GENERAL RESULTS ON THE CONVERGENCE OF MULTIPOINT HERMITE-PADÉ APPROXIMANTS OF NIKISHIN SYSTEMS

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ABSTRACT. We consider simultaneous approximation of Nikishin systems of functions by means of rational vector functions which are constructed interpolating along a prescribed table of points. We give general conditions for the uniform convergence of such approximants with a geometric rate under very weak assumptions.

1. INTRODUCTION

One of the basic results in rational approximation theory is Markov's Theorem (see [12]) on the convergence of diagonal Padé approximants to the Cauchy transform of a finite positive Borel measure supported on a compact subset of the real line. No restriction is imposed on the measure. In [9] the authors obtain a version of Markov's Theorem for so called multipoint Padé approximants in which the interpolation conditions are distributed along an arbitrary symmetric (with respect to the real line) triangular table of points bounded away from the convex hull of the support of the measure. The main objective of this paper is to produce a similar result for certain (Nikishin) systems of Markov functions. The approximating functions used are multipoint Hermite-Padé vector rational functions which extend the construction used in [9] to the vector case.

Let $S = (s_1, \cdots, s_m)$ be a system of finite Borel measures with constant sign, and bounded support $\operatorname{supp}(s_i) \subset \mathbb{R}$, $i = 1, \ldots, m$, contained in the real line consisting of infinitely many points. Let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_m)$ be the corresponding system of Markov functions, where

$$\widehat{s}_k(z) = \int \frac{ds_k(x)}{z - x}, \quad k = 1, \ldots, m. \tag{1}$$

We study the uniform convergence of multipoint Hermite-Padé approximants for a special type of system of Markov functions introduced by E.M. Nikishin in [13]. Let us define them.

Let σ_1 and σ_2 be two measures with constant sign supported on $\mathbb{R}$ and let Δ_1, Δ_2 denote the smallest intervals containing $\operatorname{supp}(\sigma_1)$ and $\operatorname{supp}(\sigma_2)$, respectively. We write $\Delta_i = \operatorname{Co}(\operatorname{supp}(\sigma_i))$. Assume that $\Delta_1 \cap \Delta_2 = \emptyset$ and define

$$\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x - t} d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x).$$

Therefore, $\langle \sigma_1, \sigma_2 \rangle$ is a measure with constant sign and support equal to that of σ_1.

DEFINITION 1. For a system of intervals $\Delta_1, \ldots, \Delta_m$ contained in $\mathbb{R}$ satisfying $\Delta_j \cap \Delta_{j+1} = \emptyset$, $j = 1, \ldots, m - 1$, and finite Borel measures $\sigma_1, \ldots, \sigma_m$ with constant sign in $\operatorname{Co}(\operatorname{supp}(\sigma_j)) = \Delta_j$ and such that, each one has infinitely many points in its support, we define by induction

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_j \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_j \rangle \rangle, \quad j = 2, \ldots, m.$$

We say that $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_1 = \langle \sigma_1 \rangle = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots \quad , s_m = \langle \sigma_1, \ldots, \sigma_m \rangle$$

is the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$.

Notice that all the measures in a Nikishin system have the same support, namely $\text{supp}(\sigma_1)$. Take an arbitrary Nikishin system of measures $S = (s_1, \ldots, s_m)$, and let $\widehat{S} = (\widehat{s}_1, \ldots, \widehat{s}_m)$ be the corresponding Nikishin system of functions. We fix a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ and denote $|\mathbf{n}| = n_1 + \cdots + n_m$. Let $\alpha_{\mathbf{n}}$ be a monic polynomial with real coefficients, whose zeros lie in $D = \overline{\mathbb{C}} \setminus \Delta_1$, and $\deg \alpha_{\mathbf{n}} \leq |\mathbf{n}| + \min\{n_i\}$. It is easy to verify that there exist polynomials $Q_{\mathbf{n}}$, $P_{\mathbf{n},i}$, $i = 1, \ldots m$, such that

$$i) \quad \deg Q_{\mathbf{n}} \leq |\mathbf{n}|, \quad Q_{\mathbf{n}} \not\equiv 0, \quad \deg P_{\mathbf{n},i} \leq |\mathbf{n}| - 1$$

$$ii) \quad \left[\frac{Q_{\mathbf{n}}\widehat{s}_i - P_{\mathbf{n},i}}{\alpha_{\mathbf{n}}} \right](z) = \mathcal{O}\left(\frac{1}{z^{n_i+1}} \right) \in H(D), \quad i = 1, \ldots, m. \tag{2}$$

Finding $Q_{\mathbf{n}}$, $P_{\mathbf{n},i}$, $i = 1, \ldots, m$, reduces to solving a system of $(m+1)|\mathbf{n}|$ homogeneous linear equations on the $(m+1)|\mathbf{n}|+1$ unknown coefficients of the polynomials. Thus, a nontrivial solution exists. $P_{\mathbf{n},i}$ interpolates the function $Q_{\mathbf{n}}\widehat{s}_i$ at the zeros of $\alpha_{\mathbf{n}}$. We call $R_{\mathbf{n}} = (R_{\mathbf{n},1}, \ldots, R_{\mathbf{n},m})$, $R_{\mathbf{n},i} = \frac{P_{\mathbf{n},i}}{Q_{\mathbf{n}}}$, $i = 1, \ldots, m$, a multi-point Hermite-Padé approximant of the system of functions $\widehat{S}$, associated to the multi-index $\mathbf{n}$ and the polynomial $\alpha_{\mathbf{n}}$. If $\alpha_{\mathbf{n}} \equiv 1$, we have a classical Hermite-Padé approximant.

We study the uniform convergence of the sequences $\{R_{\mathbf{n},i}\}_{\mathbf{n}\in\Lambda}$, $\Lambda \subset \mathbb{Z}_+^m$, $i = 1, \ldots, m$, to the functions $\widehat{s}_i$ respectively. First, we obtain convergence in a weaker sense for a wide class of sequences of multi-indices Λ. Then, restricting a bit more the type of multi-indices considered, we obtain uniform convergence on each compact subset of $\overline{\mathbb{C}} \setminus \Delta_1$.

Let us introduce the type of weak convergence to be considered. Let $E \subset \mathbb{C}$ and $\mathcal{F}$ be the set of all open coverings $\{U_\nu\}$ of E by disks U_ν. Let $|U_\nu|$ denote the radius of U_ν. We define

$$m_1(E) = \inf\left\{ \sum |U_\nu| : \{U_\nu\} \in \mathcal{F} \right\}$$

to be the 1 dimensional Hausdorff content of E. From the definition it is easy to prove that the Hausdorff content is sub-additive and monotonic. More on this concept and its properties may be found in [11].

Let f_n, $n \in \mathbb{N}$, and f be complex-valued functions defined on a region $\Omega \subset \mathbb{C}$. We write

$$\mathcal{H} - \lim_{n\to\infty} f_n = f, \qquad K \subset \Omega,$$

and say that f_n converges in 1-Hausdorff content to f on each compact subset K of Ω if for each $K \subset \Omega$ and $\epsilon > 0$

$$\lim_{n\to\infty} m_1(\{z \in K : |(f_n - f)(z)| \geq \epsilon\}) = 0.$$

In the sequel $\Lambda \subset \mathbb{Z}_+^m$ is a sequence of distinct multi-indices, and $\{\alpha_{\mathbf{n}}\}$, $\mathbf{n} \in \Lambda$, a sequence of polynomials with real coefficients whose zeros lie in a compact subset $F \subset D = \overline{\mathbb{C}} \setminus \Delta_1$. Let φ_t, $t \in \overline{\mathbb{C}} \setminus \Delta_1$, denote the conformal mapping from D onto the unit disk, such that $\varphi_t(t) = 0$ and $\varphi_t'(t) > 0$. We have

THEOREM 1. *Let $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system and $\Lambda \subset \mathbb{Z}_+^m$. We assume that there exists a positive constant c such that for all $\mathbf{n} \in \Lambda$ and $i = 1, \ldots, m$*

$$n_i \geq \frac{|\mathbf{n}|}{m} - c|\mathbf{n}|^\kappa, \qquad \kappa < 1. \tag{3}$$

Then, for each compact set $K \subset \mathbb{C} \setminus \Delta_1$, $\varepsilon > 0$, and each $i = 1, \ldots, m$, we have

$$\limsup_{\mathbf{n}\in\Lambda} \left(m_1(\{z \in K : |(\widehat{s}_i - R_{\mathbf{n},i})(z)| > \varepsilon\}) \right)^{1/2|\mathbf{n}|} \leq \delta_K < 1, \tag{4}$$

where

$$\delta_K = \max\{|\varphi_t(z)| : z \in K, \, t \in F \cup \Delta_2 \cup \{\infty\}\}.$$

In particular,

$$\mathcal{H} - \lim_{\mathbf{n}\in\Lambda} R_{\mathbf{n},i} = \widehat{s}_i, \qquad K \subset \mathbb{C} \setminus \Delta_1, \qquad i = 1, \ldots, m. \tag{5}$$

If the rational functions $R_{\mathbf{n},i}$ are holomorphic in D, one can use a lemma of A. A. Gonchar (see [8, Lemma 1]) which guarantees that then convergence in 1-Hausdorff content yields uniform convergence on each compact subset of D. When $m = 1$ we have multipoint Padé approximants and it is well known that $Q_n, n \in \mathbb{Z}_+$, has n simple zeros that lie in Δ_1. The corresponding statement was proved in [3] for all $Q_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^2$. In [5], the authors prove that $Q_{\mathbf{n}}$ has $|\mathbf{n}|$ simple zeros that lie on Δ_1 for all $\mathbf{n} \in \mathbb{Z}_+^3$. The question of whether or not this remains true when $m > 3$ remains open. For general $m \in \mathbb{Z}_+, m \geq 4$, the best known result says (see [6]): Let

$$\mathbb{Z}_+^m(*) = \{\mathbf{n} \in \mathbb{Z}_+^m \mid \nexists 1 \leq i < j < k \leq m : n_i < n_j < n_k\}.$$

Then, $Q_{\mathbf{n}}$ has $|\mathbf{n}|$ simple zeros that lie on Δ_1 for all $\mathbf{n} \in \mathbb{Z}_+^m(*)$.

Taking into consideration the previous remarks, from Theorem 1 we immediately obtain

COROLLARY 1. *Under the assumptions of Theorem 1, if $m \leq 3$ or $m > 3$ and $\Lambda \subset \mathbb{Z}_+^m(*)$, we have that for each compact set $K \subset D$*

$$\limsup_{\mathbf{n} \in \Lambda} \|\widehat{s}_i - R_{\mathbf{n},i}\|_K^{1/2|\mathbf{n}|} \leq \delta_K < 1, \qquad i = 1, \ldots, m. \tag{6}$$

and

$$\lim_{\mathbf{n} \in \Lambda} R_{\mathbf{n},i} = \widehat{s}_i, \qquad i = 1, \ldots, m, \tag{7}$$

uniformly on each compact subset K of D.

In [7] (see Theorem 7 and the remark at the end of section 5) we obtained the exact rate of convergence in terms of the solution of a vector valued equilibrium problem in the presence of an external field. There, the measures are assumed to satisfy $\sigma_i \in \mathbf{Reg}, i = 1, \ldots, m$, in the sense defined in [14], and the sequence $|\alpha_{\mathbf{n}}|^{1/|\mathbf{n}|}, \mathbf{n} \in \Lambda$, is required to have limit. As compared to the results in [7], Theorem 1 and Corollary 1 are less precise in the expression of the rate of convergence but much more general in its range of application. Taking $\kappa = 0$ in Theorem 1 and Corollary 1, we obtain the main results of [1] (for the case when the measures have compact supports).

Section 2 contains some lemmas. The last section is dedicated to the proof of Theorem 1. We end that section with two remarks on extensions.

2. AUXILIARY RESULTS

From ii), we have

$$z^{\nu} \left[\frac{Q_{\mathbf{n}} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_{\mathbf{n}}} \right] (z) = \mathcal{O}\left(\frac{1}{z^2} \right) \in H(D), \quad \nu = 0, \ldots, n_i - 1, \qquad i = 1, \ldots m.$$

Let Γ be a closed integration path with wilding number 1 for all its interior points. We denote by $\text{Int}(\Gamma)$ and $\text{Ext}(\Gamma)$ the bounded and unbounded connected components, respectively, of the complement of Γ. Take Γ so that the zeros of $\alpha_{\mathbf{n}}$ lie in $\text{Ext}(\Gamma)$ and $\text{supp}(s_i) \subset \text{Int}(\Gamma)$. Using Cauchy's Theorem

$$0 = \int_{\Gamma} z^{\nu} \left[\frac{Q_{\mathbf{n}} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_{\mathbf{n}}} \right] (z) dz = \int_{\Gamma} z^{\nu} \frac{Q_{\mathbf{n}} \widehat{s}_i}{\alpha_{\mathbf{n}}} (z) dz, \qquad \nu = 0, \ldots, n_i - 1.$$

Taking into account (1), Fubini's Theorem, and Cauchy's Integral Formula, we deduce that $Q_{\mathbf{n}}$ satisfies the following multi-orthogonal conditions with respect to the measures of the system S

$$0 = \int x^{\nu} Q_{\mathbf{n}}(x) \frac{ds_i(x)}{\alpha_{\mathbf{n}}(x)}, \quad \nu = 0, \ldots, n_i - 1, \quad i = 1, \ldots, m. \tag{8}$$

DEFINITION 2. Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. For each $i = 1, \ldots, m$, we define an associated multi-index $\mathbf{n}^i = (n_2^i, \ldots, n_m^i)$ whose $m - 1$ components satisfy

$$n_j^i = \begin{cases} \min(n_1, \ldots, n_{j-1}, n_i - 1), & \text{when} \quad j = 2, \ldots, i, \\[2mm] \min(n_i, n_j), & \text{when} \quad j = i + 1, \ldots, m. \end{cases}$$

We denote $|\mathbf{n}^i| = \sum_{j=2}^m n_j^i$.

Let $1 \leq i < j \leq m$. Set

$$s_{i,j} = \langle \sigma_i, \dots, \sigma_j \rangle \quad (s_{j,j} = \sigma_j).$$

It is well known (see the appendix in [10]) that there exists a first degree polynomial $\mathcal{L}_{i,j}$ and a finite positive Borel measure $\tau_{i,j}$, $\mathrm{Co}(\mathrm{supp}(\tau_{i,j})) \subset \mathrm{Co}(\mathrm{supp}(s_{i,j}))$ such that

$$\frac{1}{\widehat{s}_{i,j}(z)} = \mathcal{L}_{i,j}(z) + \widehat{\tau}_{i,j}(z).$$

For each $k \in \{1, \dots, m\}$ we define an auxiliary Nikishin system $S^k = (s_2^k, \dots, s_m^k) = \mathcal{N}(\sigma_2^k, \dots, \sigma_m^k)$. If $k = 1$, we take

$$S^1 = (s_2^1, \dots, s_m^1) = (d\sigma_2, w_3^1 d\sigma_2, \dots, w_m^1 d\sigma_2) = \mathcal{N}(\sigma_2, \dots \sigma_m).$$

When $2 \leq k \leq m$, then

$$S^k = (s_2^k, \dots, s_m^k) = (\tau_{2,k}, w_3^k d\tau_{2,k}, \dots, w_m^k d\tau_{2,k}) =$$

$$\mathcal{N}(\tau_{2,k}, \widehat{s}_{2,k} d\tau_{3,k}, \dots, \widehat{s}_{k-1,k} d\tau_{k,k}, \widehat{s}_{k,k} d\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_m).$$

These auxiliary systems were introduced in [1]. Subsequently, in Theorem 3.1.3 of [4] certain relations between Markov functions corresponding to different components of distinct auxiliary Nikishin systems were established. For convenience of the reader, we write these formulas according to our notation. We have:

$$\frac{1}{\widehat{s}_k^1(z)} = \mathcal{L}_k(z) + \widehat{s}_2^k(z), \tag{9}$$

$$\frac{\widehat{s}_j^1(z)}{\widehat{s}_k^1(z)} = a_j + \widehat{s}_{j+1}^k(z) + c_j \widehat{s}_j^k(z), \quad j = 2, \dots, k-1, \tag{10}$$

and

$$\frac{\widehat{s}_j^1(z)}{\widehat{s}_k^1(z)} = a_j + \widehat{s}_j^k(z), \quad j = k+1, \dots, m, \tag{11}$$

where a_j and c_j denote constants and $\mathcal{L}_k$ is a polynomial of degree 1.

Set

$$\Phi_{\mathbf{n},i}(z) = \left[\frac{Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_\mathbf{n}} \right](z).$$

Given $Q_\mathbf{n}$ satisfying ii), for each fixed $i = 1, \dots, m$, $P_{\mathbf{n},i}$ is uniquely determined by the formula

$$P_{\mathbf{n},i}(z) = \int \frac{Q_\mathbf{n}(z)\alpha_\mathbf{n}(x) - Q_\mathbf{n}(x)\alpha_\mathbf{n}(z)}{z - x} \frac{ds_i(x)}{\alpha_\mathbf{n}(x)}. \tag{12}$$

In fact, it is easy to see that the expression defined by the integral is a polynomial of degree $\leq |\mathbf{n}| - 1$ since it equals

$$\int \left(\frac{Q_\mathbf{n}(z) - Q_\mathbf{n}(x)}{z - x} \alpha_\mathbf{n}(x) - \frac{\alpha_\mathbf{n}(z) - \alpha_\mathbf{n}(x)}{z - x} Q_\mathbf{n}(x) \right) \frac{ds_i(x)}{\alpha_\mathbf{n}(x)}.$$

Let $P_{\mathbf{n},i}'$ be the polynomial defined by (12). This formula may be rewritten as

$$\left[\frac{Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i}'}{\alpha_\mathbf{n}} \right](z) = \int \frac{Q_\mathbf{n}(x)}{z - x} \frac{ds_i(x)}{\alpha_\mathbf{n}(x)} = \mathcal{O}\left(\frac{1}{z} \right) \in H(D). \tag{13}$$

If $P_{\mathbf{n},i}$ comes from ii), subtracting ii) from the previous relation it follows that

$$\left[\frac{P_{\mathbf{n},i} - P_{\mathbf{n},i}'}{\alpha_\mathbf{n}} \right](z) = \mathcal{O}\left(\frac{1}{z} \right) \in H(D).$$

Consequently, $P_{\mathbf{n},i} \equiv P_{\mathbf{n},i}'$ as stated.

LEMMA 1. *Let* $\mathbf{n} = (n_1, \dots, n_m)$ *be a multi-index. For each* $i = 1, \dots m$, *we have*

$$\int_{\Delta_2} (h_j \Phi_{\mathbf{n},i})(x_2) ds_j^i(x_2) = 0, \quad j = 2, \dots, m, \quad \deg h_j < n_j^i. \tag{14}$$

Proof. Fix $i \in \{1, \dots, m\}$. From the definition of $\Phi_{\mathbf{n},i}$ and (13), we obtain

$$\int_{\Delta_2} (h_j \Phi_{\mathbf{n},i})(x_2) ds_j^i(x_2) = \int_{\Delta_2} h_j(x_2) \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x_1)}{x_2 - x_1} \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} ds_j^i(x_2).$$

Since $\deg h_j < n_j^i \leq n_i$, from (8) and Fubini's Theorem, it follows that

$$\int_{\Delta_2} h_j(x_2) \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x_1)}{x_2 - x_1} \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} ds_j^i(x_2) = \int_{\Delta_2} \int_{\Delta_1} \frac{(h_j Q_{\mathbf{n}})(x_1)}{x_2 - x_1} \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} ds_j^i(x_2) =$$

$$= - \int_{\Delta_1} (h_j Q_{\mathbf{n}} \widehat{s}_j^i)(x_1) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} . \tag{15}$$

First, we prove the statement of the lemma for the case $i + 1 \leq j \leq m$. If $i = m$, the set of j is empty and there is nothing to prove. Let $i \leq m - 1$. Using (11), we obtain that

$$- \int_{\Delta_1} (h_j Q_{\mathbf{n}} \widehat{s}_j^i)(x_1) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} = \int_{\Delta_1} (h_j Q_{\mathbf{n}})(x_1) \left(\frac{\widehat{s}_j^1(x_1)}{\widehat{s}_i^1(x_1)} - a_j \right) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} =$$

$$- a_j \int_{\Delta_1} (h_j Q_{\mathbf{n}})(x_1) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} + \int_{\Delta_1} (h_j Q_{\mathbf{n}} \widehat{s}_j^1)(x_1) \frac{d\sigma_1(x_1)}{\alpha_{\mathbf{n}}(x_1)} .$$

In the last equality we have the sum of two terms. By hypothesis $\deg h_j < n_j^i \leq \min\{n_i, n_j\}$. Taking into account (8), we deduce that both terms vanish. Hence the first case is proved.

Now, we analyze the case when $2 \leq j \leq i$. Using several times formula (10) to make j descend to 2 and finally formula (9), we obtain the equalities

$$\widehat{s}_j^i = \frac{\widehat{s}_{j-1}^1}{\widehat{s}_i^1} - a_{j-1} - c_{j-1} \widehat{s}_{j-1}^i = \tag{16}$$

$$\frac{\widehat{s}_{j-1}^1}{\widehat{s}_i^1} - a_{j-1} - c_{j-1} \left(\frac{\widehat{s}_{j-2}^1}{\widehat{s}_i^1} - a_{j-2} - c_{j-2} \widehat{s}_{j-2}^1 \right) = \cdots = \mathcal{L}_i + \frac{1}{\widehat{s}_i^1} \sum_{k=1}^{j-1} c_{k-1} \widehat{s}_k^1 ,$$

where $\mathcal{L}_i$ denotes a polynomial of degree 1, $\widehat{s}_1^i \equiv 1$, and $c_{k-1}, k = 2, \dots, j - 1$, are constants. Substituting (16) into (15), we have that

$$\int_{\Delta_1} (h_j Q_{\mathbf{n}})(x_1) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} =$$

$$= - \int_{\Delta_1} (h_j Q_{\mathbf{n}})(x_1) \mathcal{L}_i(x_1) \frac{ds_i(x_1)}{\alpha_{\mathbf{n}}(x_1)} - \sum_{k=1}^{j-1} c_{k-1} \int_{\Delta_1} (h_j Q_{\mathbf{n}})(x_1) \frac{ds_k(x_1)}{\alpha_{\mathbf{n}}(x_1)} .$$

From (8) and the hypothesis $\deg h_j \leq \min(n_1 - 1, \dots, n_{j-1} - 1, n_i - 2)$, we obtain that all the integrals on the right hand side of this equality are zero. Hence, we have finished the proof. $\square$

In the proof of Theorem 2 below, we use the concept of AT system introduced by E. M. Nikishin in [13].

DEFINITION 3. Let ω_i, $i = 1, \dots, m$, be a continuous functions with constant sign on an interval $[a, b]$ of the real line. It is said that $(\omega_1, \dots, \omega_m)$ forms an AT system for the index $\mathbf{n} = (n_1, \dots, n_m)$ on $[a, b]$ if no matter what polynomials $h_1, \dots, h_m$ one chooses with $\deg h_i \leq n_i - 1$, $i = 1, \dots, m$, not all identically equal to zero, the function

$$\mathcal{H}_{\mathbf{n}}(x) = \mathcal{H}_{\mathbf{n}}(h_1, \dots, h_m; x) = h_1(x)\omega_1(x) + \cdots + h_m(x)\omega_m(x) \tag{17}$$

has at most $|\mathbf{n}| - 1$ zeros on $[a, b]$ ($\deg h_j \leq -1$ forces $h_i \equiv 0$).

Regarding AT systems, Theorem 2 of [5] and Theorem 2 of [6] may be jointly stated as follows.

LEMMA 2. Let $(s_2, \dots, s_m) = \mathcal{N}(\sigma_2, \dots, \sigma_m)$ be an arbitrary Nikishin system of $m - 1$ measures. If $m \in \{2, 3\}$ the system of functions $(1, \widehat{s}_2, \dots, \widehat{s}_m)$ forms an AT system for any index $\mathbf{n} \in \mathbb{Z}_+^m$ on any interval $[a, b]$ disjoint from Δ_2. When $m \geq 4$ the system of functions $(1, \widehat{s}_2, \dots, \widehat{s}_m)$ forms an AT system for any index $\mathbf{n} \in \mathbb{Z}_+^m(*)$ on any interval $[a, b]$ disjoint from Δ_2.

The observations made after the statement of Theorem 1 that when $m \leq 3$, or $m > 3$ and $\mathbf{n} \in \mathbb{Z}_+^m(*)$, all $Q_\mathbf{n}$ have exactly $|\mathbf{n}|$ simple zeros in Δ_1 is an immediate consequence of Lemma 2 (see [5] and [6]).

THEOREM 2. *Let* $S = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *be a Nikishin system and* $\mathbf{n} \in \mathbb{Z}_+^m$. *For each* $i = 1, \ldots, m$, *set* $N^i = |\mathbf{n}^i| + n_i$, *where* $\mathbf{n}^i$ *is the associated multi-index. Then, for each* $i = 1, \ldots, m$, *there exists a monic polynomial* $W_{\mathbf{n},i}$, $\deg W_{\mathbf{n},i} \geq |\mathbf{n}^i|$, *whose zeros are simple and lie in the interior of* $\mathrm{Co}(\mathrm{supp}(\sigma_2))$, *such that*

$$0 = \int x^\nu Q_\mathbf{n}(x) \frac{ds_i(x)}{(\alpha_\mathbf{n} W_{\mathbf{n},i})(x)}, \quad \nu = 0, 1, \ldots, N^i - 1. \tag{18}$$

and

$$\left[\frac{Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_\mathbf{n} W_{\mathbf{n},i}} \right](z) = \frac{1}{q(z)} \int \frac{(q Q_\mathbf{n})(x)}{(\alpha_\mathbf{n} W_{\mathbf{n},i})(x)} \frac{ds_i(x)}{z - x} = \mathcal{O}\left(\frac{1}{z^{N^i+1}} \right) \in H(D), \tag{19}$$

where q *is an arbitrary polynomial with degree* $\leq N^i$.

Proof. Proving the existence of the polynomial $W_{\mathbf{n},i}$ is equivalent to showing that the function $\Phi_{\mathbf{n},i} = [(Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i})/\alpha_\mathbf{n}]$ has at least $|\mathbf{n}^i|$ changes of sign in the interior of $\mathrm{Co}(\mathrm{supp}(\sigma_2))$.

Consider the system of functions $(\widehat{s}_{3,2}^i, \ldots, \widehat{s}_{3,m}^i)$, where $\widehat{s}_{3,2}^i \equiv 1$ and for each $j = 3, \ldots, m$, $\widehat{s}_{3,j}^i$ is the Markov function of the measure $s_{3,j}^i = \langle \sigma_3^i, \ldots, \sigma_j^i \rangle$. Notice that $ds_j^i = \widehat{s}_{3,j}^i d\sigma_2^i$, $j = 2, \ldots, m$. Rewriting with this notation the equality (14) of Lemma 1, we have

$$\int_{\Delta_2} (h_j \widehat{s}_{3,j}^i \Phi_{\mathbf{n},i})(x) d\sigma_2^i = 0, \quad j = 2, \ldots, m,$$

where for each $j = 2, \ldots, m$, h_j is an arbitrary polynomial of degree $< n_j^i$. Hence

$$\int_{\Delta_2} (\mathcal{H} \Phi_{\mathbf{n},i})(x) d\sigma_2^i = 0, \tag{20}$$

with

$$\mathcal{H}(x) = \mathcal{H}(h_2, \ldots, h_m; x) = \sum_{j=2}^m (h_j \widehat{s}_{3,j}^i)(x), \qquad \deg h_j < n_j^i.$$

Let us assume that $\Phi_{\mathbf{n},i}$ has at most N changes of sign in the interior of the interval Δ_2, where $N \leq |\mathbf{n}^i| - 1$. Then, we can choose polynomials h_j, $j = 2, \ldots, m$, such that the function $\mathcal{H}(h_2, \ldots, h_m, x)$ has a simple zero at each point where $\Phi_{\mathbf{n},i}$ changes sign in the interior of Δ_2 and a zero of multiplicity $|\mathbf{n}^i| - N - 1$ at one of the extreme points of Δ_2. According to Lemma 2, the system of functions $(1, \widehat{s}_{3,3}^i, \ldots, \widehat{s}_{3,m}^i)$ forms an AT system on Δ_2. Consequently, $\mathcal{H}$ cannot have more zeros on Δ_2 than those that we have assigned and $\mathcal{H} \Phi_{\mathbf{n},i}$ has constant sign on Δ_2. This contradicts (20). Therefore, $\Phi_{\mathbf{n},i}$ has at least $|\mathbf{n}^i|$ changes of sign in the interior of Δ_2. Let $W_{\mathbf{n},i}$ be the monic polynomial whose zeros lie at the points where $\Phi_{\mathbf{n},i}$ changes of sign in the interior of Δ_2. From ii) in (2) and taking into account that $\deg W_{\mathbf{n},i} \geq |\mathbf{n}^i|$ we obtain that

$$\left[\frac{Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_\mathbf{n} W_{\mathbf{n},i}} \right](z) = \mathcal{O}\left(\frac{1}{z^{N^i+1}} \right) \in H(D). \tag{21}$$

Now, let us prove (18). Let Γ be a closed integration path with winding number 1 for all its interior points, such that the zeros of $\alpha_\mathbf{n}$ and the interval Δ_2 are in $\mathrm{Ext}(\Gamma)$ and $\Delta_1 \subset \mathrm{Int}(\Gamma)$. From (21) and Cauchy's Theorem, we obtain for each $i = 1, \ldots, m$,

$$0 = \int_\Gamma z^\nu \left[\frac{Q_\mathbf{n} \widehat{s}_i - P_{\mathbf{n},i}}{\alpha_\mathbf{n} W_{\mathbf{n},i}} \right](z) dz, \quad \nu = 0, \ldots, N^i - 1.$$

Substituting $\widehat{s}_i$ by its integral expression, we have that

$$0 = \int_\Gamma z^\nu \left[\frac{Q_\mathbf{n}}{\alpha_\mathbf{n} W_{\mathbf{n},i}} \right](z) \int_{\Delta_2} \frac{ds_i(x)}{z - x} dz - \int_\Gamma z^\nu \left[\frac{P_{\mathbf{n},i}}{\alpha_\mathbf{n} W_{\mathbf{n},i}} \right](z) dz, \quad \nu = 0, \ldots, N^i - 1.$$

On the right hand side, the second term vanishes since $P_{\mathbf{n},i}/(\alpha_\mathbf{n} W_{\mathbf{n},i})$ is holomorphic in $\mathrm{Int}(\Gamma)$. Using Fubini's Theorem and Cauchy's Integral Formula, we obtain (18).

Assume that $\deg q \leq N^i$. From (18), we have that

$$0 = \int_{\Delta_1} \frac{q(z) - q(x)}{z - x} Q_{\mathbf{n}}(x) \frac{ds_i(x)}{\alpha_{\mathbf{n}} W_{\mathbf{n},i}(x)}.$$

Hence,

$$\int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{W_{\mathbf{n},i}(x)} \frac{ds_i(x)}{(z-x)\alpha_{\mathbf{n}}(x)} = \frac{1}{q(z)} \int_{\Delta_1} \frac{(qQ_{\mathbf{n}})(x)}{W_{\mathbf{n},i}(x)} \frac{ds_i(x)}{(z-x)\alpha_{\mathbf{n}}(x)}. \tag{22}$$

Define

$$\widehat{P}_i(z) = \int_{\Delta_1} \frac{Q_{\mathbf{n}}(z)(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x) - Q_{\mathbf{n}}(x)(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(z)}{z - x} \frac{ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)}$$

The function $\widehat{P}_i$ is a polynomial of degree $\leq |\mathbf{n}| - 1$ because

$$\widehat{P}_{\mathbf{n},i}(z) = \int_{\Delta_1} \left[\frac{Q_{\mathbf{n}}(z) - Q_{\mathbf{n}}(x)}{z - x} (\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x) - \frac{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(z) - (\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)}{z - x} Q_{\mathbf{n}}(x) \right] \frac{ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)}.$$

Using (22) we obtain

$$\left[\frac{Q_{\mathbf{n}} \widehat{s}_i - \widehat{P}_i}{\alpha_{\mathbf{n}} W_{\mathbf{n},i}} \right](z) = \int_{\Delta_1} \frac{Q_{\mathbf{n}}(x)}{W_{\mathbf{n},i}(x)} \frac{ds_i(x)}{(z-x)\alpha_{\mathbf{n}}(x)} =$$

$$\frac{1}{q(z)} \int_{\Delta_1} \frac{(qQ_{\mathbf{n}})(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} \frac{ds_i(x)}{z - x} = \mathcal{O}\left(\frac{1}{z^{N^i+1}} \right) \in H(D), \quad i = 1, \ldots, m. \tag{23}$$

Substracting (23) from (21), we have that

$$\left[\frac{P_{\mathbf{n},i} - \widehat{P}_{\mathbf{n},i}}{\alpha_{\mathbf{n}} W_{\mathbf{n},i}} \right](z) = \mathcal{O}\left(\frac{1}{z^{N^i+1}} \right) \in H(D), \quad i = 1, \ldots, m.$$

Then $\deg(P_{\mathbf{n},i} - \widehat{P}_{\mathbf{n},i}) < \deg(\alpha_{\mathbf{n}} W_{\mathbf{n},i})$ and the polynomial of the numerator vanishes at all the zeros of $\alpha_{\mathbf{n}} W_{\mathbf{n},i}$ counting multiplicities. Obviously, this implies that $P_{\mathbf{n},i} \equiv \widehat{P}_i$, and we conclude the proof. $\qquad \square$

Set $N = \max\{N^i : i = 1, \ldots, m\}$. From (18) it is obvious that $Q_{\mathbf{n}}$ changes sign at least N times in the interior of Δ_1. Hence, $Q_{\mathbf{n}}$ can be represented as the product of two monic polynomials

$$Q_{\mathbf{n}} = Q_{\mathbf{n},1} Q_{\mathbf{n},2}, \qquad \deg Q_{\mathbf{n},1} = \widetilde{N} \geq N,$$

described as follows. The zeros $\{x_{\mathbf{n},j}\}$, $j = 1, \ldots, \widetilde{N}$, of the polynomial $Q_{\mathbf{n},1}$ are simple and lie in the interior of Δ_1. The polynomial $Q_{\mathbf{n},2}$ does not change sign on Δ_1 and $\deg Q_{\mathbf{n},2} \leq |\mathbf{n}| - \widetilde{N}$.

LEMMA 3. *Fix $i \in \{1, \ldots, m\}$ and $\mathbf{n} \in \mathbb{Z}_+^m$. Then, for any polynomial $\mathcal{P}$, $\deg \mathcal{P} < \widetilde{N} + N^i$, we have*

$$\int_{\Delta_1} \mathcal{P}(x) \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} = \sum_{j=1}^{\widetilde{N}} \lambda_{\mathbf{n},j}^i \mathcal{P}(x_{\mathbf{n},j}), \tag{24}$$

where

$$\lambda_{\mathbf{n},j}^i = \int_{\Delta_1} \frac{Q_{\mathbf{n},1}(x)}{Q_{\mathbf{n},1}'(x_{\mathbf{n},j})} \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(x - x_{\mathbf{n},j})(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)}. \tag{25}$$

Furthermore, for each $i \in \{1, \ldots, m\}$, the number of coefficients $\lambda_{\mathbf{n},j}^i$ whose sign equals that of the measure $(Q_{\mathbf{n},2}/\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x) ds_i(x)$ is greater than or equal to $(\widetilde{N} + N^i)/2 \geq |\mathbf{n}^i| + n_i$. Finally, for any polynomial q, $\deg q \leq N^i$,

$$\frac{Q_{\mathbf{n},2}(z)(\widehat{s}_i - \frac{P_{\mathbf{n},i}}{Q_{\mathbf{n}}})(z)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(z)} = \int_{\Delta_1} \frac{(Q_{\mathbf{n},1} q)(x)}{(Q_{\mathbf{n},1} q)(z)} \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)(z - x)}$$

$$= \int_{\Delta_1} \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)(z - x)} - \frac{p_i(z)}{Q_{\mathbf{n},1}(z)}, \tag{26}$$

where

$$p_{\mathbf{n},i} = \int_{\Delta_1} \frac{Q_{\mathbf{n},1}(z) - Q_{\mathbf{n},1}(x)}{z - x} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)}.$$

and

$$\frac{p_{\mathbf{n},i}(z)}{Q_{\mathbf{n},1}(z)} = \sum_{j=1}^{\widetilde{N}} \frac{\lambda_{\mathbf{n},j}^i}{z - x_{\mathbf{n},j}}. \tag{27}$$

Proof. Let $\mathcal{L}$ be the Lagrange polynomial which interpolates $\mathcal{P}$ at each of the points $x_{\mathbf{n},j}$, $j = 1, \ldots, \widetilde{N}$. Then,

$$\mathcal{P} - \mathcal{L} = Q_{\mathbf{n},1}\mathcal{P}_1, \quad \deg \mathcal{P}_1 < N^i.$$

Integrating with respect to the measure $(Q_{\mathbf{n},2}/\alpha_{\mathbf{n}}W_{\mathbf{n},i}))(x)ds_i(x)$ and using (18), we arrive to (24)-(25).

Take

$$\mathcal{P}(z) = \prod^+ (z - x_{\mathbf{n},j})^2,$$

where $\prod^+$ denotes the product over all j for which $\lambda_{\mathbf{n},j}^i$ has the same sign as $Q_{\mathbf{n},2}/(\alpha_{\mathbf{n}}W_{\mathbf{n},i})ds_i$. If $\deg \mathcal{P} < \widetilde{N} + N^i$, we can substitute $\mathcal{P}$ into (24) to obtain

$$\int_{\Delta_1} \mathcal{P}(x) \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} = \sum_{j=1}^{\widetilde{N}} \lambda_{\mathbf{n},j}^i \mathcal{P}(x_{\mathbf{n},j}).$$

Since $\mathcal{P}$ has positive sign on Δ_1, the integral on the left hand side has the same sign as the measure $(Q_{\mathbf{n},2}/\alpha_{\mathbf{n}}W_{\mathbf{n},i}))(x)ds_i(x)$. On the right hand side, all the terms corresponding to $\lambda_{\mathbf{n},j}^i$ with the same sign as the measure $(Q_{\mathbf{n},2}/\alpha_{\mathbf{n}}W_{\mathbf{n},i}))(x)ds_i(x)$ vanish. Hence, this sum has a sign different to that of the measure $(Q_{\mathbf{n},2}/\alpha_{\mathbf{n}}W_{\mathbf{n},i}))(x)ds_i(x)$ (or equals zero). This contradiction implies that $\deg \mathcal{P} \geq \widetilde{N} + N^i$ and from the definition of $\mathcal{P}$, we obtain the assertion concerning the sign of the coefficients $\lambda_{\mathbf{n},j}^i$.

To prove (26) we use (19), which can be rewriten as

$$\frac{Q_{\mathbf{n},2}(z)(\hat{s}_i - \frac{P_{\mathbf{n},i}}{Q_{\mathbf{n}}})(z)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(z)} = \int_{\Delta_1} \frac{(Q_{\mathbf{n},1}q)(x)}{(Q_{\mathbf{n},1}q)(z)} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(z - x)(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)},$$

where q is an arbitrary polynomial such that $\deg q \leq N^i$. In particular, when $q \equiv 1$ we have that

$$\frac{Q_{\mathbf{n},2}(z)(\hat{s}_i - \frac{P_{\mathbf{n},i}}{Q_{\mathbf{n}}})(z)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(z)} = \int_{\Delta_1} \frac{(Q_{\mathbf{n},1})(x)}{(Q_{\mathbf{n},1})(z)} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(z - x)(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} \mp \int_{\Delta_1} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(z - x)(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} =$$

$$\int_{\Delta_1} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(z - x)(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} - \frac{p_{\mathbf{n},i}(z)}{Q_{\mathbf{n},1}(z)}.$$

Hence, (26) holds.

Now, since $p_{\mathbf{n},i} < \deg Q_{\mathbf{n},1}$, taking into account that the zeros of $Q_{\mathbf{n},1}$ are simple, using (25) and the definition of $p_{\mathbf{n},i}$, we obtain that

$$\mathrm{res}\left(\frac{p_{\mathbf{n},i}}{Q_{\mathbf{n},1}}, x_{\mathbf{n},j}\right) = \lim_{z \to x_{\mathbf{n},j}} (z - x_{\mathbf{n},j}) \frac{p_{\mathbf{n},i}(z)}{Q_{\mathbf{n},1}(z)}$$

$$= \lim_{z \to x_{\mathbf{n},j}} \frac{z - x_{\mathbf{n},j}}{Q_{\mathbf{n},1}(z)} \int_{\Delta_1} \frac{Q_{\mathbf{n},1}(z) - Q_{\mathbf{n},1}(x)}{z - x} \frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} = \lambda_{\mathbf{n},j}^i.$$

Thus, we have (27) and we finished the proof. $\square$

3. Proof of Theorem 1

Proof of Theorem 1. Without loss of generality we can assume that $Q_{\mathbf{n},2}ds_i/(\alpha_{\mathbf{n}}W_{\mathbf{n},i})$ is a positive measure. Fix $i \in \{1,\dots,m\}$. We denote

$$g_{\mathbf{n}}(z) = \overline{\prod}(z - x_{\mathbf{n},j})^2,$$

where $\overline{\prod}$ denotes the product over all the zeros of $Q_{\mathbf{n},1}$ such that $\lambda_{\mathbf{n},j}^i < 0$. From Lemma 3 and the assumptions, we have that

$$\deg g_{\mathbf{n}} \le 2\left(\widetilde{N} - \frac{\widetilde{N} + N^i}{2}\right) = \widetilde{N} - N^i \le |\mathbf{n}| - |\mathbf{n}^i| - n_i \le cm|\mathbf{n}|^{\kappa}.$$

and

$$\deg Q_{\mathbf{n},2} = |\mathbf{n}| - \widetilde{N} \le |\mathbf{n}| - |\mathbf{n}^i| - n_i \le cm|\mathbf{n}|^{\kappa}.$$

Let $\widetilde{W}_{\mathbf{n}}$ be a monic polynomial with real coefficients that divides $\alpha_{\mathbf{n}}W_{\mathbf{n},i}$. We take $\widetilde{W}_{\mathbf{n}}$ to be of degree sufficiently large so as to able to use the quadrature formula (24) on the polynomial

$$\mathcal{P} = p_{\mathbf{n}} = \frac{g_{\mathbf{n}}\alpha_{\mathbf{n}}W_{\mathbf{n},i}}{\widetilde{W}_{\mathbf{n}}}.$$

That is, we want that

$$\deg p_{\mathbf{n}} < \deg Q_{\mathbf{n},1} + |\mathbf{n}^i| + n_i.$$

Additionaly, it is convenient that the degree of $p_{\mathbf{n}}$ be as large as possible. We will see later the advantage of this. It is sufficient to take $\widetilde{W}_{\mathbf{n}}$ such that

$$3cm|\mathbf{n}|^{\kappa} < \deg \widetilde{W}_{\mathbf{n}} \le 3cm|\mathbf{n}|^{\kappa} + 2$$

In fact, using this restriction and (3), we have that

$$\deg p_{\mathbf{n}} = \deg g_{\mathbf{n}} + \deg \alpha_{\mathbf{n}} + \deg W_{\mathbf{n},i} - \deg \widetilde{W}_{\mathbf{n}} < 2(|\mathbf{n}| - cm|n|^{\kappa}) \le 2(|\mathbf{n}^i| + n_i) \le \deg Q_{\mathbf{n},1} + |\mathbf{n}^i| + n_i$$

as needed. The extra 2 in the upper bound of $\deg \widetilde{W}_{\mathbf{n}}$ is to be able to select conjugate pairs of zeros, if needed, to guarantee that $\widetilde{W}_{\mathbf{n}}$ has real coefficients.

Let $M = 2\max\{1, \max\{|z| : z \in \Delta_1\}\}$. We denote by $B_{\mathbf{n}}$ the product of the zeros of $Q_{\mathbf{n},2}$ whose absolute value is larger than M, counting multiplicities. According to the upper bound obtained for the degree of $Q_{\mathbf{n},2}$, this polynomial has at most $cm|\mathbf{n}|^{\kappa}$ zeros of absolute value greater than M.

Consider the function

$$K_{\mathbf{n}}(x,z) = \frac{p_{\mathbf{n}}(z) - p_{\mathbf{n}}(x)}{(z-x)p_{\mathbf{n}}(z)}.$$

This is a polynomial on x of degree $< \deg Q_{\mathbf{n},1} + |\mathbf{n}^i| + n_i$. Hence, appling the quadrature formula (24) and taking into account (24), (26), and (27), we have

$$\Omega_{\mathbf{n}}(z) = \frac{g_{\mathbf{n}}Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}}\widetilde{W}_{\mathbf{n}}} = \frac{(g_{\mathbf{n}}\alpha_{\mathbf{n}}W_{\mathbf{n},i})(z)}{B_{\mathbf{n}}\widetilde{W}_{\mathbf{n}}(z)}\left[\int_{\Delta_1}\frac{1}{z-x}\frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} - \frac{p_{\mathbf{n},i}(z)}{Q_{\mathbf{n},1}(z)}\right] =$$

$$\frac{(g_{\mathbf{n}}\alpha_{\mathbf{n}}W_{\mathbf{n},i})(z)}{B_{\mathbf{n}}\widetilde{W}_{\mathbf{n}}(z)}\left[\int_{\Delta_1}\left(\frac{1}{z-x} - K_{\mathbf{n}}(x,z)\right)\frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} - \right.$$

$$\left. \sum_{j=1}^{\deg Q_{\mathbf{n},1}} \lambda_{\mathbf{n},j}^i\left(\frac{1}{z-x_{\mathbf{n},j}} - K_{\mathbf{n}}(x_{\mathbf{n},j},z)\right)\right] =$$

$$\frac{(g_{\mathbf{n}}\alpha_{\mathbf{n}}W_{\mathbf{n},i})(z)}{B_{\mathbf{n}}\widetilde{W}_{\mathbf{n}}(z)}\left[\int_{\Delta_1}\frac{p_{\mathbf{n}}(x)}{(z-x)p_{\mathbf{n}}(z)}\frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} - \sum_{j=1}^{\deg Q_{\mathbf{n},1}}\lambda_{\mathbf{n},j}^i\frac{p_{\mathbf{n}}(x_{\mathbf{n},j})}{(z-x_{\mathbf{n},j})p_{\mathbf{n}}(z)}\right]$$

$$= \frac{1}{B_{\mathbf{n}}}\left[\int_{\Delta_1}\frac{p_{\mathbf{n}}(x)}{z-x}\frac{Q_{\mathbf{n},2}(x)ds_i(x)}{(\alpha_{\mathbf{n}}W_{\mathbf{n},i})(x)} - \sum_{j=1}^{\deg Q_{\mathbf{n},1}}\lambda_{\mathbf{n},j}^i\frac{p_{\mathbf{n}}(x_{\mathbf{n},j})}{z-x_{\mathbf{n},j}}\right]$$

Denote by $d(z, \Delta_1)$ the distance between z and Δ_1. Using (24) once more, we obtain that

$$\left| \sum_{j=1}^{\deg Q_{\mathbf{n},1}} \lambda_{\mathbf{n},j}^i \frac{p_{\mathbf{n}}(x_{\mathbf{n},j})}{z - x_{\mathbf{n},j}} \right| \leq \frac{1}{d(z, \Delta_1)} \sum_{j=1}^{\deg Q_{\mathbf{n},1}} |\lambda_{\mathbf{n},j}^i p_{\mathbf{n}}(x_{\mathbf{n},j})| =$$

$$\frac{1}{d(z, \Delta_1)} \left| \int p_{\mathbf{n}}(x) \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} \right| .$$

When we apply the quadrature formula, we use that all the terms of the sum have the same sign as $Q_{\mathbf{n},2}(x) ds_i(x) / (\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)$ or are equal to zero since this polynomial $p_{\mathbf{n}}$ vanishes at all the zeros of $g_{\mathbf{n}}$ on Δ_1 and $\alpha_{\mathbf{n}} W_{\mathbf{n},i}/\widetilde{W}_{\mathbf{n}}$ is a polynomial whose coefficients are real numbers and its zeros lie outside of the interval Δ_1. Reasoning analogously, we obtain that

$$\left| \int \frac{p_{\mathbf{n}}(x)}{z - x} \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} \right| \leq \frac{1}{d(z, \Delta_1)} \left| \int p_{\mathbf{n}}(x) \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} \right| .$$

Therefore,

$$|\Omega_{\mathbf{n}}(z)| \leq \left| \frac{2}{B_{\mathbf{n}} d(z, \Delta_1)} \int_{\Delta_1} p_{\mathbf{n}}(x) \frac{Q_{\mathbf{n},2}(x) ds_i(x)}{(\alpha_{\mathbf{n}} W_{\mathbf{n},i})(x)} \right| =$$

$$\left| \frac{2}{B_{\mathbf{n}} d(z, \Delta_1)} \right| \left| \int_{\Delta_1} (g_{\mathbf{n}} Q_{\mathbf{n},2})(x) \frac{ds_i(x)}{\widetilde{W}_{\mathbf{n}}(x)} \right| \leq \frac{2|s_i|}{d(z, \Delta_1)} \left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_{\Delta_1} . \tag{28}$$

To estimate $\|g_{\mathbf{n}} Q_{\mathbf{n},2} / B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}\|_{\Delta_1}$, we analyze each factor separately. The zeros of $g_{\mathbf{n}}$ lie on Δ_1, hence

$$\|g_{\mathbf{n}}\|_{\Delta_1} \leq (2\|x\|_{\Delta_1})^{\deg g_{\mathbf{n}}} \leq M^{cm|\mathbf{n}|^\kappa} .$$

The zeros of $\widetilde{W}_{\mathbf{n}}$ are on $\Delta_2 \cup F$. Set $d = \min\{1, d(\Delta_1, \Delta_2 \cup F)\}$. Obviously, $d > 0$. Then

$$\left\| \frac{1}{\widetilde{W}_{\mathbf{n}}} \right\|_{\Delta_1} \leq \left(\frac{1}{d} \right)^{\deg \widetilde{W}_{\mathbf{n}}} \leq \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa} .$$

To find a bound for the values of $Q_{\mathbf{n},2}/B_{\mathbf{n}}$, we consider each one of its zeros. Let ζ be a zero of $Q_{\mathbf{n},2}$. If $|\zeta| \leq M$, we have that

$$\|x - \zeta\|_{\Delta_1} \leq \|x\|_{\Delta_1} + |\zeta| \leq 2M .$$

On other hand, if $|\zeta| > M$, $B_{\mathbf{n}}$ includes ζ as a one of its factors, and

$$\left\| \frac{x - \zeta}{\zeta} \right\|_{\Delta_1} \leq 1 + |\zeta|^{-1} \|x\|_{\Delta_1} \leq 2M .$$

Therefore,

$$\left\| \frac{Q_{\mathbf{n},2}}{B_{\mathbf{n}}} \right\|_{\Delta_1} \leq (2M)^{\deg Q_{\mathbf{n},2}} \leq M^{2cm|\mathbf{n}|^\kappa} .$$

Finally, we arrive to the following estimate

$$\left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_{\Delta_1} \leq \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa} ,$$

which together with (28), yields

$$|\Omega_{\mathbf{n}}(z)| \leq \frac{2|s_i|}{d(z, \Delta_1)} \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa} , \qquad z \in \overline{\mathbb{C}} \setminus \Delta_1 .$$

That is

$$\frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} = \mathcal{O}\left(\frac{1}{z} \right) \in H(\overline{\mathbb{C}} \setminus \Delta_1), \quad z \to \infty .$$

and on each compact subset $K \subset D = \overline{\mathbb{C}} \setminus \Delta_1$ we have that

$$\left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_K \leq \frac{2|s_i|}{d(K, \Delta_1)} \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa} \tag{29}$$

Take φ_t as defined in the statement of the Theorem. Set $\gamma_\rho = \{z : |\varphi_\infty(z)| = \rho\}, 0 < \rho < 1$. From (29), we have that

$$\left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_{\gamma_\rho} \leq \frac{2|s_i|}{d(\gamma_\rho, \Delta_1)} \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa}. \tag{30}$$

Fix a compact set $K \subset D$. Take ρ sufficiently close to 1 so that $F \cup \Delta_2 \cup K \subset \text{Ext}(\gamma_\rho)$. Set $v_{\mathbf{n}} = \deg(\alpha_{\mathbf{n}} W_{\mathbf{n},i}/\widetilde{W}_{\mathbf{n}})$. Let $w_{\mathbf{n}}$ be the exact multiplicity of the zero which $\Omega_{\mathbf{n}}$ has at infinity. Taking into account that $\deg(g_{\mathbf{n}} Q_{\mathbf{n},2}) \leq 2cm|\mathbf{n}|^\kappa$, $\deg Q_{\mathbf{n},1} \geq |\mathbf{n}^i| + n_i \geq |\mathbf{n}| - cm|\mathbf{n}|^\kappa$, $\deg \widetilde{W}_{\mathbf{n}} \leq 4cm|\mathbf{n}|^\kappa$ and $\deg W_{\mathbf{n},i} \geq (m-1)(\frac{|\mathbf{n}|}{m} - c|\mathbf{n}|^\kappa)$, from ii) in (2), we have that there exists a constant $c' > 0$ such that

$$2|\mathbf{n}| - c'|\mathbf{n}|^\kappa \leq w_{\mathbf{n}} + v_{\mathbf{n}} \leq 2|\mathbf{n}|. \tag{31}$$

Let $\{y_{\mathbf{n},1}, \dots, y_{\mathbf{n},v_{\mathbf{n}}}\}$ be the set of zeros of the polynomial $(\alpha_{\mathbf{n}} W_{\mathbf{n}}/\widetilde{W}_{\mathbf{n}})$. We have that

$$\frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}} \varphi_\infty^{\omega_{\mathbf{n}}} \prod_{j=1}^{v_{\mathbf{n}}} \varphi_{y_{\mathbf{n},j}}} \in H(D).$$

Define

$$\kappa(\gamma_\rho) = \inf\{|\varphi_t(z)| : z \in \gamma_\rho, \, t \in F \cup \Delta_2 \cup \{\infty\}\} > 0.$$

As a function of the two variables z and t, $|\varphi_t(z)|$ is continuous on $\overline{\mathbb{C}}^2$. Hence $\kappa(\gamma_\rho) > 0$, because $\gamma_\rho \cap (F \cup \Delta_2) = \emptyset$ and $|\varphi_t(z)|$ vanishes only when $z = t$.

From (30), we obtain that

$$\left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}} \varphi_\infty^{\omega_{\mathbf{n}}} \prod_{j=1}^{v_{\mathbf{n}}} \varphi_{y_{\mathbf{n},j}}} \right\|_{\gamma_\rho} \leq \frac{2|s_i|}{d(\gamma_\rho, \Delta_1) \kappa(\gamma_\rho)^{w_{\mathbf{n}} + v_{\mathbf{n}}}} \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa}.$$

Since $K \subset \text{Ext}(\gamma_\rho)$, using the maximum principle for analytic functions, we have that this inequality holds for all $z \in K$. Then

$$\left| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right| \leq$$

$$\frac{2|s_i|}{d(\gamma_\rho, \Delta_1) \kappa(\gamma_\rho)^{w_{\mathbf{n}} + v_{\mathbf{n}}}} \left(\frac{M}{d} \right)^{4cm|\mathbf{n}|^\kappa} \left| \varphi_\infty^{\omega_{\mathbf{n}}}(z) \prod_{j=1}^{v_{\mathbf{n}}} \varphi_{y_{\mathbf{n},j}}(z) \right|, \qquad z \in K.$$

Set

$$\delta_K = \max\{|\varphi_t(z)| : z \in K, \, t \in F \cup \Delta_2 \cup \{\infty\}\} < 1.$$

By the continuity of $|\varphi_t(z)|$ on $\overline{\mathbb{C}}^2$, we can assure that $\delta_K < 1$, because it is equal to 1 only when $z \in \Delta_1$. From the definition of δ_K, the last inequality, and (31), we obtain that

$$\limsup_{\mathbf{n} \in \Lambda} \left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_K^{1/2|\mathbf{n}|} \leq \frac{\delta_K}{\kappa(\gamma_\rho)}.$$

Making ρ tend to 1, we have that $\kappa(\gamma_\rho)$ tends to 1; hence, we conclude that

$$\limsup_{\mathbf{n} \in \Lambda} \|\Omega_{\mathbf{n}}\|_K^{1/2|\mathbf{n}|} = \limsup_{\mathbf{n} \in \Lambda} \left\| \frac{g_{\mathbf{n}} Q_{\mathbf{n},2}(\widehat{s}_i - R_{\mathbf{n},i})}{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}}} \right\|_K^{1/2|\mathbf{n}|} \leq \delta_K < 1.$$

In particular, for each compact $K \subset D$ and $\delta > 0$,

$$|\Omega_{\mathbf{n}}(z)| \leq (\delta_K + \delta)^{2|\mathbf{n}|}, \quad z \in K \tag{32}$$

for all $\mathbf{n} \in \Lambda$, except a finite number of multi-indices.

Notice that

$$\widehat{s}_i - R_{\mathbf{n},i} = \frac{B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}} \Omega_{\mathbf{n}}}{g_{\mathbf{n}} Q_{\mathbf{n},2}}. \tag{33}$$

Set $R = 2 \max\{\|z\|_K, 1\}$. Let $Q_{\mathbf{n},R}$ be the monic polynomial whose zeros are those of $Q_{\mathbf{n},2}$ with absolute value less than R. Let us obtain an estimate of $\|(B_{\mathbf{n}} \widetilde{W}_{\mathbf{n}} Q_{\mathbf{n},R})/(g_{\mathbf{n}} Q_{\mathbf{n},2})\|_K$.

Since the zeros of $g_\mathbf{n}$ lie in Δ_1, then

$$\left\|\frac{1}{g_\mathbf{n}}\right\|_K \leq \left(\frac{1}{d(K,\Delta_1)}\right)^{\deg g_\mathbf{n}} \leq \left(\frac{1}{d_1}\right)^{cm|\mathbf{n}|^\kappa},$$

where $d_1 = \min\{d(K,\Delta_1),1\}$. The zeros of $\widetilde{W}_\mathbf{n}$ lie on $F \cup \Delta_2$, thus

$$\|\widetilde{W}_\mathbf{n}\|_K \leq (M(K,F \cup \Delta_2))^{\deg \widetilde{W}_\mathbf{n}} \leq M_1^{4cm|\mathbf{n}|^\kappa},$$

where $M(K,F \cup \Delta_2) = \max\{|z - \zeta| : z \in K, \zeta \in F \cup \Delta_2\}$ and $M_1 = \max\{M(K,F \cup \Delta_2),1\}$. If ζ is a zero of $Q_{\mathbf{n},2}/Q_{\mathbf{n},R}$, since ζ is a factor of $B_\mathbf{n}$, we obtain

$$\left|\frac{\zeta}{z-\zeta}\right| = \frac{1}{|1-\frac{z}{\zeta}|} \leq \frac{1}{1-|\frac{z}{\zeta}|} \leq 2 \leq R.$$

The other factors of $B_\mathbf{n}$ have absolute value less than R. Hence,

$$\left\|\frac{B_\mathbf{n}Q_{\mathbf{n},R}}{Q_{\mathbf{n},2}}\right\|_K \leq R^{cm|\mathbf{n}|^\kappa}$$

We conclude that

$$\left\|\frac{B_\mathbf{n}\widetilde{W}_\mathbf{n}Q_{\mathbf{n},R}}{g_\mathbf{n}Q_{\mathbf{n},2}}\right\|_K \leq \left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}. \tag{34}$$

Let $\delta > 0$ be an arbitrary positive number, such that $\delta_K + \delta < 1$. Using (32), (33) and (34), we obtain

$$|(\widehat{s}_i - R_{\mathbf{n},i})(z)| \leq \frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{|Q_{\mathbf{n},R}(z)|}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}, \qquad z \in K. \tag{35}$$

In case that there exists an infinite subsequence of multi-indices $\Lambda' \subset \Lambda$ such that $Q_{\mathbf{n},R} \equiv 1, n \in \Lambda'$, from the last inequality, we obtain that

$$\limsup_{\mathbf{n} \in \Lambda'} \|\widehat{s}_i - R_{\mathbf{n},i}\|_K^{1/2|\mathbf{n}|} \leq \delta_K + \delta.$$

Making δ tend to 0, we arrive to

$$\limsup_{\mathbf{n} \in \Lambda'} \|\widehat{s}_i - R_{\mathbf{n},i}\|_K^{1/2|\mathbf{n}|} \leq \delta_K.$$

This is the case for the whole sequence Λ if $m \leq 3$ or $m > 3$ and $\Lambda \subset \mathbb{Z}_+^m(*)$ as in the hypothesis of Corollary 1 as follows from Lemma 2. Thus Corollary 1 is proved. In general, if such a subsequence exists, for that subsequence we have uniform convergence, which is stronger than convergence in Hausdorff content. Therefore, without loss of generality, we can assume $\deg Q_{\mathbf{n},R} \geq 1, n \in \Lambda$.

Fix $\varepsilon > 0$. From (35), it follows that

$$\{z \in K : |(\widehat{s}_i - R_{\mathbf{n},i})(z)| > \epsilon\} \subset$$

$$\left\{z \in K : \frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{|Q_{\mathbf{n},R}(z)|}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa} > \varepsilon\right\} =$$

$$\left\{z \in K : |Q_{\mathbf{n},R}(z)| < \frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{\varepsilon}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}\right\} \subset$$

$$\left\{z \in \mathbb{C} : |Q_{\mathbf{n},R}(z)| < \frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{\varepsilon}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}\right\} \subset$$

$$\bigcup_{\{\zeta:Q_{\mathbf{n},R}(\zeta)=0\}}\left\{z \in \mathbb{C} : |z - \zeta| < \left(\frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{\varepsilon}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}\right)^{1/\deg Q_{\mathbf{n},R}}\right\}.$$

From the subadditivity and monotonicity of the Hausdorff content, we deduce that

$$m_1(\{z \in K : |(\widehat{s}_i - R_{\mathbf{n},i})(z)| > \epsilon\}) \leq \deg Q_{\mathbf{n},R}\left(\frac{(\delta_K + \delta)^{2|\mathbf{n}|}}{\varepsilon}\left(\frac{RM_1}{d_1}\right)^{4cm|\mathbf{n}|^\kappa}\right)^{1/\deg Q_{\mathbf{n},R}}.$$

Since $1 \leq \deg Q_{\mathbf{n},R} \leq cm|\mathbf{n}|^{\kappa}$, we obtain that

$$\limsup_{\mathbf{n} \in \Lambda} \left(m_1(\{z \in K : |(\widehat{s}_i - R_{\mathbf{n},i})(z)| > \varepsilon\}) \right)^{1/2|\mathbf{n}|} \leq \delta_K + \delta \,.$$

Letting $\delta \to 0$, we have (4). With this we conclude the proof. $\qquad\square$

REMARK . These results hold true if in the definition of a Nikishin system we only require that the interior (in $\mathbb{R}$) of $\Delta_j \cap \Delta_{j+1}$, $j = 2, \ldots, m-1$, be empty as long as the measures s_j, $j = 1, \ldots, m$, are finite. This allows consecutive intervals Δ_j to have a common end point (starting from the second one). If Δ_1 and Δ_2 have a common end point, we cannot rely on the extra interpolation points which appear on Δ_2 to deduce the rate of convergence because they can accumulate on Δ_1 and then $\delta_K = 1$. Disregarding these extra interpolation points and following basically the same scheme of proof, instead of (4) one can prove

$$\limsup_{\mathbf{n} \in \Lambda} \left(m_1(\{z \in K : |(\widehat{s}_i - R_{\mathbf{n},i})(z)| > \varepsilon\}) \right)^{m/|\mathbf{n}|(m+1)} \leq \delta'_K < 1 \,,$$

where

$$\delta'_K = \max\{|\varphi_t(z)| : z \in K, \, t \in F \cup \{\infty\}\} \,,$$

which is sufficient to prove (5), and in place of (6) one gets

$$\limsup_{\mathbf{n} \in \Lambda} \|\widehat{s}_i - R_{\mathbf{n},i}\|_K^{m/|\mathbf{n}|(m+1)} \leq \delta'_K < 1 \,, \qquad i = 1, \ldots, m \,,$$

from which (7) follows. Certainly, these bounds are also true when $\Delta_1 \cap \Delta_2 = \emptyset$. One has to compare δ_K^2 and $(\delta'_K)^{(m+1)/m}$ and choose the smaller one of these values to give a better estimate. For example, if Δ_2 is very close to Δ_1 the second one of these values may be smaller than the first. Assumption (3) (either when Δ_1 and Δ_2 are disjoint or have a common end point) can be replaced by

$$n_i \geq \frac{|\mathbf{n}|}{m} - o(|\mathbf{n}|) \,, \qquad i = 1, \ldots, m \,,$$

and Theorem 1 and Corollary 1 remain valid. We have restricted generality in the statements in order to simplify the arguments in the proofs, though basically they remain the same.

REMARK . The measures σ_j may be allowed to have unbounded support. If $\text{supp}(\sigma_1)$ remains a compact set nothing changes. When $\text{supp}(\sigma_1)$ is an unbounded subset of the real line (say $\Delta_1 = [0, +\infty)$), geometric rate of convergence in (4) and (6) may not take place if Δ_2 is also unbounded. Nevertheless, even in this situation one can still prove (5) and (7) assuming, for example, Carleman's condition (see [2])

$$\sum_{n=0}^{\infty} \left(\frac{1}{c_n}\right)^{1/2n} = \infty \,, \qquad c_n = \int x^n d\sigma_1(x) \,.$$

In this case the proof is substantially more difficult and additional arguments are required. See [1] to understand the methodology to be used.

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