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General Relativity Extended

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1. Introduction

We extend Einstein's General Relativity in two ways:
(1) Einstein Field Equations ("*EFE*") explain gravity by energy distributions over space-time, but they can also explain electromagnetism by charge distributions in like manner. This is not to be confused with the well-known Einstein-Maxwell equations, in which electromagnetic fields' energy contents are added onto those as attributed to the presence of matter, to account for gravitational motions; in short, we are here substituting the term "electric charge" for energy, and electromagnetism for gravity, i.e., a geometrization of the electromagnetic force.
(2) *EFE* describe *one* space-time, but we propose *two*: one for "particles" and the other for "waves;" to wit, there are two gravitational constants and we have unified the gravitational motions in a "combined space-time 4-manifold."
In Section 2, we shall prove that electromagnetic fields as produced by charges, in analogy with gravitational fields as produced by energies, cause spacetime curvatures, not because of the energy contents of the fields but because of the Coulomb potential of the charges; as a result, we shall derive a special constant of proportionality between an electromagnetic energy-momentum tensor and Einstein tensor, to arrive at

$$R_{\mu\nu,em} - \frac{1}{2} R_{em} \cdot g_{\mu\nu,em}^{att;\ rep} = -\frac{16\pi G}{\left(1 - \gamma_{grav}^{-2} \cdot g_{11,grav}\right) c^5} T_{\mu\nu,em}^{att;\ rep}. \quad (1)$$

The geodesics of the resultant electromagnetic 4-manifold represent the same dynamics as that given by the classical Lagrangian resulting in the Lorentz force law of motion.
In Section 3, we define "combined manifold" $\mathcal{M}^{[3]}$ as the graph of a diffeomorphism from one manifold $\mathcal{M}^{[1]}$ to another $\mathcal{M}^{[2]}$, akin to the idea of a diagonal map. We derive the values for: (1) the energy distribution between a particle in $\mathcal{M}^{[1]}$ and its accompanied electromagnetic wave in $\mathcal{M}^{[2]}$, for the combined entity $[particle, wave]$, and (2) the gravitational constant $G2$ for $\mathcal{M}^{[2]}$, where there exist only electromagnetic waves and gravitational forces. Because of a large $G2$, an astronomical black hole B arose in $\mathcal{M}^{[2]}$, branching out $\mathcal{M}^{[1]}$ (the *Big Bang*), with a portion of a wave energy in $\mathcal{M}^{[2]}$ transferred to $\mathcal{M}^{[1]}$ as a photon, which collectively were responsible for the subsequent formation of matter. Being within the Schwarzschild radius, B in $\mathcal{M}^{[2]}$ is a complex (sub) manifold, which furnishes exactly the geometry for the observed quantum mechanics; moreover, B provides an energy interpretation to quantum probabilities in $\mathcal{M}^{[1]}$. In brief, our $\mathcal{M}^{[3]}$ casts quantum mechanics in the framework of General Relativity.
In Section 4, we draw a summary.

2. *EFE* for Electromagnetism

2.1 Background

In this Section 2 we derive Einstein Field Equations for electromagnetism and unite it with gravity in one common explicit form of *EFE*. Since Einstein's success in geometrizing gravity in General Relativity, a major drive has been the search for a unified geometric field theory (for some of the latest many attempts, see, e.g., [14, 24, 33]). A brief account here is in order. In about 1920 Kaluza and Klein proposed a 5-dimensional manifold combining Maxwell equations with *EFE*; the idea was soon put aside due to the emergence of quantum mechanics, which revealed two other fundamental forces of nature: the strong and the weak nuclear forces. Nevertheless, the construct of a "curled-up" dimension eventually resurfaced later in string theories.

In about the same time, Weyl introduced the idea of gauge invariance of conformal Riemannian geometry, which later led to Yang-Mills theory, supersymmetry, quantum field theories, and the unified M string theory by Witten (cf. [36]). A basic premise underlying these developments has been that in order to deal with the periodic nature as inherent in electrodynamics a complex structure is indispensable, thus opening up Clifford algebra, Finsler geometry, K*ä*hler manifolds (see, e.g., [25]), and Calabi-Yau spaces, all involving dimensions higher than $\mathbb{R}^4$ - - the suitability of which in describing the physical universe has been increasingly questioned in recent literature (cf. e.g., [33]).

Amid the above intensive elaborate mathematical research, as is well known, gravity remains resistant to unification, where the electroweak theory has been established by Winberg, Glashow and Salem since the late 1960s and the electrostrong theory has been treated under the subject of quantum chromodynamics.

A distinct feature of gravity is the existence of the principle of equivalence between inertial masses and gravitational masses, so that the two cancel out and the size of the inertial mass does not need to be addressed explicitly. Here we shall solve the problem of the lack of the same principle for electromagnetism (cf. [5]) via the denominator of the constant of proportionality

$$\kappa_{em} = -\frac{16\pi G}{\left(1 - \gamma_{grav}^{-2} g_{11,grav}\right) c^5}. \tag{2}$$

In this connection, we also make a distinct identification of $T_{11,em}^{att;rep}$ with the norm of the Poynting vector (cf. [1] for a discussion of the Poynting vector), and as a result the derived geodesics correspond to the least action by Feynman. In that we have demonstrated a Poynting vector on the right-hand-side of *EFE* being in direct correspondence with a minimization of the integral of kinetic energy minus potential energy over all trajectories on the left, we see the reasons why any other identifications of $T_{\mu\nu,em}$ have resulted in difficulties in geometrizing electromagnetism or else have led to the above-mentioned other geometries. In this regard, our $T_{11,em}^{att;rep}$ has unit $joule/\left(\sec ond \cdot meter^2\right)$, representing energy flows in a specific direction across an area of square meter per second, and yet the common identification of $T_{11,em}$ with the energy densities has unit $joule/\left(meter^3\right)$ (see, e.g., [35], 45, equation (2.8.10)), representing stationary energies, but the energy-momentum tensor is defined for energy flows. Here we cite [7]: "An important problem is to determine the flow energy along a given direction for a given physical field. This description uses a 2-covariant symmetric tensor field T_{ij}, called the energy-momentum tensor. The energy flow in the X direction is given by the expression

$$T(X,X) = T_{ij}X^iX^j\text{." ([7], 75, equation 5.3.25)} \tag{3}$$

As such, it comes as no surprise that our $T_{11,em}^{att;rep}$ directly leads to the least action, from which follows the Lorentz force law governing the general nonquantum electrodynamics (see [15], II-19-7).

Our approach here in this paper is to pay careful attention to the intricate details laying the foundation of Special Relativity, General Relativity, and electromagnetism and to underscore the essential logic that connects these three topics. Following Einstein, we make use of the differential geometric property of Einstein tensor

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R \cdot g_{\mu\nu} \tag{4}$$

being proportional to energy-momentum tensor $T_{\mu\nu}$ (cf. [21], 858) and apply weak field approximations (see [12], 814-818) to establish the constant of proportionality κ_{em} as based on weakly attractive or repulsive electromagnetic fields (cf. [35], 151-157 for a derivation of EFE). As such, there will be numerous "approximately-equal" signs in our derivation of κ_{em}; nevertheless, the derived value of κ_{em} is *exact*.

The significance of our results is that the distribution of electric charges in space-time results in a 4-manifold $\mathcal{M}_{em}^4$ of curvatures and charges move along geodesics of $\mathcal{M}_{em}^4$, i.e., a geometrization of the electromagnetic force, which is a step toward a unified field theory (for related work integrating electromagnetism with EFE, cf. e.g., [29, 31]).

Our derivation below will first aim at deriving g_{em} (proving that the associated geodesics are exactly the classical electromagnetic Lagrangian), then $\mathcal{E}_{em}$, and finally

$$\frac{\mathcal{E}_{12,em}}{\mathcal{E}_{11,em}^{att;rep}} = \frac{-\left\|\bar{\mathbf{g}}\right\| \mathbf{V}_{Q,x}}{\pm\left\|\bar{\mathbf{S}}\right\|} \equiv \frac{T_{12,em}\text{ (momentum)}}{T_{11,em}^{att;rep}\text{ (energy)}}, \tag{5}$$

to obtain

$$\kappa_{em} = \frac{\mathcal{E}_{11}}{T_{11}}. \tag{6}$$

To go one step further, we will also unite electromagnetism with gravity in one set of *EFE* to arrive at

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R \cdot g_{\mu\nu} = -\frac{8\pi G}{c^2}T_{\mu\nu,grav} \mp \frac{16\pi G}{\left(1-\gamma^{-2}g_{11,grav}\right)c^5}T_{\mu\nu,em}^{att;rep^*}. \tag{7}$$

2.2 Derivations

Definition 1. *The Minkowski space*

$$\mathbb{R}^{1+3} \ : \ = \{\, (t, \mathbf{x} \equiv (x,y,z)) \in \mathbb{R}^4 \mid \textit{the inner product} \tag{8}$$

$$\left\langle \mathbf{e}_i, \mathbf{e}_j \right\rangle \ : \ = \mathbf{e}_i^T \eta \mathbf{e}_j,\ i,j = 1,2,3,4, \tag{9}$$

$$\eta \ : \ = diag\left(1, -c^{-2}, -c^{-2}, -c^{-2}\right)_{\mathbf{E}}, \tag{10}$$

$$\mathbf{E} \equiv \left(\mathbf{e}_i \equiv (Kronecker\ \delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4})\right)_{i=1}^4, \tag{11}$$

$$c \equiv \textit{the speed of light in the vacuum}\}. \tag{12}$$

$$\textit{The proper time } \tau_o \textit{ of any reference frame } O$$

$$\textit{is such that } \tau_o(O) \equiv (\tau_o, 0, 0, 0). \tag{13}$$

Remark 1. *If $\mathcal{M}^4 = \mathbb{R}^{1+3}$, then $f =$ the Lorentz transformation $\mathcal{L}$; $\mathcal{L} : S \longrightarrow \tilde{S}$ has the following matrix representation if $(t, x, y, z) = (0, 0, 0, 0) = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ and $\mathcal{L}(1, V, 0, 0) = (\tilde{t}_o, 0, 0, 0)$, $V \in \mathbb{R}$:*

$$L = \gamma \begin{pmatrix} 1 & -\frac{V}{c^2} \\ -V & 1 \end{pmatrix}_{(\mathbf{e}_1, \mathbf{e}_2)}, \tag{14}$$

where $(V, 0, 0)$ is the velocity of $\tilde{S}$ relative to S and

$$\gamma \equiv \left(1 - \left(\frac{V}{c}\right)^2\right)^{-\frac{1}{2}} \in [1, \infty) \tag{15}$$

is the Lorentz factor. Consider an emission of light at $t_o = 0 = \tilde{t}_o$ in the direction of $V \in \mathbb{R}$; then $\forall t_o, \tilde{t}_o > 0$ S observes $(t_o, t_o c)$ and $\tilde{S}$ observes $(\tilde{t}_o, \tilde{t}_o c)$; further,

$$L(t_o, t_o c)^T = \gamma \left(1 - \frac{V}{c}\right) \cdot (t_o, t_o c)^T = (\tilde{t}_o, \tilde{t}_o c)^T; \tag{16}$$

thus,

$$\frac{\tilde{t}_o}{t_o} = \gamma \left(1 - \frac{V}{c}\right) = \lambda, \text{ an eigenvalue of } L. \tag{17}$$

Note that

$$\gamma \left(1 - \frac{V}{c}\right) \cdot \gamma \left(1 + \frac{V}{c}\right) = 1; \tag{18}$$

i.e., L has two eigenvalues

$$\lambda_{\max} = \gamma \left(1 + \frac{|V|}{c}\right) > 1, \text{ and} \tag{19}$$

$$\lambda_{\min} = \gamma \left(1 - \frac{|V|}{c}\right) < 1. \tag{20}$$

Remark 2. *At this point, we alert the reader to be aware of the existence of* three *identities: (1) the reader (or the analyst), who serves as the laboratory frame O and sets up a local parametrization*

$$f : U_{(0,\mathbf{0})} \subset \mathbb{R}^{1+3} \longrightarrow \text{ the space-time } 4 - manifold\ \mathcal{M}^4, \tag{21}$$

(2) S, and (3) $\tilde{S}$.

Remark 3. *In the above Equation (14), if $V = 0$, then $\mathcal{L} = I$. Consider now $V(t) \equiv 0\ \forall t \in (-\infty, 0]$; however, $\forall t \in (0, T]$, we have $V(t) \approx at$ for some $T > 0$ and some constant acceleration $a > 0$, due to the existence of some force. Then*

$$\lambda = \frac{\tilde{t}_o}{t_o} \approx \gamma(t) \left(1 - \frac{V(t)}{c}\right) \tag{22}$$

measures the curvatures of $\mathcal{M}^4$ over $(0, T]$. This treatment of λ will play a vital role in our subsequent derivations. Since $V(t) \approx at > 0$ on $(0, T]$, we have

$$\lambda \approx \sqrt{\frac{c - V(t)}{c + V(t)}} < 1. \tag{23}$$

By Einstein's General Relativity, a clock undergoing a gravitational free fall slows down (e.g., consider a clock approaching a black hole). As such, we conclude that $\lambda < 1$ for attractive forces; by a reversal of time in the preceding dynamics, we deduce that $\lambda > 1$ for repulsive forces. We will thus make the following distinction and notation:

$$\lambda_{att} := \gamma \left(1 - \frac{|V|}{c}\right) < 1, \text{ and} \tag{24}$$

$$\lambda_{rep} := \gamma \left(1 + \frac{|V|}{c}\right) > 1. \tag{25}$$

Further, note that $\forall \left(\frac{V}{c}\right) \approx 0$, one uses

$$\frac{m_o}{\lambda_{att}} \approx m_o \gamma \text{ and} \tag{26}$$

$$\frac{m_o}{\lambda_{rep}} \approx m_o \gamma^{-1} \tag{27}$$

for (Special) relativistic adjustment of a mass. Also, a metric g on $\mathcal{M}^4$ by definition is such that

$$g_{11} \approx \left(\frac{\tilde{t}_o}{t_o}\right)^2 \approx \left(\lambda_{att;\,rep}\right)^2 = \lambda_{att}^{\pm 2}. \tag{28}$$

Remark 4. *Let $p_1, p_2 \in \mathcal{M}^4$; then a maximization of*

$$\int_{f^{-1}(p_1)}^{f^{-1}(p_2)} \frac{d\tilde{t}_o}{dt_o} dt_o \tag{29}$$

over all trajectories $\{(t, x(t), y(t), z(t))\}$ derives the geodesic from p_1 to p_2 maximizing the proper time elapsed in $\tilde{S}$.

Proposition 1. *Let g be a local metric of $\mathcal{M}^4$ and express g as a matrix in the basis of $\mathbf{B} \equiv \left\{\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\}$; if $f \approx \mathcal{L}$ (i.e., $\mathcal{M}^4$ is near flat), then*

$$\frac{d\tilde{t}_o}{dt_o} = (1, 0, 0, 0)\ g_{\mathbf{B}}\ \left(\mp 1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z\right)^T. \tag{30}$$

Proof. Without loss of generality, consider

$$L = \gamma \begin{pmatrix} 1 & \pm\frac{\mathbf{V}}{c^2} \\ \pm\mathbf{V} & 1 \end{pmatrix} \tag{31}$$

and calculate $(1,0)\ g_{\mathbf{B}}\ (\mp 1, \mathbf{V})$

$$= (1,0)\left(\left(L^{-1}\right)^{T}\right)^{-1}\left[\left(L^{-1}\right)^{T} g_{\mathbf{B}}\, L^{-1}\right] L\left(\mp 1, \mathbf{V}\right)^{T} \tag{32}$$

$$\approx (1,0)\left(\gamma\begin{pmatrix} 1 & \pm\mathbf{V} \\ \pm\frac{\mathbf{V}}{c^2} & 1\end{pmatrix}\right)\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{c^2}\end{pmatrix}\begin{pmatrix}\Delta\tilde{t}_o \\ 0\end{pmatrix} \tag{33}$$

$$= \left(\gamma, \mp\frac{\gamma\mathbf{V}}{c^2}\right)\begin{pmatrix}\Delta\tilde{t}_o \\ 0\end{pmatrix} \text{ (observe that } L : (\mp 1, \mathbf{V})^T \longmapsto (\Delta\tilde{t}_o, 0)^T, \tag{34}$$

where $\Delta\tilde{t}_o$ is the proper time of $\tilde{S}$ by definition)

$$= \frac{\Delta\tilde{t}_o}{\sqrt{1-\left(\frac{\mathbf{V}}{c}\right)^2}} = \frac{\Delta\tilde{t}_o}{\left\|(\mp 1, -\mathbf{V})^T\right\|_{\eta}} = \frac{\Delta\tilde{t}_o}{\left\|L^{-1}(\mp 1, -\mathbf{V})^T\right\|_{\eta}} \tag{35}$$

$$= \frac{\Delta\tilde{t}_o}{\Delta t_o} \approx \frac{d\tilde{t}_o}{dt_o}, \text{(where } L^{-1} : (\mp 1, -\mathbf{V})^T \longmapsto (\Delta t_o, 0)^T, \tag{36}$$

analogous to the above Equation (34)).

■

The Setup - -

We consider the dynamics of a charge Q at $(0,0,0,0) \in U$ that attracts or repels a charge q at $(0,x,y,z) \in U$, where

$$r_\infty \equiv \sqrt{(x^2+y^2+z^2)} \text{ is such that } r_\infty^{-1} \approx 0. \tag{37}$$

Theorem 1. *(Feynman [15], II-28-2) The field momentum produced by Q is*

$$\mathbf{P}(t) = \frac{Q^2}{4\pi\epsilon_o r_o c^2}\mathbf{V}_Q(t), \tag{38}$$

where $\epsilon_o \equiv$ the permittivity constant $\approx \frac{1}{9\times 4\pi} \times 10^{-9} \times \frac{coulomb^2\cdot second^2}{kilogram\cdot meter^3}$, $r_o \equiv$ the "classical electron radius" $\approx 2.82\times 10^{-15}$ meter, and $\mathbf{V}_Q(t) << c$ is the velocity of Q at t.

Remark 5. *We note that the above Equation (38) was derived in [15] by an integration over the (continuous) field energy densities (cf. [15], II-28-2 and II-8-12). Thus, to apply Equation (38) to the above* Setup *of exactly two (discrete) point charges, we must have*

$$Q = q = \text{ the smallest charge } = \text{ an electron.} \tag{39}$$

Definition 2.

$$\textit{The } \text{average } \textit{field momentum density } \bar{\mathbf{g}}(t) := \mathbf{P}(t) \Big/ \left(\frac{4\pi r_\infty^3}{3}\right). \tag{40}$$

Theorem 2. *(Feynman [15], II-27-9) The Poynting vector $\mathbf{S}$ is related to the momentum density $\mathbf{g}$ by*

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S}. \tag{41}$$

Corollary 1.

$$\mathbf{P}(t) = \left(\frac{4\pi r_\infty^3}{3}\right)\bar{\mathbf{g}}(t) \tag{42}$$

$$= \left(\frac{4\pi r_\infty^3}{3}\right)\frac{\bar{\mathbf{S}}(t)}{c^2}. \tag{43}$$

$$\textit{where } \bar{\mathbf{S}}(t) \equiv \textit{the } \text{average } \textit{field energy flow in the direction} \tag{44}$$

$$\textit{of } \mathbf{V}_Q(t)\textit{, with unit equal to } \left(\frac{joule}{second \cdot meter^2}\right). \tag{45}$$

Theorem 3. *(Feynman* [15], *II-27-11: Conservation of the total momentum of particles and field)*

$$\mathbf{P}(t) \equiv m_{Q,o}\mathbf{V}_Q(t) = -m_{q,o}\mathbf{V}(t), \tag{46}$$

where $m_{Q,o}$ and $m_{q,o}$ are respectively the rest masses of Q and q.

Remark 6. *The Newton's law of motion as adjusted for the effect of Special Relativity is*

$$\mathbf{F}^{att;\, rep} = \left(\gamma^{\pm 1} m_o\right)\left(\gamma^{\pm 2}\mathbf{a}\right) \tag{47}$$

respectively for attractive and repulsive force $\mathbf{F}^{att;\, rep}$ if $\mathbf{a}$ is in the direction of $\mathbf{V}$ (cf. [23], *Equation* (13.31)*, 272-273; also, Equations* (26),(27) *above).*

Proposition 2. *Let $v(t) := \|\mathbf{V}(t)\|$ and $v_Q(t) := \left\|\mathbf{V}_Q(t)\right\|$; then*

$$\gamma^{\pm 2}\left(\frac{v(t)}{c}\right) = \frac{\textit{the electric potential energy } PE_e \textit{ of } Q \textit{ and } q}{\textit{the rest energy } RE \textit{ of } q}. \tag{48}$$

Proof. By Theorems 1 and 3,

$$\left(\frac{v(t)}{c}\right) = \left(\frac{1}{m_{q,o}c^2}\right)\cdot q\left(\frac{Q}{q}\frac{v_Q(t)}{c}\frac{r_\infty}{r_o}\right)\cdot\frac{Q}{4\pi\epsilon_o r_\infty} \tag{49}$$

$$\equiv \frac{1}{RE}\cdot K\cdot\frac{qQ}{4\pi\epsilon_o r_\infty}, \tag{50}$$

where

$$K \equiv \frac{Q}{q}\frac{v_Q(t)}{c}\frac{r_\infty}{r_o} = \frac{v_Q(t)\cdot\left(\frac{r_\infty}{c}\right)}{r_o} \text{ (cf. Remark 5)} \tag{51}$$

is an electrodynamic adjustment factor of the electrostatic potential (cf. [15], II-15-14, 15);

$$K = 1 \text{ if } v_Q(t)\cdot\left(\frac{r_\infty}{c}\right) \equiv v_Q(t)\cdot t = r_o, \tag{52}$$

i.e., the point charge Q travels to the boundary of the "classical electron," or equivalently, Q is a stationary electron. Thus, taking into account the effect of Special Relativity, we have

$$\gamma^{\pm 2}\left(\frac{v(t)}{c}\right) = \frac{\gamma^{\pm 2}KQq/4\pi\epsilon_o r_\infty}{RE} = \frac{PE_e}{RE}. \tag{53}$$

■

Corollary 2.

$$-\gamma^{\pm 2}\left(\frac{v(t)}{c}\right)\left(\frac{v(t)\,v_Q(t)}{c^2}\right)=\frac{q\mathbf{V}(t)\cdot\mathbb{A}(t)}{RE}, \tag{54}$$

where $\mathbb{A}(t)$:= the vector potential, or curl $\mathbb{A}(t)$ = the magnetic field $\mathbb{B}$.

Proof. Since

$$-v(t)\,v_Q(t)=\mathbf{V}(t)\cdot\mathbf{V}_Q(t)\ \text{ and} \tag{55}$$

$$\frac{\gamma^{\pm 2}KQ\mathbf{V}_Q(t)}{4\pi\epsilon_o r_\infty c^2}=\mathbb{A}(t)\ \ (\,[15]\,,\text{II-14-4}), \tag{56}$$

we have

$$-\gamma^{\pm 2}\left(\frac{v(t)}{c}\right)\left(\frac{v(t)\,v_Q(t)}{c^2}\right) \tag{57}$$

$$=\frac{\gamma^{\pm 2}KQ\,q\mathbf{V}(t)\cdot\mathbf{V}_Q(t)}{RE\cdot 4\pi\epsilon_o r_\infty c^2}=\frac{q\mathbf{V}(t)\cdot\mathbb{A}(t)}{RE}. \tag{58}$$

■

Definition 3. *We call an electromagnetic field attractive if the total potential energy is negative, and repulsive if the total potential energy is positive.*

Proposition 3. *For any weakly attractive or repulsive electromagnetic field, the metric $g_{em}^{att;\,rep}$ has the following matrix representation in the basis of* **B** *(refer to Proposition 1 above):*

$$g_{em}^{att;\,rep}=\begin{pmatrix}\lambda_{em}^{\pm 2} & -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_x}{c^3} & -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_y}{c^3} & -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_z}{c^3}\\ -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_x}{c^3} & o\left(\frac{v}{c}\right)-c^{-2} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)^3\\ -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_y}{c^3} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)-c^{-2} & o\left(\frac{v}{c}\right)^3\\ -\frac{2\gamma^{\pm 2}v_Q\mathbf{V}_z}{c^3} & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)^3 & o\left(\frac{v}{c}\right)-c^{-2}\end{pmatrix}. \tag{59}$$

Proof. First, we note that besides being symmetric, $g_{em}^{att;\,rep}\longrightarrow\eta$, as $\mathbf{V},\mathbf{V}_Q\longrightarrow\mathbf{0}$. Second,

$$g_{11,em}^{att;\,rep}=\lambda_{em}^{\pm 2}\approx\left(\frac{\tilde{t}_o}{t_o}\right)^2_{att;\,rep}\quad(\text{cf. Equation }(28)\,). \tag{60}$$

Third, by Proposition 1 we have

$$\frac{d\tilde{t}_o}{dt_o}=(1,0,0,0)\ g_{em}^{att;\,rep}\ \left(\mp 1,\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_z\right)^T \tag{61}$$

$$=\mp\lambda^{\pm 2}-\frac{2\gamma^{\pm 2}v_Q v^2}{c^3} \tag{62}$$

$$\approx\mp\gamma^{\pm 2}\left(1\mp\frac{2v}{c}\right)+\frac{2q\mathbf{V}\cdot\mathbb{A}}{RE}\ \text{(by Corollary 2)} \tag{63}$$

$$=\mp\gamma^{\pm 2}+2\gamma^{\pm 2}\left(\frac{v}{c}\right)+\frac{2q\mathbf{V}\cdot\mathbb{A}}{RE} \tag{64}$$

$$\equiv\mp\left(\frac{1}{1-\left(\frac{v}{c}\right)^2}\right)^{\pm 1}+\frac{2\left(PE_e+q\mathbf{V}\cdot\mathbb{A}\right)}{RE}\text{(by Proposition 2);} \tag{65}$$

here we note that $q\mathbf{V}\cdot\mathbb{A}$ is not to be identified with the magnetic potential energy since the magnetic force being always orthogonal to the velocity of q does not do any work; nevertheless, we will henceforth set $PE_e + q\mathbf{V}\cdot\mathbb{A} \equiv PE_{em}$ for presentation brevity (cf. [30], 84, where PE_{em} is noted for the term "generalized potential" energy). To continue, we thus have

$$\frac{d\tilde{t}_o}{dt_o} \approx \mp\left(\frac{1}{1-\left(\frac{v}{c}\right)^2}\right)^{\pm 1} + \frac{2\left(PE_e + q\mathbf{V}\cdot\mathbb{A}\right)}{RE} \tag{66}$$

$$\approx \mp\left(1 \pm \left(\frac{v}{c}\right)^2\right) + \frac{2PE_{em}}{RE} \tag{67}$$

$$= \mp 1 - \frac{m_o v^2}{m_o c^2} + \frac{2PE_{em}}{RE} \tag{68}$$

$$= \mp 1 - \frac{2\left(\text{kinetic energy } KE - PE_{em}\right)}{RE}, \tag{69}$$

which is equivalent to Feynman's least action for the classical electrodynamics since a maximization of

$$\int_{f^{-1}(p_1)}^{f^{-1}(p_2)} \frac{d\tilde{t}_o}{dt_o} dt_o = \int \left(PE - KE\right)\, dt_o \tag{70}$$

is equivalent to a minimization of $\int \left(KE - PE\right)\, dt_o$ (cf. Equation (29), and [15], II-19-7). ■

Remark 7. *Applying the same proof as above, we can also incidentally derive for any weak gravitational field the following results (which will be used later):*

$$g_{grav} \approx diag\left(\lambda_{grav}^2, -c^{-2}, -c^{-2}, -c^{-2}\right)_{\mathbf{B}}, \tag{71}$$

and

$$\begin{aligned}&(1,0,0,0)\circ g_{grav,4\times4,\mathbf{B}}\circ\left(-1,\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_z\right)^T\\ &= -1 - 2\cdot\left(\frac{KE_{grav} - PE_{grav}}{RE}\right), \textit{with}\end{aligned} \tag{72}$$

$$\frac{PE_{grav}}{RE} \;:=\; \frac{m_o\cdot\left(\frac{\gamma^2 GM}{r^2}\right)\cdot r}{m_o c^2} = \gamma^2\left(\frac{a_{grav}}{c}\right)t \tag{73}$$

$$= \gamma^2\left(\frac{v}{c}\right). \tag{74}$$

We note that in the literature (e.g., [23], 288, 294) one finds that

$$\frac{d\tilde{t}_o}{dt} = \left(1,\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_z\right)\circ g_{4\times4,\mathbf{E}}\circ\left(1,\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_z\right)^T, \tag{75}$$

where $g_{4\times4,\mathbf{E}}$ measures the norm of the motion $\left(1,\mathbf{V}_x,\mathbf{V}_y,\mathbf{V}_z\right)$ on the parameter domain U and pass it onto $\|\cdot\|_{T_p\mathcal{M}^4} \equiv \left\|(\Delta\tilde{t}_o,0,0,0)^T_{\tilde{\mathbf{E}}}\right\|_{T_p\mathcal{M}^4}$; we instead adhere to the standard treatment in differential geometry to express g as $g_{4\times4,\mathbf{B}}$ on $T_p\mathcal{M}^4$, to project $\frac{\partial f}{\partial t}$ onto the proper time $\Delta\tilde{t}_o$ in the tangent space.

Corollary 3. *The Einstein tensor*

$$\mathcal{E}_{em}^{att;\, rep} \approx \begin{pmatrix} \mp\frac{6v}{r_k^2 c} & -\frac{6v_Q \mathbf{V}_x}{r_k^2 c^3} & -\frac{6v_Q \mathbf{V}_y}{r_k^2 c^3} & -\frac{6v_Q \mathbf{V}_z}{r_k^2 c^3} \\ -\frac{6v_Q \mathbf{V}_x}{r_k^2 c^3} & -O\left(r_k^{-2}\right) & O\left(r_k^{-2}c^{-4}\right) & O\left(r_k^{-2}c^{-4}\right) \\ -\frac{6v_Q \mathbf{V}_y}{r_k^2 c^3} & O\left(r_k^{-2}c^{-4}\right) & -O\left(r_k^{-2}\right) & O\left(r_k^{-2}c^{-4}\right) \\ -\frac{6v_Q \mathbf{V}_z}{r_k^2 c^3} & O\left(r_k^{-2}c^{-4}\right) & O\left(r_k^{-2}c^{-4}\right) & -O\left(r_k^{-2}\right) \end{pmatrix}_{\mathbf{B}}. \tag{76}$$

Proof. $\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R \cdot g_{\mu\nu}$; $\forall \mathcal{M}^4 \approx \mathbb{R}^{1+3}$ we have

$$\left(R_{\mu\nu}\right) \approx diag\left(-\frac{3}{r_K^2}, -\frac{1}{r_K^2}, -\frac{1}{r_K^2}, -\frac{1}{r_K^2}\right) \text{ and} \tag{77}$$

$$R \approx -\frac{6}{r_K^2}, \tag{78}$$

where $r_K \equiv$ the radius of sectional curvatures (cf. [21], 860; [35], 154). Thus, substituting Equation (59) into $\left(g_{\mu\nu}\right)$ in $\left(\mathcal{E}_{\mu\nu}\right)$, we arrive at the conclusion. ∎

Lemma 4. *Denote the mass density of q by*

$$\bar{m}_{q,o} \equiv \frac{m_{q,o}}{\left(4\pi r_\infty^3/3\right)}; \tag{79}$$

then we have

$$\bar{m}_{q,o} r_\infty^2 \approx \left(1 - \gamma_{grav}^{-2} g_{11,grav}\right) \cdot \frac{3c^2}{8\pi G}, \tag{80}$$

where

$$g_{11,grav} \approx \lambda_{grav}^2 \approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{V}_\alpha}{c}\right), \tag{81}$$

with $\mathbf{V}_\alpha \equiv$ *the radial velocity* (> 0) *of any arbitrary particle* α *gravitating toward q at a distance of* r_∞*, and* $G \equiv$ *the universal gravitational constant.*

Proof.

$$g_{11,grav} \approx \lambda_{grav}^2 \approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{V}_\alpha}{c}\right) \text{ (refer to Eq. (24), (28))} \tag{82}$$

$$\approx \gamma_{grav}^2 \left(1 - \frac{2\mathbf{a}_\alpha t}{c}\right) \text{ (cf. Remark 2)} \tag{83}$$

$$= \gamma_{grav}^2 \left(1 - \frac{2G\bar{m}_{q,o}}{r_\infty^2 c} \cdot \frac{4\pi r_\infty^3}{3} \cdot \frac{r_\infty}{c}\right); \tag{84}$$

thus,

$$\bar{m}_{q,o} r_\infty^2 \approx \left(1 - \gamma_{grav}^{-2} g_{11,grav}\right) \cdot \frac{3c^2}{8\pi G}. \tag{85}$$

∎

Remark 8. *The above lemma expresses the gravitating mass density of q in terms of its effect on* $\mathcal{M}^4$ *as measured by* $g_{11,grav}$*; by the principle of equivalence,* $\bar{m}_{q,o}$ *is also the inertial mass density, and in the next theorem* $\bar{m}_{q,o}$ *is to be treated as such. Also, note that as* $r_\infty^{-1} \longrightarrow 0$*, we have* $\left|r_\infty^{-2} - r_K^{-2}\right| \longrightarrow 0$.

Theorem 5.

$$\mathcal{E}_{\mu\nu,em}^{att;\,rep} := R_{\mu\nu,em} - \frac{1}{2}R_{em} \cdot g_{\mu\nu,em}^{att;\,rep} = -\frac{16\pi G}{\left(1-\gamma_{grav}^{-2} \cdot g_{11,grav}\right) c^5} T_{\mu\nu,em}^{att;\,rep}. \tag{86}$$

Proof.

$$\frac{\mathcal{E}_{12,em}}{\mathcal{E}_{11,em}^{att;\,rep}} = \pm\frac{1}{c^2}\left(\frac{v_Q}{v}\right)\mathbf{V}_x \text{ (by Equation (76))} \tag{87}$$

$$= \pm\frac{1}{c^2}\cdot\left(\frac{m_{q,o}}{m_{Q,o}}\right)\cdot\left(-\frac{m_{Q,o}}{m_{q,o}}\mathbf{V}_{Q,x}\right) \text{ (by Equation (46))} \tag{88}$$

$$= \frac{-\frac{\|\bar{\mathbf{s}}\|}{c^2}\mathbf{V}_{Q,x}}{\pm\|\bar{\mathbf{S}}\|} = \frac{-\|\bar{\mathbf{g}}\|\,\mathbf{V}_{Q,x}}{\pm\|\bar{\mathbf{S}}\|} \text{ (by Equation (43))} \tag{89}$$

$$\equiv \frac{T_{12,em}}{T_{11,em}^{att;\,rep}}, \tag{90}$$

where $T_{11,em}^{att;\,rep}$ and $T_{1j,em}$, $j = 2,3,4$, are respectively the energy-flow and the momentum densities. Thus,

$$\mathcal{E}_{em}^{att;\,rep} = \kappa_{em} T_{em}^{att;\,rep} \text{ has} \tag{91}$$

$$\kappa_{em} = \frac{\mathcal{E}_{11,em}^{att;\,rep}}{T_{11,em}^{att;\,rep}} = \mp\frac{6v}{r_k^2 c} / \pm\|\bar{\mathbf{S}}\| \text{ (by Equations (76), (90))}, \tag{92}$$

but

$$\|\bar{\mathbf{S}}\| = \frac{3c^2}{4\pi r_\infty^3}\cdot m_{q,o} v \text{ (by Equations (43), (46))}, \tag{93}$$

so

$$\kappa_{em} = -\frac{6}{r_K^2 c}\cdot\frac{4\pi r_\infty^3}{3c^2 m_{q,o}} \tag{94}$$

$$= -\frac{6}{r_\infty^2 c}\cdot\frac{1}{c^2 \bar{m}_{q,o}} \text{ (cf. Remark 8)} \tag{95}$$

$$= -\frac{6}{c^3}\cdot\frac{8\pi G}{\left(1-\gamma_{grav}^{-2} g_{11,grav}\right)\cdot 3c^2} \text{ (by the preceding Lemma 4)} \tag{96}$$

$$= -\frac{16\pi G}{\left(1-\gamma_{grav}^{-2}\cdot g_{11,grav}\right)c^5}. \tag{97}$$

■

Remark 9. $T_{11,em}^{att;\,rep} \equiv \pm\|\bar{\mathbf{S}}\|$ *has unit (recalling from Equation (45))*

$$\frac{joule}{second\cdot meter^2} \tag{98}$$

$$= \frac{kilogram\cdot meter^2}{second^2}\cdot\frac{1}{second\cdot meter^2} \tag{99}$$

$$= \frac{kilogram}{second^3}, \tag{100}$$

so that $\left(\kappa_{em} \cdot T_{11,em}^{att;rep}\right)$ *has unit*

$$= \frac{[G]}{[c^5]} \cdot \frac{kilogram}{second^3} \tag{101}$$

$$= \frac{meter^3}{kilogram \cdot second^2} \cdot \frac{second^5}{meter^5} \cdot \frac{kilogram}{second^3} \tag{102}$$

$$= \frac{1}{meter^2} = \left[\frac{1}{r_k^2}\right], \tag{103}$$

measuring the local curvatures of $\mathcal{M}_{em}^4$. *We emphasize that our* $T_{11,em}$ *represents energy flows in a specific direction across an area of square meter per second, which is different from the common identification of* $T_{11,em}$ *with stationary energy densities with unit:* $\left[joule/\left(meter^3\right)\right]$ *(see, e.g., [35], 45, equation (2.8.10)).*

Remark 10. *We can now obtain a geometric union of gravitation and electromagnetism to arrive at*

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R \cdot g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu,grav} \mp \frac{16\pi G}{\left(1 - \gamma^{-2} g_{11,grav}\right) c^5} T_{\mu\nu,em}^{att;rep^*}, \tag{104}$$

where for expository neatness we set:

$$g_{\mu\nu,em}^{rep^*} \equiv g_{\mu\nu,em}^{rep} \; \forall \mu\nu \neq 1, \; g_{11,em}^{rep^*} \equiv -g_{11,em}^{rep} = -\lambda_{em}^{-2}; \tag{105}$$

$$T_{\mu\nu,em}^{rep^*} \equiv T_{\mu\nu,em}^{rep} \; \forall \mu\nu \neq 1, \; T_{11,em}^{rep^*} \equiv -T_{11,em}^{rep} = \left\| \bar{\mathrm{S}}\left(t\right) \right\|. \tag{106}$$

Theorem 6. *The set of Einstein Field Equations*

$$\mathcal{E}_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R \cdot g_{\mu\nu} = -\frac{8\pi G}{c^2} T_{\mu\nu,grav} \mp \frac{16\pi G}{\left(1 - \gamma^{-2} g_{11,grav}\right) c^5} T_{\mu\nu,em}^{att;rep^*} \tag{107}$$

has solutions:

$$R_{\mu\nu} = R_{\mu\nu,grav} \pm R_{\mu\nu,em}, \tag{108}$$

$$R = R_{grav} + R_{em}, \tag{109}$$

$$\text{and } g_{\mu\nu} = w_{grav} \cdot g_{\mu\nu,grav} \pm w_{em} \cdot g_{\mu\nu,em}^{att;rep^*}, \tag{110}$$

$$\text{with } w_{grav} \equiv \frac{R_{grav}}{R} \text{ and } w_{em} \equiv \frac{R_{em}}{R} \equiv 1 - w_{grav}. \tag{111}$$

Proof. Consider the operation $\mathcal{E}_{\mu\nu,grav} \pm \mathcal{E}_{\mu\nu,em}^{att;rep}$ and denote

$$\frac{R_{grav} \cdot g_{\mu\nu,grav}}{R_{grav} + R_{em}} \pm \frac{R_{em} \cdot g_{\mu\nu,em}^{att;rep}}{R_{grav} + R_{em}} \tag{112}$$

by $g_{\mu\nu} \left(\equiv w_{grav} \cdot g_{\mu\nu,grav} \pm w_{em} \cdot g_{\mu\nu,em}^{att;rep}\right)$; we see that the operation of $\mathcal{E}_{\mu\nu,grav} \pm \mathcal{E}_{\mu\nu,em}^{att;rep}$ is valid if and only if $g_{\mu\nu}$ is form-invariant with respect to measuring geodesics, possessing the same energy interpretations as g_{grav} and $g_{em}^{att;rep}$. Here we have:

$$
\begin{aligned}
& (1,0,0,0) \circ \left(w_{grav} \cdot g_{grav} + w_{em} \cdot g_{em}^{att}\right) \circ \left(-1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z\right)^T && (113)\\
& \text{(cf. Equation (30) in Proposition 1)} && (114)\\
= & \; w_{grav} \cdot \left(-1 - 2 \cdot \left(\frac{KE_{grav}}{RE}\right) + 2 \cdot \left(\frac{PE_{grav}}{RE}\right)\right) && \\
& + w_{em} \cdot \left(-1 - 2 \cdot \left(\frac{KE_{em}^{att}}{RE}\right) + 2 \cdot \left(\frac{PE_{em}^{att}}{RE}\right)\right) && (115)\\
& \text{(by equations (72) and (69))} && (116)\\
\equiv & \; -1 - \frac{2KE_{gravem}^{att}}{RE} + \frac{2PE_{gravem}^{att}}{RE}, && (117)
\end{aligned}
$$

where

$$
\begin{aligned}
KE_{gravem}^{att} &\equiv w_{grav} \cdot KE_{grav} + w_{em} \cdot KE_{em}^{att}, \text{ and} && (118)\\
PE_{gravem}^{att} &\equiv w_{grav} \cdot PE_{grav} + w_{em} \cdot PE_{em}^{att}. && (119)
\end{aligned}
$$

Now since

$$
(-R_{11,em}) - \frac{1}{2} R \cdot g_{11,em}^{rep^*} = \frac{-16\pi G}{\left(1 - \gamma^{-2} g_{11,grav}\right) c^5} T_{11,em}^{rep^*} \qquad (120)
$$

and

$$
\begin{aligned}
& (1,0,0,0) \circ g_{em}^{rep^*} \circ \left(-1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z\right)^T && \\
\equiv & \; (1,0,0,0) \circ g_{em}^{rep} \circ \left(1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z\right)^T, && (121)
\end{aligned}
$$

we have

$$
\begin{aligned}
& (1,0,0,0) \circ \left(w_{grav} \cdot g_{grav} - w_{em} \cdot g_{em}^{rep^*}\right) \circ \left(-1, \mathbf{V}_x, \mathbf{V}_y, \mathbf{V}_z\right)^T && (122)\\
= & \; w_{grav} \cdot \left(-1 - 2 \cdot \left(\frac{KE_{grav}}{RE}\right) + 2 \cdot \left(\frac{PE_{grav}}{RE}\right)\right) && \\
& - w_{em} \cdot \left(1 - 2 \cdot \left(\frac{KE_{em}^{rep}}{RE}\right) + 2 \cdot \left(\frac{PE_{em}^{rep}}{RE}\right)\right) \text{ (equation (69))} && (123)\\
\equiv & \; -1 - \frac{2KE_{gravem}^{rep}}{RE} + \frac{2PE_{gravem}^{rep}}{RE}, && (124)
\end{aligned}
$$

where

$$
\begin{aligned}
KE_{gravem}^{rep} &\equiv w_{grav} \cdot KE_{grav} - w_{em} \cdot KE_{em}^{rep}, \text{ and} && (125)\\
PE_{gravem}^{rep} &\equiv w_{grav} \cdot PE_{grav} - w_{em} \cdot PE_{em}^{rep}. && (126)
\end{aligned}
$$

Consequently, $g_{\mu\nu} = w_{grav} \cdot g_{\mu\nu,grav} \pm w_{em} \cdot g_{\mu\nu,em}^{att;rep^*}$ is form-invariant in measuring geodesics, with identical interpretations of energies to that of $g_{\mu\nu,grav}$ and $g_{\mu\nu,em}^{att;rep}$. I.e.,

$$
\mathcal{E} := \mathcal{E}_{grav} \pm \mathcal{E}_{em}^{att;rep} = -\frac{8\pi G}{c^2} T_{grav} \mp \frac{16\pi G}{\left(1 - \gamma^{\mp 2} g_{11,grav}\right) c^5} T_{em}^{att;rep^*} \qquad (127)
$$

results in a metric $g_{\mu\nu}$ that renders

$$g_{1\cdot} \circ (-1, \mathbf{V})^T = -1 - \frac{2KE_{gravem}}{RE} + \frac{2PE_{gravem}}{RE}. \tag{128}$$

■

Corollary 4.

$$\frac{\tilde{t}_o}{t_o} \approx 1 + \frac{KE_{gravem}}{RE} - \frac{PE_{gravem}}{RE}, \tag{129}$$

where

$$KE_{gravem} \equiv w_{grav} \cdot KE_{grav} \pm w_{em} \cdot KE_{em}^{att;rep} \tag{130}$$

$$and\ PE_{gravem} \equiv w_{grav} \cdot PE_{grav} \pm w_{em} \cdot PE_{em}^{att;rep}. \tag{131}$$

Proof. By Equation (28), $\lambda^2_{att;\,rep} \approx \left(\frac{\tilde{t}_o}{t_o}\right)^2$, but

$$g_{11,grav} \approx \lambda^2_{grav} \approx 1 + 2 \cdot \frac{KE_{grav}}{RE} - 2 \cdot \frac{PE_{grav}}{RE} \quad \text{(cf. equation (72))} \tag{132}$$

and

$$g_{11,em}^{att;rep^*} \approx \pm\lambda_{em}^{\pm 2} \ \text{(cf. equation (59) and notation (105))} \tag{133}$$

$$= \pm\left(1 \pm 2 \cdot \frac{KE_{em}^{att;rep}}{RE} \mp 2 \cdot \frac{PE_{em}^{att;rep}}{RE}\right) \quad \text{(cf. equation (69));} \tag{134}$$

thus,

$$\left(\frac{\tilde{t}_o}{t_o}\right)^2 \approx g_{11} = w_{grav} \cdot g_{11,grav} \pm w_{em} \cdot g_{11,em}^{att;rep^*} \tag{135}$$

$$= w_{grav} \cdot \lambda^2_{grav} \pm w_{em} \cdot \left(\pm\lambda_{em}^{\pm 2}\right) \tag{136}$$

$$= w_{grav} \cdot \left(1 + 2 \cdot \frac{KE_{grav}}{RE} - 2 \cdot \frac{PE_{grav}}{RE}\right) + w_{em} \cdot \left(1 \pm 2 \cdot \frac{KE_{em}^{att;rep}}{RE} \mp 2 \cdot \frac{PE_{em}^{att;rep}}{RE}\right) \tag{137}$$

$$= 1 + 2 \cdot \frac{w_{grav} \cdot KE_{grav} \pm w_{em} \cdot KE_{em}^{att;rep}}{RE} - 2 \cdot \frac{w_{grav} \cdot PE_{grav} \pm w_{em} \cdot PE_{em}^{att;rep}}{RE}, \tag{138}$$

so that

$$\frac{\tilde{t}_o}{t_o} \approx 1 + \frac{KE_{gravem}}{RE} - \frac{PE_{gravem}}{RE}. \tag{139}$$

■

Remark 11. *In General Relativity the spacetime proportionality* $\left(\frac{\tilde{t}_o}{t_o}\right)$ *is a major point of interest, and we have derived the above analogous equation that integrates gravity with electromagnetism.*

3. *EFE* for the Quantum Geometry

3.1 Description

In this section we construct a "combined space-time 4-manifold $\mathcal{M}^{[3]}$" as the graph of a diffeomorphism from one manifold $\mathcal{M}^{[1]}$ to another $\mathcal{M}^{[2]}$, akin to the idea of a diagonal map. $\mathcal{M}^{[2]}$ consists solely of electromagnetic waves as described by Maxwell Equations for a free space (from matter), which with all its (continuous) field energy can exist independently; $\mathcal{M}^{[2]}$ predates $\mathcal{M}^{[1]}$. Due to a large gravitational constant $G^{[2]}$ in $\mathcal{M}^{[2]}$, an astronomical black hole $\mathbf{B} \subset \mathcal{M}^{[2]}$ came into being (cf. e.g., [10, 34], for formation of space-time singularities in Einstein manifolds), and resulted in $\mathcal{M}^{[1]} \times \mathbf{B}$ (i.e., the *Big Bang* - - when $\mathcal{M}^{[2]}$ branched out $\mathcal{M}^{[1]}$; cf. e.g., [16], for how a black hole may give rise to a macroscopic universe): photons then emerged in $\mathcal{M}^{[1]}$ with their accompanied electromagnetic waves existing in $\mathbf{B}$. Any energy entity j in $\mathcal{M}^{[1]}$ is a particle resulting from a superposition of electromagnetic waves in $\mathbf{B}$ and

$$\text{the combined entity} \equiv [particle, wave] \tag{140}$$

$$\text{has energy } E_j^{[3]} = E_j^{[1]} + E_j^{[2]} \tag{141}$$

(where the term "particle wave" was exactly used in Feynman [15], "ghost wave - - Gespensterfelder" by Einstein [28, p. 287-288], and "pilot wave" by de Broglie). Particles in $\mathcal{M}^{[1]}$ engage in electromagnetic, (nuclear) weak, or strong interactions via exchanging virtual particles. Both particles and waves engage in gravitational forces separately and respectively in $\mathcal{M}^{[1]}$ and $\mathcal{M}^{[2]}$. Being within the Schwarzschild radius, $\mathbf{B}$ in $\mathcal{M}^{[2]}$ is a complex (sub) manifold, which furnishes exactly the geometry for the observed quantum mechanics in $\mathcal{M}^{[3]}$; moreover, $\mathbf{B}$ provides an energy interpretation to quantum probabilities in $\mathcal{M}^{[1]}$. In summary, $\mathcal{M}^{[3]}$ casts quantum mechanics in the framework of General Relativity and honors the most venerable tenet in physics - - the conservation of energy - - from the Big Bang to mini black holes.

3.2 Derivations

Definition 4. *Let* $j \in \mathbb{N}$*; a combined energy entity is* $E_j^{[3]} := E_j^{[1]} + E_j^{[2]}$*, where* $\forall i \in \{1,2\}$ $E_j^{[i]}$ *exerts and receives gravitational forces on and from* $\left\{E_k^{[i]} \mid k \in \mathbb{N} - \{j\}\right\}$.

Lemma 7. $\forall i \in \{1,2\}$ $\left\{E_j^{[i]} \mid j \in \mathbb{N}\right\}$ *form a space-time 4-manifold* $\mathcal{M}^{[i]}$ *that observes EFE:*

$$R_{\mu\nu}^{[i]} - \frac{1}{2}R^{[i]}g_{\mu\nu}^{[i]} = -\frac{8\pi G^{[i]}}{c^2}T_{\mu\nu}^{[i]}. \tag{142}$$

Proof. (By General Relativity.) ■

Remark 12. *The long existing idea of dual mass is fundamentally different from that of our* $[particle, wave]$*; dual mass (see* [22, 27]*) is a solution of the above EFE for* $i = 1$ only.

Definition 5. *A combined space-time 4-manifold is*

$$\mathcal{M}^{[3]} := \left\{ \left(p^{[1]}, p^{[2]}\right) \in \mathcal{M}^{[1]} \times \mathcal{M}^{[2]} \mid h\left(p^{[1]}\right) = p^{[2]}, h = \textit{any diffeomorphism} \right\}. \tag{143}$$

Proposition 4. $\left\{ E_j^{[3]} \mid j \in \mathbb{N} \right\}$ *form* $\mathcal{M}^{[3]}$.

Proof. $\forall j \in \mathbb{N}$ $E_j^{[3]}$ can be assigned with a coordinate point $\mathbf{u}_j \in U \subset \mathbb{R}^{1+3} \equiv$ the Minkowski space. Since $\forall i \in \{1,2\}$ $\mathcal{M}^{[i]}$ is a manifold, there exists a diffeomorphism $f^{[i]} : U \longrightarrow f^{[i]}(U) \subset \mathcal{M}^{[i]}$; i.e., $f^{[i]}\left(\mathbf{u}_j\right) = \mathbf{p}_j^{[i]} \in \mathcal{M}^{[i]}$, so that $\mathbf{p}_j^{[2]} = f^{[2]}\left(\mathbf{u}_j\right) = f^{[2]}\left(f^{[1]^{-1}}\left(\mathbf{p}_j^{[1]}\right)\right) = h\left(\mathbf{p}_j^{[1]}\right)$, with $h \equiv f^{[2]} \circ f^{[1]^{-1}}$ being a diffeomorphism. ∎

Theorem 8. *Any metric* $g_{\mu\nu}^{[3]}$ *for* $\mathcal{M}^{[3]}$ *is such that*

$$g_{\mu\nu}^{[3]} = \frac{G^{[2]}}{G^{[1]} + G^{[2]}} \cdot g_{\mu\nu}^{[1]} + \frac{G^{[1]}}{G^{[1]} + G^{[2]}} \cdot g_{\mu\nu}^{[2]}. \tag{144}$$

Proof. Since $g_{\mu\nu}^{[3]}$ is the inner product of the direct sum of the tangent spaces: $T_{p^{[1]}}\mathcal{M}^{[1]} \oplus T_{p^{[2]}}\mathcal{M}^{[2]}$, we have $g_{\mu\nu}^{[3]} = a \cdot g_{\mu\nu}^{[1]} + b \cdot g_{\mu\nu}^{[2]}$ for some $a, b \in \mathbb{R}$. Since $\forall i \in \{1,2,3\}$ $g_{11}^{[i]}$ is the *time* $\times$ *time* component of $g^{[i]}$, we have the well-known relation

$$g_{11}^{[i]} = 1 - \frac{2G^{[i]}M^{[i]}}{rc^2}, \tag{145}$$

implying at once that $a = w_1 \in (0,1)$ and $b = 1 - w_1$. Thus,

$$g_{11}^{[3]} = 1 - \frac{2G^{[3]}M^{[3]}}{rc^2} \tag{146}$$

$$= w_1\left(1 - \frac{2G^{[1]}M^{[1]}}{rc^2}\right) + (1 - w_1)\left(1 - \frac{2G^{[2]}M^{[2]}}{rc^2}\right) \tag{147}$$

$$= 1 - \frac{2w_1G^{[1]}M^{[1]} + 2(1 - w_1)G^{[2]}M^{[2]}}{rc^2}, \tag{148}$$

implying that

$$G^{[3]}M^{[3]} \equiv G^{[3]}M^{[1]} + G^{[3]}M^{[2]} \tag{149}$$

$$= w_1G^{[1]}M^{[1]} + (1 - w_1)G^{[2]}M^{[2]}. \tag{150}$$

Since $M^{[1]}$ and $M^{[2]}$ are arbitrary, we have

$$w_1G^{[1]} = G^{[3]} = (1 - w_1)G^{[2]}, \tag{151}$$

$$\text{i.e., } w_1\left(G^{[1]} + G^{[2]}\right) = G^{[2]}, \tag{152}$$

$$\text{or } w_1 = \frac{G^{[2]}}{G^{[1]} + G^{[2]}} \text{ and } 1 - w_1 = \frac{G^{[1]}}{G^{[1]} + G^{[2]}}. \tag{153}$$

∎

Corollary 5.

$$G^{[3]} = \left(\frac{G^{[1]} G^{[2]}}{G^{[1]} + G^{[2]}} \right). \tag{154}$$

Corollary 6. *If* $\frac{G^{[1]}}{G^{[2]}} \approx 0$, *then:*

(1) $G^{[3]} \approx G^{[1]}$ and $w_1 \approx 1$;
(2) if $\left\{ E_j^{[2]} \mid j \in \mathbb{N} \right\}$ are contained within a radius R such that

$$g_{11}^{[2]} = 1 - \frac{2G^{[2]} \sum_j E_j^{[2]}}{Rc^4} < 0, \tag{155}$$

then the proper time ratio

$$\frac{\Delta t_0^{[2]}}{\Delta t_0^{[1]}} = \sqrt{g_{11}^{[2]}} \in \mathbb{C}, \tag{156}$$

i.e., $t_0^{[2]}$ carries the unit of $\sqrt{-1}$ second (by analytic continuation).

Remark 13. *If in addition to* $\left\{ E_j^{[3]} = E_j^{[1]} + E_j^{[2]} \mid j \in \mathbb{N} \right\}$ *there exist dark energies as defined by*

$$\left\{ \left(0, E_l^{[2]} \right) \mid l \in \mathbb{N} \right\} \tag{157}$$

in $\mathcal{M}^{[2]}$, *then the above Schwarzschild radius R is even larger.*

Remark 14. *Without our setup of* $\mathcal{M}^{[2]}$, *the subject of black holes necessarily has been about gravitational collapses within* $\mathcal{M}^{[1]}$ *due to high concentrations of matter. By contrast, our geometry is about a large* $G^{[2]}$ *that causes* $g_{11}^{[2]} < 0$ *over* $\mathbf{B} \subset \mathcal{M}^{[2]}$; *in* [16] *the authors showed the possibility that the interior of a black hole could "give rise to a new macroscopic universe;" that macroscopic universe is just our* $\mathcal{M}^{[1]}$, *and the black hole is* $\mathbf{B} \subset \mathcal{M}^{[2]}$. *As such, studies of the black hole interior are of great relevance to our construct of* $\mathcal{M}^{[1]} \times \left(\mathbf{B} \subset \mathcal{M}^{[2]} \right)$ *provided however that the analytic framework is free from the familiar premise of material crushing, or particles entering/escaping a black hole (as in Hawking radiation, see, e.g.,* [23]; *for a review of some of the research in the black hole interior, see, e.g.,* [2, 6, 8, 17]*).*

Corollary 7. $\forall \left\{ M^{[3]}, m^{[3]} \right\}$ *one has the following Newtonian limit:*

$$\begin{aligned} m^{[3]} \mathbf{a}^{[3]} = -[& \left(\frac{G^{[2]}}{G^{[1]} + G^{[2]}} \right) \left(\frac{G^{[1]} M^{[1]} m^{[1]}}{\|\mathbf{r}\|^2} \right) \\ & + \left(\frac{G^{[1]}}{G^{[1]} + G^{[2]}} \right) \left(\frac{G^{[2]} M^{[2]} m^{[2]}}{\|\mathbf{r}\|^2} \right)] \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|}, \end{aligned} \tag{158}$$

or

$$\mathbf{a}^{[3]} = -\frac{G^{[3]} M^{[3]}}{\|\mathbf{r}\|^2} \left(\frac{M^{[1]}}{M^{[3]}} \cdot \frac{m^{[1]}}{m^{[3]}} + \frac{M^{[2]}}{M^{[3]}} \cdot \frac{m^{[2]}}{m^{[3]}} \right) \frac{\mathbf{r}}{\|\mathbf{r}\|}. \tag{159}$$

Corollary 8. *If* $\frac{M^{[1]}}{M^{[3]}} = \frac{m^{[1]}}{m^{[3]}} \equiv \mu_1 \in (0,1)$, *then the laboratory-measured mass as denoted by* $\hat{M}$ *is such that*

$$\hat{M} = M^{[3]} \left({\mu_1}^2 + (1-\mu_1)^2 \right). \tag{160}$$

Proof.

$$\mathbf{a}^{[3]} = -\frac{G^{[3]}\hat{M}}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \tag{161}$$

$$= -\frac{G^{[3]}M^{[3]}\left({\mu_1}^2 + (1-\mu_1)^2\right)}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} \text{ (by Equation (159)).} \tag{162}$$

■

Corollary 9.

$$M^{[3]} = \frac{\hat{M}}{{\mu_1}^2 + (1-\mu_1)^2}, \tag{163}$$

$$M^{[1]} = \frac{\hat{M}\mu_1}{{\mu_1}^2 + (1-\mu_1)^2} \equiv \hat{M}\phi^{[1]}, \textit{ and} \tag{164}$$

$$M^{[2]} = \frac{\hat{M}(1-\mu_1)}{{\mu_1}^2 + (1-\mu_1)^2} \equiv \hat{M}\phi^{[2]}. \tag{165}$$

Notation 1. *The above notation of an overhead caret, e.g.,* $\hat{E} = E^{[3]}({\mu_1}^2 + (1-\mu_1)^2)$ *for a laboratory-measured energy, will be used throughout the remainder of our Chapter; note in particular that a quantity multiplied by* $\phi^{[2]} \equiv \frac{(1-\mu_1)}{{\mu_1}^2+(1-\mu_1)^2}$, *e.g.,* $\hat{E}\phi^{[2]}$, *indicates a conversion from a laboratory established quantity into that part of the quantity as contained in* $\mathbf{B} \subset \mathcal{M}^{[2]}$.

Hypotheses (We will assume the following in our subsequent derivations:)

(1) $\frac{G^{[1]}}{G^{[2]}} \approx 0$ is such that

$$(a)\ g^{[3]}_{\mu\nu} = \frac{G^{[2]}}{G^{[1]}+G^{[2]}} \cdot g^{[1]}_{\mu\nu} + \frac{G^{[1]}}{G^{[1]}+G^{[2]}} \cdot g^{[2]}_{\mu\nu} \approx g^{[1]}_{\mu\nu}, \text{ and} \tag{166}$$

$$(b)\ g^{[2]}_{11} = \left(\frac{\Delta t^{[2]}_0}{\Delta t^{[1]}_0}\right)^2 < 0 \text{ throughout } \mathbf{B} \subset \mathcal{M}^{[2]}, \tag{167}$$

implying that $\Delta t^{[2]}_0$ has unit $\sqrt{-1}$ second (by analytic continuation; cf. e.g., [4] for the inherent necessity of the unit of i in standard quantum theory, and [20] for analytic continuation of Lorentzian metrics).

(2) $\forall j \in \mathbb{N}$ $E^{[2]}_j$ is either a single electromagnetic wave of length λ_j or a superposition of electromagnetic waves, and $E^{[2]}_j$ engages in gravitational forces with $\left\{E^{[2]}_k \mid k \in \mathbb{N} - \{j\}\right\}$ only.

(3) $\forall j \in \mathbb{N}$ $E^{[1]}_j$ is a particle (a photon if $E^{[2]}_j$ is a single electromagnetic wave) and engages in gravitational forces with $\left\{E^{[1]}_k \mid k \in \mathbb{N} - \{j\}\right\}$; in addition, $E^{[1]}_j$ may engage in electromagnetic, weak, or strong interactions with $E_{k\neq j}^{[1]}$ via exchanging virtual particles in $\mathcal{M}^{[1]}$.

Notation 2. *$\ddot{h} \equiv \frac{h}{second^2}$, $h \equiv$ Planck constant; NLT $\equiv$ nonlinear terms.*

Theorem 9.

$$G^{[2]} = \frac{c^5}{4\ddot{h}\phi^{[2]}}.$$

Proof. In order to apply General Relativity in our derivation, we set the Planck length as the lower limit of electromagnetic wave lengths under consideration, i.e., $\lambda \geq \lambda_P :\approx 10^{-35}$meter, or equivalently, $\nu \equiv \frac{c}{\lambda} \in \left(0 \text{ Hz}, 10^{43}\text{Hz} \equiv \nu_P\right)$ (which covers a spectrum from infrared to ultraviolet, to well beyond gamma rays, $\nu_{gamma} \approx 10^{21}$Hz). Thus, let $E_j^{[1]}$ be a photon with frequency $\nu_j^{[1]} \in (0 \text{ Hz}, \nu_P)$ as observed from a laboratory frame $S^{[1]}$ (in $\mathcal{M}^{[1]}$). Consider $E_j^{[2]}$ ($\equiv \hat{E}_j\phi^{[2]}$) within its wave length λ_j, i.e., $E_j^{[2]}$ as contained in a ball B of radius $\frac{\lambda_j}{2}$, and consider a reference frame $S^{[2]}$ on the boundary of B. Since the gravitational effect of $E_j^{[2]}$ on $S^{[2]}$ is as if the ball B of energy $E_j^{[2]}$ were concentrated at the ball center, we have

$$g_{11}^{[2]} = 1 - \frac{2G^{[2]}E_j^{[2]}}{\frac{\lambda_j}{2} \cdot c^4} \equiv 1 - \frac{4G^{[2]}E_j^{[2]}\nu_j^{[1]}}{c^5}. \tag{168}$$

Since the frequency $\nu_j^{[2]}$ of $E_j^{[2]}$ relative to frame $S^{[2]}$ is exactly 1 cycle and by Hypothesis (1)(b) the unit of $t_0^{[2]}$ is $\sqrt{-1}$ second, we have

$$\nu_j^{[2]} = \frac{1 \text{ (cycle)}}{i \cdot \text{second}}, \tag{169}$$

so that

$$g_{11}^{[2]} \ : = \left(\frac{\partial t_0^{[2]}}{\partial t_0^{[1]}}\right)^2 := \lim_{\Delta t_0^{[1]} \to 0} \left(\frac{\Delta t_0^{[2]}}{\Delta t_0^{[1]}}\right)^2 \tag{170}$$

$$= \left(\frac{\Delta t_0^{[2]}}{\Delta t_0^{[1]} = 1 \text{ second}}\right)^2 - NLT \text{ (where the nonlinear terms} \tag{171}$$

$$NLT > 0 \text{ due to the gravitational attraction of } S^{[2]} \text{ toward } E_j^{[2]})$$

$$\equiv \left(\frac{\nu_j^{[1]}}{\nu_j^{[2]}}\right)^2 - NLT \equiv \left(\frac{\nu_j^{[1]}}{1/\left(i \cdot \text{second}\right)}\right)^2 - NLT \tag{172}$$

$$= -\nu_j^{[1]2}\text{second}^2 - NLT \tag{173}$$

$$= 1 - \frac{4G^{[2]}E_j^{[2]}\nu_j^{[1]}}{c^5} \text{ (from Equation (168));} \tag{174}$$

by the preceding Equations, (173) and (174), we have

$$-\nu_j^{[1]}\text{second}^2 - \frac{NLT+1}{\nu_j^{[1]}} = -\frac{4G^{[2]}E_j^{[2]}}{c^5}, \tag{175}$$

or

$$\frac{c^5}{4G^{[2]}}\left(\nu_j^{[1]}\text{second}^2 + \frac{NLT+1}{\nu_j^{[1]}}\right) = E_j^{[2]} \equiv \hat{E}_j\phi^{[2]}, \tag{176}$$

or

$$\hat{E}_j = \left(\frac{c^5\text{second}^2}{4G^{[2]}\phi^{[2]}}\right)\cdot\nu_j^{[1]} + \left(\frac{c^5}{4G^{[2]}\phi^{[2]}}\right)\cdot\frac{NLT+1}{\nu_j^{[1]}} \tag{177}$$

$$\equiv h\nu_j^{[1]} + \ddot{h}\cdot\frac{NLT+1}{\nu_j^{[1]}} \text{ (refer to Notation 2)}, \tag{178}$$

where

$$\ddot{h}\cdot\frac{NLT+1}{\nu_j^{[1]}} \equiv \ddot{h}\cdot\frac{NLT+1}{c}\cdot\lambda_j \equiv \Delta\hat{E}_j \tag{179}$$

$$\text{is the uncertainty energy.} \tag{180}$$

Thus, comparing Equations (177) with (178), we have

$$G^{[2]} = \frac{c^5}{4\ddot{h}\phi^{[2]}}. \tag{181}$$

■

Remark 15. *The above factor* $\left(1/\phi^{[2]}\right) \equiv \left[\mu_1{}^2 + (1-\mu_1)^2\right]/(1-\mu_1)$ *from Corollary 9 and Equation* (165) *has a U-shaped graph as a function of* $\mu_1 \equiv m^{[1]}/m^{[3]}$*: as* μ_1 *increase from 0 to 0.29* $\left(\approx 1-\frac{\sqrt{2}}{2}\right)$*, 0.5 and 1,* $\left(\frac{1}{\phi^{[2]}}\right)$ *decreases from 1 to the minimum 0.83* $\left(\approx 2\left(\sqrt{2}-1\right)\right)$*, then rises to 1 and approaches* ∞*. Incidentally, we have also provided a derivation of* $\hat{E} = h\nu$ *from the above Equation* (178)*; we note that* $g_{11}^{[2]} = 1 - \frac{4G^{[2]}E_j^{[2]}\nu_j^{[1]}}{c^5}$*, being a derivative, contains quantum uncertainties as* $\Delta t_0^{[1]} \to 0$.

We now cast quantum mechanics in General Relativity.

Claim Let $U \subset \mathbb{R}^{1+3}$ be a parameter domain of a laboratory frame; let $\rho : U \longrightarrow [0,\infty)$ be the probability density function of a particle $E_j^{[1]}$, and let $\mathbb{E} : U \longrightarrow \mathbb{C}^3$ be the electric field that contains $E_j^{[2]}$ in $\mathbf{B} \subset \mathcal{M}^{[2]}$ (which is complex by Hypothesis (1)(b)). Assume that ρ is of a positive constant proportionality β (of unit $\left(\frac{1}{\text{joule}}\right)$) to the electromagnetic field energy density of $\mathbb{E}$ (over U). Then the wave function $\psi : U \longrightarrow \mathbb{C}$ of $E_j^{[1]}$ is such that

$$\psi(t, \mathbf{x}) = z_0 \cdot \|\mathbb{E}(t, \mathbf{x})\|_{\mathbb{C}^3}, \tag{182}$$

where $z_0 \in \mathbb{C}$ is a constant and the complex norm (cf. e.g., [18], p. 221)

$$\|(z_1, z_2, z_3)\|_{\mathbb{C}^3}^2 := z_1^2 + z_2^2 + z_3^2 \in \mathbb{C}. \tag{183}$$

We back up the above *Claim* as follows: By the assumption in the *Claim*,

$$|\psi(t, \mathbf{x})|^2 = \rho(t, \mathbf{x}) = \beta \cdot \left| \|\mathbb{E}(t, \mathbf{x})\|_{\mathbb{C}^3} \right|^2 \cdot \epsilon_o \phi^{[2]}, \tag{184}$$

where $\epsilon_o \equiv$ the permittivity constant. Thus,

$$\psi(t, \mathbf{x}) = \sqrt{\beta \epsilon_o \phi^{[2]}} \cdot e^{i\theta} \|\mathbb{E}(t, \mathbf{x})\|_{\mathbb{C}^3} \tag{185}$$

$$= z_0 \cdot \|\mathbb{E}(t, \mathbf{x})\|_{\mathbb{C}^3}. \tag{186}$$

Remark 16. *From Hypotheses (1), (2), and (3), the $\mathbb{E}(t, \mathbf{x})$ in $\mathbf{B} \subset \mathcal{M}^{[2]}$ of the above Claim is only the effect or the consequence of the dynamics in $\mathcal{M}^{[1]}$; i.e., $\mathbb{E}(t, \mathbf{x})$ is formed by the forces in $\mathcal{M}^{[1]}$.*

Remark 17. *Also from Hypotheses (1), (2), and (3), any particle $p_i \in \mathcal{M}^{[1]}$ is formed by a superposition of electromagnetic fields in $\mathbf{B} \subset \mathcal{M}^{[2]}$; i.e., p_i has its distinct identity $\mathbb{E}_{p_i}(t, \mathbf{x})$, with*

$$\mathbb{E}_{p_i}(t, \mathbf{x}) = \sum_j \mathbb{E}_{i,j}(t, \mathbf{x_j}) = \begin{pmatrix} z_1(t, \mathbf{x}) \\ z_2(t, \mathbf{x}) \\ z_3(t, \mathbf{x}) \end{pmatrix}_i \in \mathbb{C}^3(t, \mathbf{x}), \tag{187}$$

i.e., composed of electromagnetic propagations through $(t, \mathbf{x})$ of multiple directions $\{\mathbf{x_j}\} \subset \mathbb{R}^3$, multiple frequencies $\{\omega_j\}$, and multiple phases $\{\theta_j\}$. This assertion is supported by the following three considerations:

(1) As is well known, traveling waves can sum to standing waves, and the sum of standing waves can approximate arbitrary functions by Fourier series.
(2) Physically, the pair creation process of antiparticles by photons such as

$$\gamma + \gamma \longrightarrow \textit{electron } e^- + \textit{positron } e^+, \tag{188}$$

has been well established (cf. [19], 164).
(3) We also note the possibility of engendering a new particle $\tilde{p}_i$ from an existing particle p_i via a field transformation

$$\Phi : \mathbb{E}_{p_i}(t, \mathbf{x}) \in \mathbb{C}^3(t, \mathbf{x}) \longmapsto \mathbb{E}_{\tilde{p}_i}(t, \mathbf{x}) \in \mathbb{C}^3(t, \mathbf{x}), \tag{189}$$

especially by the general principle of symmetry as associated with electric charge, spatial parity, and time direction.

Remark 18. *Historically Schrödinger had initially interpreted his* $\left|\Psi_{p_i}(t,\mathbf{x})\right|^2$ *as the electric charge density (cf. e.g.,* [15] *, III-21-6). Now the above Equation* (184) *shows that his interpretation was not too different from ours. In fact, the vector potential* $\mathbb{A}$ *in classical electrodynamics is the same as the wave function* Ψ *in quantum mechanics, so that the solutions of Maxwell Equations are identical to those of Schrödinger's Equation (cf.* [15], *II-15-8 and 20-3, also III-21-6). In short, Maxwell Equations, as applied to free spaces, already gave a description of the (quantum) fields* $\subset \mathbf{B} \subset \mathcal{M}^{[2]}$*, even though the way by which Maxwell derived his equations in 1861 was based on the electrodynamics of charges in* $\mathcal{M}^{[1]}$ *(see, e.g.,* [23], *40-47); i.e., his electromagnetic fields* $(\mathbb{E},\mathbb{B})$ *have always been in the complex* $\mathbf{B} \subset \mathcal{M}^{[2]}$*. That the complex quantum electrodynamics can assume a real classical form is simply due to the isomorphism*

$$\mathbb{R}/\left\langle 2\pi\right\rangle \approx \textit{the group of rotations; i.e.,} \tag{190}$$

$$\mathbb{E}_{oj}\cdot\cos\left(\omega_j t-\mathbf{k}_j\cdot\mathbf{x_j}+\theta_j\right)\approx\mathbb{E}_{oj}\cdot e^{-i\left(\omega_j t-\mathbf{k}_j\cdot\mathbf{x_j}+\theta_j\right)}=\mathbb{E}_j\left(t,\mathbf{x_j}\right). \tag{191}$$

Remark 19. *By the same assumption of* $\rho(t,\mathbf{x})=\beta\cdot\left|\|\mathbb{E}(t,\mathbf{x})\|_{\mathbb{C}^3}\right|^2\cdot\epsilon_o\phi^{[2]}$ *(Equation* (184)*) as in the above Claim, i.e., quantum probability density in* $\mathcal{M}^{[1]}\equiv$ *electromagnetic field energy density in* $\mathbf{B}\subset\mathcal{M}^{[2]}$ $(\bmod\ \textit{joule of energy})$*, we have analogously, probability current density in* $\mathcal{M}^{[1]}\equiv$ *the Poynting vector in* $\mathbf{B}\subset\mathcal{M}^{[2]}$ $(\bmod\ \textit{joule of energy})$*, i.e.,*

$$\mathbf{j}(t,\mathbf{x})=\beta\cdot\mathbb{S}^{[2]}(t,\mathbf{x}). \tag{192}$$

We formalize this assertion by the following proposition.

Proposition 5. *The probability current density of a particle*

$$\mathbf{j}(t,\mathbf{x})\ :=\left(\frac{\hbar}{2\hat{m}i}\right)\left(\bar{\psi}(t,\mathbf{x})\cdot\nabla\psi(t,\mathbf{x})-\psi(t,\mathbf{x})\cdot\nabla\bar{\psi}(t,\mathbf{x})\right) \tag{193}$$

$$=\beta\cdot\mathbf{S}^{[2]}(t,\mathbf{x}), \tag{194}$$

where $\hbar\equiv\frac{h}{2\pi}$*,* $\hat{m}\equiv m^{[3]}_{measured}$ $\equiv$ *the measured mass of the* $[particle, wave]$*, and* $\mathbf{S}^{[2]}(t,\mathbf{x})$ *is the Poynting vector apportioned to* $\mathbf{B}\subset\mathcal{M}^{[2]}$.

Proof. Without loss of generality as based on (linear) superpositions of fields, consider a free photon that travels in the direction of $(x>0,0,0)$ with

$$\psi(t,\mathbf{x})=z_0\cdot\|\mathbb{E}(t,\mathbf{x})\|_{\mathbb{C}^3} \tag{195}$$

$$=z_0\cdot\left\|\left(0,e^{-i(\omega t-kx)},0\right)^T\right\|_{\mathbb{C}^3} \tag{196}$$

$$=z_0e^{-i(\omega t-kx)}. \tag{197}$$

Then

$$\nabla\psi=\left(z_0e^{-i(\omega t-kx)}\cdot ki,0,0\right)^T\ \text{ and} \tag{198}$$

$$\nabla\bar{\psi}=\left(z_0e^{i(\omega t-kx)}\cdot(-ki),0,0\right)^T, \tag{199}$$

so that $\mathbf{j} := \left(\frac{\hbar}{2\hat{m}i}\right)(\bar{\psi}\cdot\nabla\psi - \psi\cdot\nabla\bar{\psi}) = \frac{1}{2\hat{m}}\left(\bar{\psi}\cdot\frac{\hbar}{i}\nabla\psi - \psi\cdot\frac{\hbar}{i}\nabla\bar{\psi}\right)$

$$= \frac{1}{2\hat{m}}\left(\bar{\psi}\psi\cdot(\hbar k,0,0)^T + \psi\bar{\psi}\cdot(\hbar k,0,0)^T\right) \equiv \frac{1}{\hat{m}}\cdot|\psi|^2\cdot\hat{\mathbf{p}} \tag{200}$$

(where $\hat{\mathbf{p}}$ denotes the measured momentum vector of unit $[\frac{\text{kilogram}\cdot\text{meter}}{\text{second}}]$)

$$= \frac{1}{\hat{m}}\cdot\left(\beta\cdot\left(\hat{u}\phi^{[2]}\right)\right)\cdot\frac{\hat{\mathbf{S}}\cdot\text{meter}^3}{c^2} \tag{201}$$

(where $|\psi|^2$ equal to $\beta\cdot\left(\hat{u}\phi^{[2]}\right)$ is from the above Equation (184),

and $\hat{\mathbf{S}}$ denotes the measured Poynting vector, cf. [15], II-27-9, so that

$\frac{\hat{\mathbf{S}}}{c^2}$ equals the momentum density of unit $\left[\frac{\text{kilogram}}{\text{second}\cdot\text{meter}^2}\right]$)

$$= \left(\frac{\hat{u}}{\hat{m}c^2/\text{meter}^3}\right)\cdot\beta\cdot\left(\hat{\mathbf{S}}\phi^{[2]}\right) = 1\cdot\beta\cdot\mathbf{S}^{[2]} \tag{202}$$

(due to the uniform probability density for a free photon).

■

Remark 20. *Our geometry of $\mathcal{M}^{[1]}\times\mathbf{B}$ serves to explain the following.*

(1) Quantum tunneling: A particle in $\mathcal{M}^{[1]}$ enters a mini black hole A, turns completely into a wave in $\mathbf{B}\subset\mathcal{M}^{[2]}$, and continue to travel in $\mathbf{B}\subset\mathcal{M}^{[2]}$ until mini black hole B, where it re-emerges in $\mathcal{M}^{[1]}$ (cf. [32]; for a recent study on wormholes, see [9]). Of course the above event is subject to the WKB probability approximation

$$\exp\left\{\left[-\frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m\left(V\left(x\right)-E\right)}dx\right]\left[1+O\left(\hbar\right)\right]\right\}, \tag{203}$$

so that tunneling does not occur outside the quantum domain; however, as one of "the Top Ten Physics Newsmakers of the Decade" (*APS News*, February 2010), quantum teleportation of information between two atoms separated by more than one meter was achieved in February 2009.

(2) Vacuum polarization: Here we provide a different geometric structure for this phenomenon from that of the "infinite sea of invisible negative energy particles" by Dirac. We consider the negative spectrum $\left(-\infty,-mc^2\right]$ of the Dirac operator $D := \left(-ic\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+mc^2\beta\right)$ a pure mathematical artifact, and we claim that instead of being negative energies traveling backward in time, antiparticles differ from their (counterpart) ordinary particles in the order of the cross product $\mathbb{B}\times\mathbb{E}$ of the electromagnetic fields in $\mathbf{B}\subset\mathcal{M}^{[2]}$. Accordingly, we identify a vacuum (in $\mathcal{M}^{[1]}$) with a pre-existing electromagnetic field in $\mathbf{B}\subset\mathcal{M}^{[2]}$ (for a recent study on vacuum energy, see [11]).

(3) The existence of dark matter and energy $\left[0,E^{[2]}\right]$: The above (1) suggests that if a particle can engage in long-distance tunneling from point A to B, then between A and B the particle becomes dark matter/energy with total energy

$$E^{[3]} = \sqrt{\left(m_0^{[3]}\right)^2c^4+p^2c^2}, \tag{204}$$

where $m_0^{[3]} = m_0^{[2]}$ = the rest mass of the dark matter, and pc = the dark energy. Here we remark that whether $m_0^{[3]} > 0$ or $m_0^{[3]} = 0$ depends on the superposition of the electromagnetic waves in $\mathbf{B} \subset \mathcal{M}^{[2]}$ forming a standing wave or not.

Remark 21. *In* [16] *Frolov et al. showed the possibility that a black hole can give rise to a macroscopic universe. Our model of $\mathcal{M}^{[3]}$ precisely claims that our recognized universe of matter $\mathcal{M}^{[1]}$ is a black hole $\mathbf{B}$ in $\mathcal{M}^{[2]}$; this geometry renders the possibility that any black hole in $\mathcal{M}^{[1]}$ leads back to $\mathbf{B} \subset \mathcal{M}^{[2]}$ (as in the Kruskal-Szekeres scheme); i.e., geometric singularities in $\mathcal{M}^{[1]}$ serve to transfer energies between $\mathcal{M}^{[1]}$ and $\mathbf{B} \subset \mathcal{M}^{[2]}$ (see* [32], *also cf.* [3] *about the subject of how quantum gravity takes over a "naked singularity"), so that a point particle does not have an infinite mass density. As such, we claim that electrons are point particles in $\mathcal{M}^{[1]}$ that carry their electromagnetic waves in $\mathbf{B} \subset \mathcal{M}^{[2]}$ and hence they do not have self-interactions as implied in, e.g., the Maxwell-Dirac system (cf.* [13]*)*

$$\left\{ \begin{array}{c} \left(i\gamma^{\mu}\partial_{\mu} - \gamma^{\mu}A_{\mu} - 1\right)\Psi = 0, \\ \partial\mu A^{\mu} = 0,\ 4\pi\partial_{\mu}\partial^{\mu}A_{\nu} = \left(\bar{\Psi}, \gamma^{\nu}\Psi\right) \end{array} \right\} \tag{205}$$

or the Klein-Gordon-Dirac system

$$\left\{ \begin{array}{c} \left(i\gamma^{\mu}\partial_{\mu} - \chi - 1\right)\Psi = 0, \\ \partial^{\mu}\partial_{\mu}\chi + M^2\chi = \frac{1}{4\pi}\left(\bar{\Psi}, \Psi\right) \end{array} \right\}. \tag{206}$$

Thus, our $\mathcal{M}^{[1]} \times \mathbf{B} \subset \mathcal{M}^{[1]} \times \mathcal{M}^{[2]}$ resolves the pervasive problem of singularities at $r = 0$ in both the classical and the quantum domains by considering a neighborhood N of $r = 0$ that transfers uncertainty energies between $\mathcal{M}^{[1]}$ and $\mathbf{B} \subset \mathcal{M}^{[2]}$, so that in calculating the electromagnetic energy of e^-, one stops at Bdry N.

Remark 22. *We also note that an electromagnetic field (being periodic in $\mathbf{B}$) renders itself a quotient space, displaying the phenomenon of "instantaneous communication," a feature serving as potential reference for quantum computing. To elaborate, the complex electric field related to a photon γ_j, $\mathbb{E}_j\left(t, \mathbf{x_j}\right) = \mathbb{E}_{oj} \cdot e^{-i\left(\omega_j t - \mathbf{k}_j \cdot \mathbf{x_j} + \theta_j\right)}$, results in a quotient space, i.e., $\forall\,(t, \mathbf{x}) \in U - \{0, \mathbf{0}\}$ we have*

$$t \equiv t_0 \left(\operatorname{mod} \frac{2\pi}{\omega_j} \equiv \frac{1}{\nu_j}\right) \tag{207}$$

for some $t_0 \in \left[0, \frac{1}{\nu_j}\right]$, and

$$\mathbf{x} \equiv \mathbf{x_0} \left(\operatorname{mod} \left(\frac{2\pi}{k_j}\right)\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \equiv \lambda_j \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right) \tag{208}$$

for some $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \in \left[0, \lambda_j\right]$; as such, $\forall \lambda_j \gtrsim 0$ we have

$$t \cong 0 \text{ and } \mathbf{x} \cong \mathbf{0}, \tag{209}$$

resulting in "instantaneous communication" across U, which, among other things, accounts for the double-slit phenomenon: That is, to propagate γ_j along the direction of $(0, \mathbf{0})$ to, say,

$$\left(\frac{\sqrt{1+d^2}}{c}, 1 \text{ meter}, |d| \text{ meter (the "upper slit")}, 0\right) \tag{210}$$

is nearly the same as via $(0,\mathbf{0})$ *to* $(0,0,y>0,0)$, *with*

$$\left|\left\|\mathbb{E}_{oj}\cdot e^{-i\left(\omega_j t-k_j y\right)}\right\|\right|^2=\left\|\mathbb{E}_{oj}\right\|^2>0, \tag{211}$$

i.e., a nonzero (constant) probability density ρ *along the y-axis for* γ_j *to be observed; similarly, a switch to the lower slit*

$$\left(\frac{\sqrt{1+d^2}}{c}, 1\ meter, -|d|\ \ meter, 0\right) \tag{212}$$

is to result in the same conclusion. However, if both slits are open, then there exists a superposition of fields

$$\cos\left(\omega_j t-k_j y\right)+\cos\left(\omega_j t+k_j y\right)=2\cos\omega_j t\ \ \cos k_j y \tag{213}$$

and the probability density of γ_j *equals zero* $\forall y$ *such that* $\cos k_j y=0$.

Also, $\mathbb{F}(t,\mathbf{x}) \equiv \sum_i \mathbb{E}_{p_i}(t,\mathbf{x}) =$ *the aggregate quantum field* in $\mathcal{M}^{[2]}$ (over all particles $\{p_i\}$ in $\mathcal{M}^{[1]}$) presents itself as *one* quantum field; as such, $\left\{\mathbb{E}_{p_i}(t,\mathbf{x})\right\}$ are correlated or "entangled," displaying global behavior such as the celebrated Einstein-Podolski-Rosen ("EPR") phenomenon.

Remark 23. *Our geometry of* $\mathcal{M}^{[1]}\times\mathbf{B}$ *thus has contributed physical logic to quantum mechanics, in particular, providing an energy interpretation to probabilities; as yet another demonstration, consider the fine structure constant,*

$$\alpha:=\frac{\frac{e^2}{4\pi\epsilon_o}}{\hbar c}=\frac{\frac{e^2}{4\pi\epsilon_o}}{\frac{h}{2\pi}\cdot\nu\lambda}=\frac{\frac{e^2}{4\pi\epsilon_o\lambda}}{E^{[3]}_{measured}/2\pi} \tag{214}$$

= *(the electrostatic potential energy between two electrons separated by a distance of* λ*) / (the energy* $E^{[3]}_{measured}$ *of the virtual photon needed to mediate the two electrons divided by* 2π*)* = *the constant* α, *or,* $E^{[3]}_{measured}\cdot\lambda=$ *constant, i.e., a uniform probability for any two electrons to interact across all space.*

Although in the above we derived an expression for $G^{[2]}$, it contained an undetermined parameter

$$\phi^{[2]}\equiv\frac{1-\mu_1}{{\mu_1}^2+(1-\mu_1)^2}. \tag{215}$$

Concerning $\mu_1\equiv\frac{M^{[1]}}{M^{[3]}}$, we consider the discrepancy in the electromagnetic mass of an electron as measured in a stationary state versus in a moving state with a constant velocity of $\|\mathbf{V}\|<<c$. In Feynman ([15], II-28-4), one finds (cf. e.g., [26], for this well-known problem)

$$m_{\mathbf{V}=\mathbf{0}}=\frac{3}{4}m_{\mathbf{V}\neq\mathbf{0}}; \tag{216}$$

by our Hypotheses (2) and (3), electromagnetic forces take place only in $\mathcal{M}^{[1]}$, but motions necessarily take place in $\mathcal{M}^{[3]}$; as such, it appears reasonable to attribute $m_{\mathbf{V}=\mathbf{0}}$ to $\mathcal{M}^{[1]}$ and $m_{\mathbf{V}\neq\mathbf{0}}$ to $\mathcal{M}^{[3]}$, i.e.,

$$\mu_1=\frac{3}{4}. \tag{217}$$

If so, then

$$\phi^{[1]} = 1.2, \tag{218}$$

$$\phi^{[2]} = 0.4, \text{ and thus by Equation (181)} \tag{219}$$

$$G^{[2]} = \frac{c^5}{1.6\ddot{h}} \approx 2.3 \times 10^{75} \times \frac{\text{meter}^3}{\text{kilogram} \cdot \text{second}^2} \tag{220}$$

$$\approx 10^{85} G^{[1]} \approx 10^{85} G^{[3]}. \tag{221}$$

Here we note that the generally recognized Schwarzschild radius for $\mathcal{M}^{[1]}$ (readily found in textbooks) is:

$$g_{11}^{[1]} = 0 = 1 - \frac{2G^{[1]}M^{[1]}}{R^{[1]}c^2} \tag{222}$$

$$\approx 1 - \frac{2 \times 6.7 \times 10^{-11} \times 10^{51}}{R^{[1]} \times (3 \times 10^8)^2}, \tag{223}$$

i.e.,

$$R^{[1]} \approx 10^{24} \text{ meter} \tag{224}$$

$$< 10^{26} \text{ meter (the actual radius of } \mathcal{M}^{[1]}\text{);} \tag{225}$$

thus, with $\mu_1 = \frac{3}{4}$ and $G^{[2]} \approx 10^{85} G^{[1]}$, we have

$$g_{11}^{[2]} = 0 = 1 - \frac{2 \times 6.7 \times 10^{-11} \times 10^{85} \times 10^{51} \times (1/3)}{R^{[2]} \times 9 \times 10^{16}}, \tag{226}$$

i.e.,

$$R^{[2]} \approx 10^{108} \text{ meter} \tag{227}$$

$$> > 10^{26} \text{ meter,} \tag{228}$$

so that $\mathcal{M}^{[2]}$ could give rise to $\mathcal{M}^{[1]}$.

4. Summary

In Section 2 above, we have shown that the classical electromagnetic least action is a geodesic of our $\mathcal{M}^4_{em}$, but as Feynman indicated ([15], II-19-8,9), the least action in quantum electrodynamics is the same as that of the classical; thus, we have contributed a geometric underpinning of both the classical and the quantum electrodynamics.

Then in Section 3, we have shown that our construct of the combined space-time 4-manifold $\mathcal{M}^{[3]}$ provides quantum mechanics with a more complete geometric framework, which can resolve many outstanding conceptual and analytical problems. In this regard, we envision a further development of our theory, to furnish more detailed analyses such as when a dark matter or energy $\left(0, E^{[2]}\right)$ becomes a combined particle of $\left(\frac{3}{4}E^{[2]}, \frac{1}{4}E^{[2]}\right) \in \mathcal{M}^{[3]}$ and how one may put our $\mathcal{M}^{[3]}$ to laboratory tests.

Thus, we have extended Einstein's General Relativity. We close our chapter with the following comment. In our view, the most crucial period in the development of modern physics was the decade from 1920 to 1930, when the new Einstein's General Relativity met the new quantum mechanics. What happened then was that those ten years was too short for General Relativity to be thoroughly digested and explored. For example, as mentioned in Section 2 the attempt of unifying electromagnetism with gravity in one set of *EFE* failed simply due to a hasty error in the identification of the electromagnetic energy-momentum tensor T. More fundamentally though, as mentioned in Section 3 the particle-wave duality as observed in the material space-time simply could not be explained satisfactorily, due to the self-imposed geometric constraint of a single set of *EFE* - - for the visible $\mathcal{M}^{[1]}$. As a result, waves became probabilities and concepts like "probability current" became necessities. Our $\mathcal{M}^{[3]}$ here interprets waves as energies and provides quantum mechanics with a more satisfying geometry.

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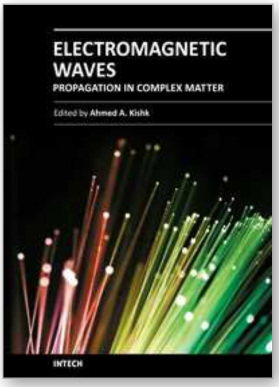

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