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GENERAL SOLUTIONS OF AXISYMMETRIC PROBLEMS IN TRANSVERSELY ISOTROPIC BODY

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Abstract

In this paper we solve axisymmetric problems by stress and deduce a series of valuable general solutions by unified method. Some of them are well-known solutions, and others have not appeared in the literature. We also prove the completeness of these general solutions.

I. Introduction

Lekhniskii[1,2] was the first one who considered axisymmetric deformation of rotating bodies for transversely isotropic media, extended Love's solution and calculated many examples. Elliott[3] expressed the three-dimensional solution with two quasi-harmonic functions. This solution especially suits to the case of axisymmetric deformation, and can be used to deduce Lekhniskii's solution. Elliott and Shield applied Elliott's solution to solve a series of important problems[4,5]. The general solution of three-dimension problems given by Hu[6] can be reduced to Lekhniskii's solution and Elliott's solution in case of axisymmetric deformation. Based on the equilibrium equations in the form of displacement, Eubanks and Sternberg[7] obtained Lekhniskii's solution and proved its completeness. They also extended Almansi's theorm, which enabled them to prove the completeness of Elliott's solution. In the present paper we solve axisymmetric problems by stress and deduce a series of general solutions by unified method. Two of them are the well-known solutions mentioned above, and others have not appeared in the literature. The deduction is elementary, and the completeness is obtained obviously.

We only consider the solid of revolution which is defined in [7], i.e. on the meridian plane any straight line parallel to the coordinate axes intersects the boundary at two points.

II. Stress Functions and General Solutions

In a transversely isotropic body the equilibrium equations, constitutive equations and geometric equations as well as the strain compatibility equations in terms of the stress are (78.8), (78.1), (78.7) and (78.10) in [1], respectively. It is easy to verify the general solution of the equilibrium equations without body force[10]:

$$\tau_{rz}=-\frac{\partial^2 F}{\partial r\partial z},\quad \sigma_z=\nabla_*^2 F,\quad \sigma_r=\frac{\partial^2 F}{\partial z^2}+\frac{\partial G}{r\partial r},\quad \sigma_\theta=\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 G}{\partial r^2} \tag{2.1}$$

$$\nabla_*^2=\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r} \tag{2.2}$$

where F and G are the functions called stress functions. Substituting from (2.1) in compatibility equations, we get

$$(a_{11}-a_{12})\left(\frac{\partial G}{r\partial r}-\frac{\partial^2 G}{\partial r^2}\right)=r\frac{\partial}{\partial r}\left[a_{12}\left(\frac{\partial^2 F}{\partial z^2}+\frac{\partial G}{r\partial r}\right)+a_{11}\left(\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 G}{\partial r^2}\right)+a_{13}\nabla_*^2 F\right] \tag{2.3}$$

$$-a_{44}\frac{\partial^3 F}{\partial r\partial z^2}-\frac{\partial}{\partial r}\left[a_{13}\nabla_*^2 G+2a_{13}\frac{\partial^2 F}{\partial z^2}+a_{33}\nabla_*^2 F\right]$$
$$=r\frac{\partial^2}{\partial z^2}\left[a_{11}\left(\frac{\partial^2 F}{\partial z^2}+\frac{\partial^2 G}{\partial r^2}\right)+a_{12}\left(\frac{\partial^2 F}{\partial z^2}+\frac{\partial G}{r\partial r}\right)+a_{13}\nabla_*^2 F\right] \tag{2.4}$$

Integrating (2.3) with respect to r, we obtain

$$a_{11}\nabla_*^2 G+(a_{11}+a_{12})\,\partial^2 F/\partial z^2+a_{13}\nabla_*^2 F=B(z) \tag{2.5}$$

where $B(z)$ is an arbitrary function of z.

Equation (2.5) can be used to eliminate the terms containing F in (2.4), and then integrating it with respect to r, we obtain

$$a_{13}\nabla_*^2 G+(2a_{13}+a_{44})\frac{\partial^2 F}{\partial z^2}+a_{33}\nabla_*^2 F=(a_{11}-a_{12})\frac{\partial^2 G}{\partial z^2}-\frac{1}{2}r^2 B''(z)+C(z) \tag{2.6}$$

where $C(z)$ is an arbitrary function of z.

Eliminating $\nabla_*^2 G$ in (2.6) by (2.5), Eqn. (2.6) becomes

$$a_{11}(a_{11}-a_{12})\frac{\partial^2 G}{\partial z^2}-[a_{11}a_{44}+a_{13}(a_{11}-a_{12})]\frac{\partial^2 F}{\partial z^2}-(a_{11}a_{33}-a_{13}^2)\nabla_*^2 F$$
$$=a_{13}B(z)+a_{11}r^2 B''(z)/2-a_{11}C(z) \tag{2.7}$$

Assume that

$$G=eH \tag{2.8}$$

where

$$e=(a_{11}+a_{12})/a_{11} \tag{2.9}$$

Substituting (2.8) into (2.5) and (2.7) and rearrange them, we have

$$d\nabla_*^2 H+a\nabla_*^2 F+d(\partial^2 F/\partial z^2)=dP(z) \tag{2.10}$$

$$d\frac{\partial^2 H}{\partial z^2}-\nabla_*^2 F-c(\partial^2 F/\partial z^2)=fP(z)+gr^2P''(z)-dQ(z) \tag{2.11}$$

where

$$P(z)=B(z)/(a_{11}+a_{12}),\quad Q(z)=a_{11}C(z)/(a_{11}^2-a_{12}^2) \tag{2.12}$$

and

$$\left.\begin{aligned} &a=a_{13}(a_{11}-a_{12})/(a_{11}a_{33}-a_{13}^2),\quad d=(a_{11}^2-a_{12}^2)/(a_{11}a_{33}-a_{13}^2)\\ &c=[a_{11}a_{44}+a_{13}(a_{11}-a_{12})]/(a_{11}a_{33}-a_{13}^2)\\ &f=a_{13}(a_{11}+a_{12})/(a_{11}a_{33}-a_{13}^2),\quad g=a_{11}(a_{11}+a_{12})/[2(a_{11}a_{33}-a_{13}^2)] \end{aligned}\right\} \tag{2.13}$$

In (2.13) the meaning of a, c, d is the same as in paper [1].

Now the stress formula (2.1) becomes

$$\tau_{rz}=-\frac{\partial^2 F}{\partial r\partial z},\quad \sigma_z=\nabla_*^2 F,\quad \sigma_r=\frac{\partial^2 F}{\partial z^2}+e\frac{\partial H}{r\partial r},\quad \sigma_\theta=\frac{\partial^2 F}{\partial z^2}+e\frac{\partial^2 H}{\partial r^2} \tag{2.14}$$

The deduction above shows that the equilibrium equations, constitutive and geometric equations are reformulated in the form of (2.10), (2.11) and (2.14). The stress functions H and F consist of two parts for each, one is the solution of the corresponding homogeneous equations of (2.10), (2.11), and the other is a special solution of Eqs. (2.10), (2.11) themselves. It is easy to find that a special solution of Eqn. (2.10), (2.11) can be written as follows

$$H=gr^2H_2(z)/d+H_0(z),\quad F=F_0(z) \tag{2.15}$$

Substituting (2.15) into (2.10) and (2.11), a set of differential equations of H_2, H_0 and F_0 is obtained

$$\left.\begin{aligned}&H_2''(z)=P''(z)\\&dH_0''(z)-cF_0''(z)=fP(z)-dQ(z)\\&4gH_2(z)+dF_0''(z)=dP(z)\end{aligned}\right\} \tag{2.16}$$

from which we can get

$$\left.\begin{aligned}&H_2(z)=P(z)+k_0z+k_1\\&dF_0''(z)=(d-4g)P(z)-4gk_0z-4gk_1\\&dH_0''(z)=fP(z)-dQ(z)-cF_0''(z)\end{aligned}\right\} \tag{2.17}$$

Here k_0, k_1 are integral constants. The corresponding stress components are

$$\left.\begin{aligned}&\tau_{rz}^*=0,\quad \sigma_z^*=0\\&\sigma_r^*=\sigma_\theta^*=(d-4g+2ge)P(z)+2gk_0(e-2)z+2gk_1(e-2)\end{aligned}\right\} \tag{2.18}$$

Becuse $d-4g+2ge=0$, (2.18) becomes

$$\tau_{rz}^*=\sigma_z^*=0,\quad \sigma_r^*=\sigma_\theta^*=m_1+m_0z \tag{2.19}$$

where

$$m_0=2g(e-2)k_0,\quad m_1=2g(e-2)k_1$$

Now we prove the fact the stress state (2.19) is also contained by the general solution of the homogeneous equations corresponding to Eqs. (2.10) and (2.11). In fact, the stress state (2.19) makes (2.14)

$$\partial^2F/\partial r\partial z=0,\quad \nabla_*^2F=0 \tag{2.20}$$

$$\frac{\partial^2 F}{\partial z^2}+e\frac{\partial H}{r\partial r}=\frac{\partial^2 F}{\partial z^2}+e\frac{\partial^2 H}{\partial r^2}=m_1+m_0z \tag{2.21}$$

The first equation of (2.20) gives

$$F=f_1(r)+f_2(z) \tag{2.22}$$

and substituting it into the second one and integrating the resultant we obtain

$$f_1(r)=c_0\ln r+c_1 \tag{2.23}$$

where c_0, c_1 are integral constants. Eqn. (2.21) gives

$$\frac{\partial^2 H}{\partial r^2}-\frac{1}{r}\frac{\partial H}{\partial r}=r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial H}{\partial r}\right)=0$$

from which

$$H=r^2h_1(z)/2+h_2(z) \tag{2.24}$$

and

$$f_2''(z)+eh_1(z)=m_0z+m_1 \tag{2.25}$$

To determine functions h_1, h_2 and f_2, we substitute (2.22) and (2.24) into the homogeneous corresponding to (2.10) and (2.11), then

$$2h_1(z)+f_2''(z)=0 \tag{2.26}$$

$$dr^2h_1''(z)/2+dh_2''(z)-cf_2''(z)=0 \tag{2.27}$$

Considering $e\neq 2$ and unite Eqs. (2.25), (2.26) and (2.27), we may obtain h_1, h_2 and f_2, and then substituting them and (2.23) into (2.22) and (2.24), the expressions for F and H are obtained, which contain five integral constants. Without losing the generality, we can assume that $P(z)=Q(z)=0$ in (2.10), (2.11), i.e.

$$d\nabla_*^2H+a\nabla_*^2F+d(\partial^2F/\partial z^2)=0 \tag{2.28}$$

$$d\frac{\partial^2 H}{\partial z^2}-\nabla_*^2F-c(\partial^2F/\partial z^2)=0 \tag{2.29}$$

It means that Eqs. (2.10), (2.11) and (2.14) are equivalent to Eqs. (2.28), (2.29), (2.14) when we only consider the stress distribution.

Utilizing the relationship of displacement and strain, constitutive equations as well as Eqs. (2.14), (2.28), (2.29), it is easy to obtain the displacement expression

$$u=-e(a_{11}-a_{12})\partial H/\partial r,\quad w=(\partial/\partial z)[e(a_{11}-a_{12})H-a_{44}F] \tag{2.30}$$

Therefore, H and F can also be called displacement functions. From the displacement point of view the arbitrariness of H, F reduces to four integral constants.

We have proved that Eqs. (2.10), (2.11), (2.14) are equivalent to Eqs. (2.28), (2.29), (2.14) in expressing the stress, therefore, the displacements deduced from them differ at most a rigid displacement (i.e. the translation in the axial direction). It is easy to verify that (2.30) includes this rigid displacement, so formula (2.30) contains any real displacement. Because every step in this section is convertable, displacement (2.30) in which H and F satisfy (2.28), (2.29) must be the solution of elasticity problem. Thus, we have proved that (2.30) is the general and complete solution.

III. The General Solution Expressed by Two Quasi-Harmonic Functions

Equations (2.28), (2.29) which functions H and F should satisfy are still complicated because unknown functions are coupled. We want to find such two functions that they can construct the general solution, but the differential equations satisfied by these functions are simpler and uncoupled. From (78.14) in [1] we know that

$$s_1^2+s_2^2=(a+c)/d,\quad s_1^2s_2^2=1/d \tag{3.1}$$

Assume

$$H=H_1-(a/d)F+(s_1^2+s_2^2)F/2 \tag{3.2}$$

and substituting it into (2.28) and (2.29),

$$\nabla_*^2 H_1+\frac{1}{2}\left(s_2^2\nabla_*^2 F+\frac{\partial^2 F}{\partial z^2}\right)+\frac{1}{2}\left(s_1^2\nabla_*^2 F+\frac{\partial^2 F}{\partial z^2}\right)=0 \tag{3.3}$$

$$\frac{\partial^2 H_1}{\partial z^2}-\frac{1}{2}s_1^2\left(s_2^2\nabla_*^2 F+\frac{\partial^2 F}{\partial z^2}\right)-\frac{1}{2}s_2^2\left(s_1^2\nabla_*^2 F+\frac{\partial^2 F}{\partial z^2}\right)=0 \tag{3.4}$$

is obtained.

When $s_1^2\neq s_2^2$, we multiply (3.3) by s_i^2, and add it to (3.4), then the result

$$\nabla_i^2 F_i=0 \tag{3.5}$$

is obtained, where

$$\nabla_i^2=\nabla_*^2+\partial^2/s_i^2\partial z^2 \qquad (i=1,2)$$

$$F_1=H_1+(s_1^2-s_2^2)F/2,\quad F_2=H_1-(s_1^2-s_2^2)F/2 \tag{3.6}$$

Based upon (3.6), H_1 and F can be expressed by F_1, F_2, then F, H are written as follows

$$F=(F_1-F_2)/(s_1^2-s_2^2),\ H=\frac{1}{2}\left[1+\frac{c-a}{d(s_1^2-s_2^2)}\right]F_1+\frac{1}{2}\left[1-\frac{c-a}{d(s_1^2-s_2^2)}\right]F_2 \tag{3.7}$$

Substitute them into (2.30) and denote

$$F_1=-\frac{2\rho d(s_1^2-s_2^2)\phi_1}{e[d(s_1^2-s_2^2)+c-a]},\quad F_2=-\frac{2\rho d(s_1^2-s_2^2)\phi_2}{e[d(s_1^2-s_2^2)-c+a]} \tag{3.8}$$

then

$$\left.\begin{aligned} u&=\rho(a_{11}-a_{12})\frac{\partial(\phi_1+\phi_2)}{\partial r}\\ w&=-\rho(a_{11}-a_{12})\frac{\partial(\phi_1+\phi_2)}{\partial z}+\frac{a_{44}\rho d}{e}\left(\frac{1}{ds_1^2-a}\cdot\frac{\partial\phi_1}{\partial z}-\frac{1}{a-ds_2^2}\cdot\frac{\partial\phi_2}{\partial z}\right)\end{aligned}\right\} \tag{3.9}$$

where

$$\rho=e(d-ac)/d \tag{3.10}$$

Obviously, ϕ_i must satisfy the quasi-harmonic equation

$$\nabla_i^2\phi_i=0 \qquad (i=1,2) \tag{3.11}$$

It is easy to verify that (3.9) is the same as (2.4.8) and (2,4.9) in [3], i.e. Elliott's solution.

When $s_1^2=s_2^2=s^2$, (3.2), (3.3) and (3.4) may be simplified as follows:

$$H=H_1+F(c-a)/2d \tag{3.12}$$

$$\nabla_*^2 H_1+s^2\nabla_*^2 F+\partial^2 F/\partial z^2=0 \tag{3.13}$$

$$\partial^2 H_1/\partial z^2-s^2(s^2\nabla_*^2 F+\partial^2 F/\partial z^2)=0 \tag{3.14}$$

Multiplying (3.13) by s^2 and adding it to (3.14), we have

$$\nabla_s^2 H_1=0 \tag{3.15}$$

where $\nabla_s^2=\nabla_*^2+\partial^2/s^2\partial z^2$. It is shown that H_1 is a quasi-harmonic function. To obtain F, we can use either (3.14), or (3.13), so the general solution of F can be written in two forms:

$$F=\frac{1}{s^2}\left(F_0+\frac{1}{2}z\frac{\partial H_1}{\partial z}\right)=\frac{2d}{a+c}\left(F_0+\frac{1}{2}z\frac{\partial H_1}{\partial z}\right) \tag{3.16}$$

$$F=\frac{1}{s^2}\left(F_0-\frac{1}{2}r\frac{\partial H_1}{\partial r}\right)=\frac{2d}{a+c}\left(F_0-\frac{1}{2}r\frac{\partial H_1}{\partial r}\right) \tag{3.17}$$

where F_0 satisfies the quasi-harmonic equation

$$\nabla_s^2F_0=0 \tag{3.18}$$

Substituting (3.16) and (3.12) into (2.30), the displacement is

$$u=-\frac{c-a}{4(c+a)}\frac{\partial}{\partial r}(\varphi_0+z\varphi_2),\quad w=\varphi_2-\frac{c-a}{4(c+a)}\frac{\partial}{\partial z}(\varphi_0+z\varphi_2) \tag{3.19}$$

where

$$\varphi_2=2e(a_{11}-a_{12})\partial H_1/\partial z,\quad \varphi_0=4e(a_{11}-a_{12})[(c+a)H_1/(c-a)+F_0] \tag{3.20}$$

Obviously, φ_0 and φ_2 should satisfy

$$\nabla_s^2\varphi_i=0\qquad(i=0,2) \tag{3.21}$$

Substituting (3.17) and (3.12) into (2.30), the displacement is

$$u=\varphi_1-\frac{c-a}{4(c+a)}(\varphi_0+r\varphi_1),\quad w=-\frac{c-a}{4(c+a)}(\varphi_0+r\varphi_1) \tag{3.22}$$

where

$$\varphi_1=-2e(a_{11}-a_{12})\partial H_1/\partial r,\quad \varphi_0=4e(a_{11}-a_{12})(F_0-(c+a)H_1/(c-a)) \tag{3.23}$$

According to (3.15), (3.18), φ_0 should satisfy Eqn. (3.21), and because of

$$\frac{\partial}{\partial r}\nabla_s^2H_1=\left(\nabla_s^2-\frac{1}{r^2}\right)\frac{\partial H_1}{\partial r}$$

φ_1 should satisfy

$$(\nabla_s^2-1/r^2)\varphi_1=0 \tag{3.24}$$

Thus, we use a simple method, but without losing generality, to prove that the general solution of axisymmetric problem can be expressed by two quasi-harmonic functions: when $s_1^2\neq s_2^2$, the solution is (3.9), where ϕ_i satisfies (3.11); when $s_1^2=s_2^2$, the solution is (3.19) or (3.22), where φ_i satisfies (3.21) and (3.24), respectively.

In the case of isotropic media, where μ is Poisson's ratio, we have

$$(c+a)/(c-a)=1-\mu,\quad s^2=1 \tag{3.25}$$

then (3.19) and (3.22) become Papkovich-Neuber's Solution[9,10].

IV. The General Solution Expressed by a Quasi-Biharmonic Function

Multiplying (2.29) by a and adding it to Eqn. (2.28), we obtain the equation

$$d\left(\nabla_*^2+a\frac{\partial^2}{\partial z^2}\right)H+(d-ac)\frac{\partial^2 F}{\partial z^2}=0$$

Introducing such a function ψ that

$$H=(d-ac)\frac{\partial^2\psi}{\partial z^2},\quad F=-d\left(\nabla_*^2+a\frac{\partial^2}{\partial z^2}\right)\psi \tag{4.1}$$

the equation above is satisfied. Substitute (4.1) into (2.14) and denote

$$d\frac{\partial\psi}{\partial z}=\varphi \tag{4.2}$$

then

$$\left.\begin{aligned}&\tau_{rz}=\frac{\partial}{\partial r}\left(\nabla_*^2+a\frac{\partial^2}{\partial z^2}\right)\varphi,\quad \sigma_z=\frac{\partial}{\partial z}\left(c\nabla_*^2+d\frac{\partial^2}{\partial z^2}\right)\varphi\\ &\sigma_r=-\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial r^2}+b\frac{\partial}{r\partial r}+a\frac{\partial^2}{\partial z^2}\right)\varphi,\quad \sigma_\theta=-\frac{\partial}{\partial z}\left(b\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+a\frac{\partial^2}{\partial z^2}\right)\varphi\end{aligned}\right\} \tag{4.3}$$

where

$$b=1+e(ac-d) \tag{4.4}$$

Substituting (4.1) into (2.28) or (2.29) and considering (3.1) and (4.2), we find that function φ satisfies the quasi-biharmonic equation

$$\nabla_1^2\nabla_2^2\varphi=0 \tag{4.5}$$

Comparing (4.3), (4.5) with (78.17) in [1], the stress expression (4.3) is just Lekhniskii's solution. In the case of isotropic media, (4.3) degenerate into Love's solution, and Eqn. (4.5), into the biharmonic equation.

Now we derive another form of the solution. First, we write (2.30) in the following form

$$u=-\frac{\partial S}{\partial r},\quad w=\frac{\partial}{\partial z}\left(S-\frac{c-a}{d}T\right) \tag{4.6}$$

where

$$S=e(a_{11}-a_{12})H,\quad T=e(a_{11}-a_{12})F \tag{4.7}$$

therefore, Eqs. (2.28) and (2.29) become

$$d\nabla_*^2S+a\nabla_*^2T+d\frac{\partial^2T}{\partial z^2}=0,\quad d\frac{\partial^2S}{\partial z^2}-\nabla_*^2T-c\frac{\partial^2T}{\partial z^2}=0 \tag{4.8}$$

Multiplying the first equation of (4.8) by a/d^2, the second one by $1/d$, and then adding them together, the resultant is

$$\frac{\partial^2}{\partial z^2}\left(S-\frac{c-a}{d}T\right)+\frac{1}{d^2}\nabla_*^2[daS+(a^2-d)T]=0 \tag{4.9}$$

Introducing such a function U that

$$S-\frac{c-a}{d}T=\nabla_*^2U,\quad da\,S+(a^2-d)T=-d^2\frac{\partial^2U}{\partial z^2} \tag{4.10}$$

Eqn. (4.9) is satisfied automatically. Eqn. (4.10) gives

$$S=\frac{1}{(ac-d)}\left[(a^2-d)\nabla_*^2U-d(c-a)\frac{\partial^2U}{\partial z^2}\right],\ T=-\frac{d}{(ac-d)}\left[a\nabla_*^2U+d\frac{\partial^2U}{\partial z^2}\right] \tag{4.11}$$

Substituting the first ones of (4.11), (4.10) into (4.6), we get

$$u=-\frac{1}{(ac-d)}\frac{\partial}{\partial r}\left[(a^2-d)\nabla_*^2U-d(c-a)\frac{\partial^2U}{\partial z^2}\right],\quad w=\frac{\partial}{\partial z}\nabla_*^2U \tag{4.12}$$

Eliminating S and T with (4.11) and (4.8), we obtain the equation

$$\nabla_1^2\,\nabla_2^2U=0 \tag{4.13}$$

We denote $\partial U/\partial r=\Phi$, and formula (4.12) can be written as

$$u=-\frac{1}{(ac-d)}\left[(a^2-d)\frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial r}+\frac{\Phi}{r}\right)-d(c-a)\frac{\partial^2\Phi}{\partial z^2}\right],\quad w=\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial r}+\frac{\Phi}{r}\right) \tag{4.14}$$

Because of

$$\frac{\partial}{\partial r}\nabla_i^2=\left(\nabla_i^2-\frac{1}{r^2}\right)\frac{\partial}{\partial r}\qquad (i=1,2) \tag{4.15}$$

function Φ should satisfy

$$\left(\nabla_1^2-\frac{1}{r^2}\right)\left(\nabla_2^2-\frac{1}{r^2}\right)\Phi=0 \tag{4.16}$$

In the case of isotropic media, (4.14) and (4.16) become Michell's solution:

$$\left.\begin{aligned} u&=\frac{1}{\mu(2-\mu)}\left[(2\mu-1)\frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial r}+\frac{\Phi}{r}\right)-2(1-\mu)\frac{\partial^2\Phi}{\partial z^2}\right]\\ w&=\frac{\partial}{\partial z}\left(\frac{\partial\Phi}{\partial r}+\frac{\Phi}{r}\right)\end{aligned}\right\} \tag{4.17}$$

$$\left(\nabla^2-\frac{1}{r^2}\right)\left(\nabla^2-\frac{1}{r^2}\right)\Phi=0 \tag{4.18}$$

We can get other general solutions by different methods to simplify and solve Eqs. (2.28) and (2.29). The character of these general solutions is that highest derivatives in the displacement expression are at most second order.

V. Conclusion

First, we give the general solutions (2.30), (2.28) and (2.29) to the axisymmetric problems in transversely isotropic media, and prove them equivalent to the equilibrium equations and compatibility relationship; then, based on these equations, we use the transform and simplification technique without losing generality and obtain a series of valuable solutions: (3.9), (3.19), (3.22), (4.3) and (4.14), among them (4.14) has not been found in other papers, Papkovich-

Neuber's solution is the special case of (3.19) and (3.22); (3.9) and (4.3) is the well-known Elliott's and Lekhniskii's solutions and (4.3) will degenerate into Love's solution in the case of isotropic media.

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