=0$ is

$$l=e^{\frac{1}{2}g(t)}\int_0^r \frac{dr}{1+r^2/4R_0^2}.$$

Its radial velocity is therefore

$$\dot{l}=\tfrac{1}{2}\dot{g}l \tag{39}$$

so that $$|\dot{l}|=c \text{ when } l=\frac{2c}{|\dot{g}|}.$$

Thus $$R=\pm\frac{2c}{\dot{g}} \quad \text{according as } \dot{g}\gtrless 0 \tag{40}$$

and $$\dot{l}=\pm\frac{cl}{R}. \tag{41}$$

The importance for Mach's principle is that R is related to σ, the cosmological density of gravitational mass. In general relativity theory, gravitational mass density is defined so as to lead to Gauss' flux theorem for small regions of space (see, for example, Synge (4)), and for the isotropic cosmological models $\sigma=\rho+3p/c^2$ where p is the pressure. The gravitational "force" on unit mass due to the field is in this case the proper acceleration relative to the space origin. With these definitions McCrea (5) has shown that the equations of general relativity for the isotropic models are consistent with the variation of the gravitational force according to the Newtonian inverse square law, using proper radial distance but Euclidean geometry, for spatial regions of any size. The field equations applied to the metric (38) give, with $\Lambda=0$, in general units,

$$\left.\begin{aligned}\frac{8\pi G}{c^2}p &= -\frac{c^2}{R_0^2}e^{-g(t)}-\ddot{g}-\frac{3}{4}\dot{g}^2\\ 8\pi G\rho &= \frac{3c^2}{R_0^2}e^{-g(t)}+\frac{3}{4}\dot{g}^2\end{aligned}\right\} \tag{42}$$

so that

$$8\pi G\sigma=-3(\ddot{g}+\tfrac{1}{2}\dot{g}^2).$$

Now for $\dot{g}>0$, $\dot{g}=\frac{2c}{R}$, $\ddot{g}=-2c\dot{R}/R^2$ by (40). Hence

$$8\pi G\sigma=-\frac{6c^2}{R^2}(1-\dot{R}/c)$$

whence, if $\dot{g}>0$, $$G\sigma R^2=-\frac{3c^2}{4\pi}(1-\dot{R}/c)$$

and, if $\dot{g}<0$, $$G\sigma R^2=-\frac{3c^2}{4\pi}(1+\dot{R}/c) \tag{43}$$

which are the required relations between σ and R, at time t. For $\dot{g}>0$ we see that $\sigma\gtrless 0$ according as $\dot{R}\gtrless c$. If $\dot{R}>c$ matter is entering the region bounded by the defined horizon; if $\dot{R}<c$ matter is passing beyond this horizon. For $\dot{g}<0$, $\sigma\gtrless 0$ according as $\dot{R}\lessgtr c$.

By (41) and (43) we get the Newtonian type equation

$$\ddot{l}=-\tfrac{4}{3}\pi G\sigma l \tag{44}$$

relating gravitational force and proper distance at time t.

Comparing equation (44) with the pondermotive equation (16) we see that, for a fundamental observer whose radial space coordinate is the proper distance l and whose time is the cosmological time t, $\mathbf{A}=0$ and $-\text{grad}\,\Phi=-\frac{4\pi}{3}\pi G\sigma\mathbf{l}$. Equation (44) however holds for all $l\leqslant R$ and not just in the neighbourhood of the origin. Consider therefore the gravitational "work" done by the field when a particle of unit mass is moved from its actual position at time t to the horizon and therefore beyond influence of the origin. This will be

$$\Phi_l=-\frac{4}{3}\pi G\int_l^R \sigma l dl. \tag{45}$$

This may be regarded as the analogue of the Newtonian potential at the distance l, in the gravitational field as witnessed by an observer at the origin. Both σ and R will vary with l in this integral as the motion proceeds, according to (42), (43). However Φ_l may be evaluated as

$$\Phi_l=\int_l^R \ddot{l} dl=\tfrac{1}{2}c^2-\tfrac{1}{2}\dot{l}^2$$
$$=\frac{c^2}{2}(1-l^2/R^2). \tag{46}$$

This is the potential at $\mathbf{l}$ at time t.

Putting $l=0$ we get $$\Phi_0=c^2/2 \tag{47}$$

which therefore provides a physical identification, in a natural way, of the potential Φ_0 arising in equation (19).

The result given by (47) for Φ_0 is got as the limit of Φ_l when l tends to zero irrespectively of the sign of σ or $\dot{g}$. For instance if $\sigma>0$ and $\dot{g}>0$ then $\dot{R}>c$, so that for the field to carry the particle to the horizon of the space origin would mean going backwards in time. The discrepancy in sign between our Φ_0 and Sciama's, referred to at the end of Section 4, arises because of Sciama's arbitrary definition of Φ for an expanding universe. His definition appears to ignore the above considerations and in particular to presuppose the identity of the cosmological gravitational mass density and the inertial density.

For cogent reasons which have been put forward elsewhere (**6**) a stationary cosmological solution is to be preferred. The only known stationary solution which does not contradict observational results (expansion, spatial isotropy) is the steady state theory proposed by Bondi and Gold (**6**). This has the metric

$$ds^2=c^2dt^2-e^{2ct/R}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) \tag{48}$$

where R is a constant which is the effective radius for the model. The steady state model, unlike the general cosmological models of metric given by (38) for which equation (16) vanishes identically, allows a static metric to be used so that the motion of a particle, relative to the observer at the origin, is measured by the rate of change of the spatial coordinates. This is the De Sitter metric

$$ds^2=c^2(1-l^2/R^2)d\tau^2-\frac{dl^2}{1-l^2/R^2}-l^2d\theta^2-l^2\sin^2\theta d\phi^2 \tag{49}$$

connected to (48) by a well known transformation. In this metric l is the distance from the origin, in the simultaneity of the fundamental observers, of our general analysis. The theory of Section 4 defines the Φ involved in (16) as $\frac{1}{2}g_{44}$ which in the case of the metric (49) gives

$$\Phi=\frac{c^2}{2}(1-l^2/R^2) \tag{50}$$

agreeing with (46) and therefore having the physical interpretation associated with (46).

It is to be noted that the steady state forms a natural cosmological background to mass concentrations. For instance the *exact* solution of the field equations for an isolated mass m superimposed on the steady state is

$$ds^2 = c^2\left(1 - \frac{2mG}{c^2 l} - \frac{l^2}{R^2}\right)d\tau^2 - \frac{dl^2}{1 - \frac{2mG}{c^2 l} - \frac{l^2}{R^2}} - l^2 d\theta^2 - l^2 \sin^2\theta d\phi^2. \quad (51)$$

For this metric

$$\Phi = \frac{c^2}{2}(1 - l^2/R^2) - \frac{mG}{l} \quad (52)$$

to be interpreted physically as the work done by the field in removing unit mass from the point in question to the horizon of the model, regarding mG/R as negligible.

A Newtonian type integral for Φ in terms of the distribution of the mass does not follow simply in the case of the general models because of the stated dependence of σ and R on cosmological epoch. However, for the steady state, σ and R are constant and we may write for the potential of unit mass, at distance l from the mass σdV constantly in the volume element dV,

$$\begin{aligned} d\Phi &= -G\int_l^R \frac{\sigma dV}{l^2}\,dl \\ &= -G\sigma dV\left(\frac{1}{l} - \frac{1}{R}\right). \end{aligned} \quad (53)$$

Thus

$$\begin{aligned} \Phi_0 &= -G\sigma\int_0^R \left(\frac{1}{l} - \frac{1}{R}\right) 4\pi l^2 dl \\ &= -\frac{2\pi}{3} G\sigma R^2. \end{aligned}$$

Equation (43) gives for the steady state $G\sigma R^2 = -3c^2/4\pi$ so that

$$\Phi_0 = c^2/2$$

in agreement with (47) and justifying our physical interpretation of Φ_0 as the gravitational potential of all the matter in the universe apparent to an observer at the origin and having influence there.

The quantity **A** of our theory defined in equation (15) of Section 4 is zero for the cosmological metric (38). On making a transformation such as that of the first example in Section 5, equation (22), a non-zero **A** arises by transformation of the $g_{\mu\nu}$. If we accept the association of the Galilean values of the $g_{\mu\nu}$ with world gravitation, according to the tentative analysis presented above, the association of **A** with the relative momentum of the universe would also follow. While the Galilean g_{44}, viz. c^2, is associated with Φ_0 as $2\Phi_0$, the spatial Galilean $g_{\mu\nu}$ are associated with **A**. Thus in the first example in Section 5 to get **A** in the particle's rest frame we have to multiply the Galilean g_{11} in the original frame, viz. $-1 = g_{11}(0)$, by v. According to the subsequent application of the pondermotive equation $-g_{11}(0)$ is proportional to the inertial mass of the particle, the whole equation indicating equality of gravitational and inertial mass in the case of a particle. Since $-g_{11}(0) = g_{44}(0)/c^2 = 2\Phi_0/c^2$ we see that inertial mass can therefore be associated with the influence of the whole universe, in accordance with Mach's principle.

7. *Comparison with Sciama's theory.*—In this final section we shall remark briefly on the three principal differences claimed by Sciama between his theory and general relativity, enumerated (i), (ii), (iii) in the introduction to this paper.

(i) It is evident from the analysis in this paper that a knowledge of G, occurring in the integral (45) leading to (47), together with R given as cT where T is the reciprocal of the Hubble parameter, leads to an estimate of the amount of matter in the universe in general relativity as well as in Sciama's theory.

(ii) Sciama states that in general relativity the principle of equivalence predicts that one gravitating mass in an otherwise empty universe produces the same inertial effects as in his theory, and since there is no universe in this case to give rise to the inductive field "it is difficult to see why the principle of equivalence should be true". Such an argument however implies a solution of the field equations involving the use of coordinates for all points of space-time in a universe which, except for the isolated mass, is empty. Such coordinates are purely conceptual, defined without reference to matter and restoring to space an objective substance, independent of matter, which general relativity has sought to deny. The logical course for general relativity, according to the field equations, is to relate the Galilean $g_{\mu\nu}$ of special relativity to world gravitation in a full universe. That general relativity may be capable of doing so has been indicated in this paper, where, independently of the value of ρ as long as it does not vanish, $\frac{1}{2}\,g_{44}(0)$ has been identified as $\Phi_0 = c^2/2$, the potential of the universe. The case of the empty universe can only logically be approached as a limit where $\rho \to 0$ and $R \to \infty$ (equation (43)), so that inertia is always accounted for.

(iii) Sciama's determination of the sign of the field in his theory is of doubtful significance as it depends on his Φ as defined turning out to be negative. Reasons for questioning this arbitrary definition of Φ have been given in Section 6.

In general relativity the coefficient of $T^{\mu\nu}$ in the field equations is chosen so as to make gravitation attractive on the small scale (the pressure being then relatively negligible). However, on the cosmological scale this leads to gravitational mass being interpreted as negative if the expansion is accelerating (equation (44)). Thus there would appear to be no intrinsic importance to be attached to the sign of the field in any theory that allows for factors at present unknown, that is whether the gravitational effect of a lump of matter is more primary than that of a cosmological region containing "zero-point" stress.

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References

(1) D. W. Sciama, *M.N.*, **113**, 34, 1953.
(2) A. Einstein, *The Meaning of Relativity* (Methuen) 5th Edition, p. 97, 1951.
(3) A. Einstein, *op. cit.*, p. 83.
(4) J. L. Synge, *Proc. Edin. Math. Soc.*, 2nd Series, **5**, 93, 1937.
(5) W. H. McCrea, *Proc. Roy. Soc.* A, **206**, 562, 1951.
(6) H. Bondi and T. Gold, *M.N.*, **108**, 252, 1948.