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General Structure of Non-Equilibrium Thermo Field Dynamics

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Non-equilibrium thermo field dynamics (NETFD) is constructed in a compact form upon several basic requirements (axioms) without referring to the existence of the reservoir. The dissipation is involved in NETFD through the axioms, preserving most properties of the usual quantum field theory, e.g., the operator formalism, the time-ordered formulation of the Green's functions, the Feynman diagram method in real time. NETFD with this general and compact form appear to be fundamental in physics.

§ 1. Introduction

In previous papers,[1),2)] we constructed a fundamental framework for a quantum field theory for nonequilibrium systems, which was called *non-equilibrium thermo field dynamics* (NETFD), this construction was based upon two concepts. One was the *thermal state* in the *thermal-Liouville space.* This state concept made it possible for us to specify the thermal situation in terms of the *thermal state condition.* The other was *coarse graining* which was realized by projecting out some partial space from the complete thermal-Liouville space. By eliminating the reservoir variables (i.e., coarse graining in the time axis) we obtained the "Hamiltonian" $\hat{H}$ in the thermal-Liouville space which describes the dissipation effect. The thermal state condition and the "Hamiltonian" $\hat{H}$ give us the *quasi-particle superoperators* and the *thermal vacuum ket-vector* automatically, with which NETFD preserves most properties of the usual quantum field theory.

In this paper, we will show that NETFD can be formulated in an extremely compact form on several basic requirements (axioms) *without referring to the existence of the reservoir.* The framework of NETFD can be said to become completely a self-contained one. Under any situation of the symmetry of the system, we can determine, by the basic requirements, the basic structure of $i\hat{H}$ the real part of which reveals the dissipation effect. The determination of $\hat{H}$ in this formalism becomes extremely simple.

In order to show the structure of the theory much more vividly, we confine ourselves to the argument written by only four superoperators, a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$, which satisfy the canonical commutation relations:

$$[a, a^\dagger]_\sigma=[\tilde{a}, \tilde{a}^\dagger]_\sigma=1\,, \tag{1·1}$$

while the other commutation relations vanish, where the *σ-commutator* is defined for arbitrary superoperators A and B by

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$$[A, B]_\sigma = AB - \sigma BA \tag{1·2}$$

with

$$\sigma = \begin{cases} +1 & \text{for boson}, \\ -1 & \text{for fermion}, \end{cases} \tag{1·3}$$

and the *tilde conjugate* $\sim$ is defined by

$$(AB)^{\sim} = \tilde{A}\tilde{B}\,, \tag{1·4a}$$

$$(c_1 A + c_2 B)^{\sim} = c_1{}^* \tilde{A} + c_2{}^* \tilde{B} \tag{1·4b}$$

with complex c-numbers c_1 and c_2. The tilde conjugate satisfies the double tilde conjugate rule

$$\tilde{\tilde{A}} = \sigma^{\nu_A} A\,, \tag{1·5}$$

where ν_A is the fermion number of A. As particular cases for (1·5), we have

$$\tilde{\tilde{a}} = \sigma a\,, \qquad \tilde{\tilde{a}}^\dagger = \sigma a^\dagger\,. \tag{1·6}$$

It is straightforward to rephrase the whole argument in this paper into a quantum field theoretical form by regarding the superoperators as the field operators and the thermal-Liouville space as the state vector space. It will be given in a forthcoming paper.

In the next section, the basic requirements (axioms) of NETFD are given with some remarks. The procedure of the coarse graining in time axis to obtain the dissipation effect is replaced by the basic requirements. In § 3, the whole structure of NETFD is reviewed upon the basic requirements. In § 4, the theory is applied to a phase-invariant bilinear model to show how it works. In § 5, the two-point Green's function of the semi-free field is explicitly obtained, which is the basic element of the Feynman diagram in the perturbational calculation of NETFD. Some discussions are given in § 6.

§ 2. Basic requirements (axioms) of NETFD

We list the basic requirements to construct NETFD in the most refined and compact manner.

A1. The equation of motion for the *thermal vacuum ket-vector in the Schrödinger representation* $|W(t)\rangle\rangle$ is given by

$$\partial_t |W(t)\rangle\rangle = -i\hat{H}|W(t)\rangle\rangle\,, \tag{2·1}$$

where $\hat{H}$ is a superoperator consisting of $a, a^\dagger, \tilde{a}$ and $\tilde{a}^\dagger$. Equation (2·1) will be called the *Schrödinger equation* or the *master equation.*

A2. The superoperator $\hat{H}$ should satisfy the relation

$$(i\hat{H})^{\sim} = i\hat{H}\,. \tag{2·2}$$

Any superoperator which satisfies the relation (2·2), i.e., the tilde conjugate invariance, will be called *Tildian.*

A3. The requirement of the conservation of the inner-product between the *thermal vacuum bra-vector* $\langle\langle 1|$ and the thermal vacuum ket-vector $|W(t)\rangle\rangle$ reads

$$\langle\langle 1|\hat{H}=0\,. \tag{2·3}$$

A4. The *thermal state condition for the thermal vacuum bra-vector* is given by

$$\langle\langle 1|a^{\dagger}=\langle\langle 1|\tilde{a}\,, \tag{2·4a}$$

$$\langle\langle 1|a=\langle\langle 1|\tilde{a}^{\dagger}\sigma\,. \tag{2·4b}$$

A5. The requirement of the existence of the *stationary thermal vacuum ket-vector in the Schrödinger representation* reads

$$\hat{H}|W(\infty)\rangle\rangle=0\,. \tag{2·5}$$

A6. The *thermal state condition for the stationary thermal vacuum ket-vector* $|W(\infty)\rangle\rangle$. The unperturbed part of the thermal state condition is given in the form

$$a|W(\infty)\rangle\rangle=\bar{f}\tilde{a}^{\dagger}|W(\infty)\rangle\rangle\,, \tag{2·6a}$$

$$\tilde{a}|W(\infty)\rangle\rangle=\sigma\bar{f}a^{\dagger}|W(\infty)\rangle\rangle\,, \tag{2·6b}$$

where $\bar{f}$ is some c-number function.

A7. The *thermal state condition for the thermal vacuum ket-vector in the Heisenberg representation* $|W(t_0)\rangle\rangle$. The unperturbed part of the thermal state condition is given in the form

$$a|W(t_0)\rangle\rangle=f\tilde{a}^{\dagger}|W(t_0)\rangle\rangle\,, \tag{2·7a}$$

$$\tilde{a}|W(t_0)\rangle\rangle=\sigma fa^{\dagger}|W(t_0)\rangle\rangle\,, \tag{2·7b}$$

where f is some c-number function.

Remarks

R1. By inspecting the symmetric property of a system, we can write down the general form of $\hat{H}$ in terms of the superoperators a, $a^{\dagger}$, $\tilde{a}$ and $\tilde{a}^{\dagger}$. Then the most basic structure of the Tildian Hamiltonian $\hat{H}$, which includes the dissipation effect of the system, is determined by the basic requirements *A2*～*A6*. This will be illustrated in § 4.

R2. In the basic requirement *A6,* we can add information about the symmetry of the stationary thermal vacuum ket-vector in the Schrödinger representation $|W(\infty)\rangle\rangle$, which reveals itself in the structure of the real part of $i\hat{H}$. In other words, the symmetry breaking effect can be added to the structure of the real part of the $i\hat{H}$ through *A6.*

R3. In the basic requirement *A7,* we can add information about the symmetry of the thermal vacuum ket-vector in the Heisenberg representation $|W(t_0)\rangle\rangle$. The information reveals itself in the definition of the quasi-particle superoperators which are determined by *A7* (see § 3 for the detailed definition of the quasi-particle superoperators). Some preliminary investigation of this point has been progressed in Ref. 3). Note that the basic requirement *A7* is determined by the initial condition of the system at $t=t_0$ (i.e., by the experimental set-up of the system at the initial time t_0).

R4. Although the thermal state conditions (2·6) in *A6* and (2·7) in *A7* are invariant under the phase transformation $a \to a \exp(i\theta)$, this does not exclude the spontaneous breakdown of phase symmetry. For example, in the case of superconductivity, a stands for the annihilation operator of electron with energy gap, and therefore, a is a linear combination of annihilation and creation operators with opposite spin (i.e., the Bogoliubov transformation), and in this way the phase symmetry of original electron field is broken.

§ 3. Structure of NETFD

The master equation (2·1) in *A1* is solved formally as

$$|W(t)\rangle\rangle = \hat{S}(t-s)|W(s)\rangle\rangle\,, \tag{3·1}$$

where

$$\hat{S}(t) = \exp[-i\hat{H}t]\,. \tag{3·2}$$

Note that as the Tildian Hamiltonian $\hat{H}$ is not Hermitian generally, $\hat{S}$ is not necessarily unitary. In the following, we require the normalization of the thermal vacuum ket-vector to be

$$\langle\langle 1|W(t)\rangle\rangle = 1\,. \tag{3·3}$$

Then the thermal average (i.e., thermal vacuum expectation value) of a superoperator $\hat{A}$ is given by $\langle\langle 1|\hat{A}|W(t)\rangle\rangle$.

The thermal average $\langle\langle 1|\hat{A}|W(t)\rangle\rangle$ can be expressed as the thermal expectation value with respect to the thermal vacuum ket-vector in the Heisenberg representation $|W(t_0)\rangle\rangle$

$$\langle\langle 1|\hat{A}|W(t)\rangle\rangle = \langle\langle 1|\hat{A}(t)|W(t_0)\rangle\rangle\,, \tag{3·4}$$

where

$$\hat{A}(t) = \hat{S}^{-1}(t-t_0)\hat{A}\hat{S}(t-t_0)\,. \tag{3·5}$$

In deriving the relation (3·4), we used *A3*. The superoperator $\hat{A}(t)$ will be called the Heisenberg representation of the superoperator $\hat{A}$. Note that the thermal vacuum ket-vector in the Heisenberg representation $|W(t_0)\rangle\rangle$ is specified by *A7*. Note also that the Heisenberg equation of motion for the superoperators is

$$\partial_t\hat{A}(t) = i[\hat{H}, \hat{A}(t)]\,. \tag{3·6}$$

As particular cases of (3·5), we have

$$a(t) = \hat{S}^{-1}(t-t_0)a\hat{S}(t-t_0)\,, \tag{3·7a}$$

$$a^{\dagger\dagger}(t) = \hat{S}^{-1}(t-t_0)a^{\dagger}\hat{S}(t-t_0)\,. \tag{3·7b}$$

Superoperators $\tilde{a}(t)$ and $\tilde{a}^{\dagger\dagger}(t)$ are given by taking the tilde conjugation of (3·7a) and (3·7b), respectively (remember *A2*):

$$\tilde{a}(t)=\hat{S}^{-1}(t-t_0)\tilde{a}\hat{S}(t-t_0), \tag{3·7c}$$

$$\tilde{a}^{\dagger\dagger}(t)=\hat{S}^{-1}(t-t_0)\tilde{a}^{\dagger}\hat{S}(t-t_0). \tag{3·7d}$$

It should be noted that $a^{\dagger\dagger}(t)$ and $\tilde{a}^{\dagger\dagger}(t)$ are not Hermitian conjugation to $a(t)$ and $\tilde{a}(t)$, respectively, when $\hat{S}$ is not unitary, although they satisfy the canonical commutation relations:

$$[a(t), a^{\dagger\dagger}(t)]_\sigma=[\tilde{a}(t), \tilde{a}^{\dagger\dagger}(t)]_\sigma=1, \tag{3·8}$$

while the other commutation relations vanish.

We use the interaction representation (i.e., the perturbative expansion) to introduce quasi-particle superoperators. The interaction representation is specified by the *unperturbed Tildian Hamiltonian* $\hat{H}_0$, which is defined by a bilinear form consisting of the superoperators a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$, and by the unperturbed part of the thermal state condition for $|W(t_0)\rangle\rangle$ given by (2·7) in *A7*. The superoperators a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$ in this representation will be called *semi-free*. Let $\hat{S}_0(t)$ denote $\hat{S}(t)$ with $\hat{H}$ being replaced by $\hat{H}_0$. In this interaction representation, (3·7) becomes

$$a(t)=\hat{S}_0^{-1}(t-t_0)a\hat{S}_0(t-t_0), \tag{3·9a}$$

$$\tilde{a}^{\dagger\dagger}(t)=\hat{S}_0^{-1}(t-t_0)\tilde{a}^{\dagger}\hat{S}_0(t-t_0), \quad \text{etc}. \tag{3·9b}$$

Since the thermal state conditions, (2·6a) in *A6* for $|W(\infty)\rangle\rangle$ and (2·7a) in *A7* for $|W(t_0)\rangle\rangle$, are linear in a and $\tilde{a}^\dagger$, and since $\hat{H}_0$ is of the bilinear form, the thermal state condition for $|W(t)\rangle\rangle$ should be linear in $a(t)$ and $\tilde{a}^{\dagger\dagger}(t)$ with arbitrary t:

$$a(t)|W(t_0)\rangle\rangle=f(t-t_0)\tilde{a}^{\dagger\dagger}(t)|W(t_0)\rangle\rangle. \tag{3·10a}$$

On the other hand, (2·3) in *A3* and (2·4a) in *A4* give

$$\langle\langle 1|a(t)=\langle\langle 1|\tilde{a}^{\dagger\dagger}(t)\sigma. \tag{3·10b}$$

These are the thermal state condition at arbitrary time. Comparing (3·10a) with (2·6a) in *A6* we find $\bar{f}=f(\infty)$, and with (2·7a) in *A7* we find $f=f(0)$.

We define the annihilation and creation quasi-particle superoperators by

$$\gamma(t)=Z^{1/2}(t-t_0)[a(t)-f(t-t_0)\tilde{a}^{\dagger\dagger}(t)], \tag{3·11a}$$

$$\tilde{\gamma}^{\natural}(t)=Z^{1/2}(t-t_0)[\tilde{a}^{\dagger\dagger}(t)-\sigma a(t)], \tag{3·11b}$$

respectively, because, then, (3·10) gives

$$\gamma(t)|W(t_0)\rangle\rangle=0, \qquad \langle\langle 1|\tilde{\gamma}^{\natural}(t)=0. \tag{3·12a}$$

The tilde conjugation of (3·12a) leads to

$$\tilde{\gamma}(t)|W(t_0)\rangle\rangle=0, \qquad \langle\langle 1|\gamma^{\natural}(t)=0. \tag{3·12b}$$

The normalization factor $Z^{1/2}(t)$ is determined by the canonical commutation relation

$$[\gamma(t), \gamma^{\dagger}(t)]_\sigma=[\tilde{\gamma}(t), \tilde{\gamma}^{\dagger}(t)]_\sigma=1\,, \tag{3·13}$$

while the other commutation relations vanish. The result is

$$Z(t)=1+n_\sigma(t)\,, \tag{3·14}$$

where

$$n_\sigma(t)=f_\sigma(t)/[1-f_\sigma(t)] \tag{3·15}$$

with

$$n_\sigma(t)=\sigma n(t)\,, \tag{3·16a}$$

$$f_\sigma(t)=\sigma f(t)\,. \tag{3·16b}$$

Using the relations (3·11) and (3·12), we obtain

$$n(t-t_0)=\langle\!\langle 1|a^{\dagger\dagger}(t)a(t)|W(t_0)\rangle\!\rangle\,. \tag{3·17}$$

The above argument shows one of the most significant roles played by the thermal state condition; *the latter condition specifies the thermal vacuum and creation and annihilation superoperators for the quasi-particles.*

Note that although we have used the same notations $a(t)$, $\tilde{a}^{\dagger\dagger}(t)$ and $|W(t_0)\rangle\!\rangle$ both for the Heisenberg and the interaction representations, we expect that one can distinguish between them by the context.

We now define the *thermal-Liouville space* in which NETFD is constructed. The thermal-Liouville space is nothing but the linear vector space spanned by the set of bra and ket vectors which are generated, respectively, by cyclic operations of the annihilation superoperators $\gamma(t)$ and $\tilde{\gamma}(t)$ on the thermal vacuum $\langle\!\langle 1|$, and of the creation superoperators $\gamma^{\dagger}(t)$ and $\tilde{\gamma}^{\dagger}(t)$ on the thermal vacuum $|W(t_0)\rangle\!\rangle$.

Both the deviation of $\hat{H}$ from the unperturbed Tildian Hamiltonian $\hat{H}_0$ and the deviation of the thermal state condition from its unperturbed linear form [i.e., (2·6) and (2·7)] are considered as perturbative effects. By adopting the usual definition of the normal product for the quasi-particle superoperators [i.e., when a product has a form in which all the creation superoperators ($\gamma^{\dagger}$ and $\tilde{\gamma}^{\dagger}$) stand to the left of the annihilation superoperators (γ and $\tilde{\gamma}$), it is called a normal product], we obtain a Wick-type formula for NETFD. This Wick-type formula leads us to Feynman-type diagrams for multi-time Green's functions in the interaction representation. We then obtain a Feynman-type diagram method for perturbative calculations for NETFD when a perturbative interaction is introduced in $\hat{H}$. We can also formulate the generating functional methods in NETFD.[4)]

Note that the perturbational calculation leads us to an expression of the Heisenberg superoperators in terms of product of quasi-particle superoperators. This is an extension of the concept of the dynamical map in the usual quantum field theory to NETFD.

According to Refs. 5) and 6), the entropy for the nonequilibrium state in the thermal-Liouville space is given by

$$S(t)=-\frac{1}{2}\ln\Omega(t)/\Omega(\infty) \tag{3·18}$$

with

$$\Omega(t)=\langle\!\langle 1|W^\dagger(t)|W(t)\rangle\!\rangle$$
$$=\langle\!\langle W^\dagger(t)|W(t)\rangle\!\rangle\,. \tag{3·19}$$

The time derivatives of $\Omega(t)$ are given by

$$d_t\Omega(t)=-\langle\!\langle W^\dagger(t)|[(i\hat{H})^\dagger+i\hat{H}]|W(t)\rangle\!\rangle\,, \tag{3·20}$$

$$d_t^2\Omega(t)=\langle\!\langle W^\dagger(t)|[(i\hat{H})^\dagger+i\hat{H}]^2|W(t)\rangle\!\rangle-\langle\!\langle W^\dagger(t)|[i\hat{H},(i\hat{H})^\dagger]|W(t)\rangle\!\rangle\,. \tag{3·21}$$

The sign of $d_t\Omega(t)$ should be determined by the boundary condition of the system (i.e., the system is open or closed). Note that if $\hat{H}$ is Hermitian $\Omega(t)$ remains constant in time.

§4. A phase-invariant bilinear model

Since the unperturbed thermal state condition (2·6) in *A6* and (2·7) in *A7* are invariant under the phase transformation $a\to a\exp(i\theta)$, $\hat{H}$ should assume this phase invariance. Thus, the general form of $\hat{H}$ is written in the form

$$\hat{H}=h_1 a^\dagger a+h_2\tilde{a}^\dagger\tilde{a}+h_3\tilde{a}^\dagger a+h_4\tilde{a}^\dagger a^\dagger+h_0\,, \tag{4·1}$$

where $h=h'+ih''$ with real quantities h' and h''. The basic requirement *A2* makes $\hat{H}$ Tildian. Then (4·1) reduces to

$$\hat{H}=h_1'(a^\dagger a-\tilde{a}^\dagger\tilde{a})+ih_1''(a^\dagger a+\tilde{a}^\dagger\tilde{a})+ih_3''\tilde{a}a+ih_4''\tilde{a}^\dagger a^\dagger+ih_0''\,. \tag{4·2}$$

The basic requirements *A3* and *A4* give us relations between the h'' terms as

$$h''_0+\sigma h_4''=0\,, \tag{4·3a}$$

$$2h_1''+h_3''+h_4''=0\,. \tag{4·3b}$$

The basic requirements *A5* and *A6* give us another relation between the h'' terms as

$$h_0''+\sigma h_3''\bar{f}_\sigma=0\,, \tag{4·4a}$$

$$2h_1''+h_3''\bar{f}_\sigma+h_4''\bar{f}_\sigma^{-1}=0\,, \tag{4·4b}$$

where

$$\bar{f}_\sigma=\sigma\bar{f}\,. \tag{4·5}$$

In deriving (4·4), we used the unperturbed thermal state condition (2·6) in *A6* as it may be consistent with the phase-invariant bilinear model. From (4·4a), we see that $\bar{f}$ is a real quantity. Note that (4·3) and (4·4) are not mutually independent. They reduce to

$$h_0''=-\sigma h_4''\,, \tag{4·6a}$$

$$h_1''=-\frac{1}{2}(h_3''+h_4''), \tag{4·6b}$$

$$h_4''=h_3''\bar{f}_\sigma. \tag{4·6c}$$

If we introduce real quantities ε, $\varkappa_1$ and $\varkappa_2$ by the definitions

$$\varepsilon=h_1', \tag{4·7a}$$

$$2\varkappa_1=h_3'', \tag{4·7b}$$

$$2\varkappa_2=h_4'', \tag{4·7c}$$

we finally obtain the general form of the Tildian Hamiltonian $\hat{H}$ for the phase-invariant bilinear model, which satisfies the basic requirement *A2*～*A6*, as

$$\hat{H}=\varepsilon(a^\dagger a-\tilde{a}^\dagger\tilde{a})-i(\varkappa_1+\varkappa_2)(a^\dagger a+\tilde{a}^\dagger\tilde{a})+i2\varkappa_1\tilde{a}a+i2\varkappa_2\tilde{a}^\dagger a^\dagger-i2\sigma\varkappa_2 \tag{4·8}$$

with the relation

$$\varkappa_2=\varkappa_1\bar{f}_\sigma. \tag{4·9}$$

By using the Heisenberg equations of motion:

$$\begin{aligned}\partial_t a(t)&=i[\hat{H}, a(t)]\\&=-i[\varepsilon-i(\varkappa_1+\varkappa_2)]a(t)+2\sigma\varkappa_2\tilde{a}^{\dagger\dagger}(t),\end{aligned} \tag{4·10a}$$

$$\begin{aligned}\partial_t a^{\dagger\dagger}(t)&=i[\hat{H}, a^{\dagger\dagger}(t)]\\&=i[\varepsilon-i(\varkappa_1+\varkappa_2)]a^{\dagger\dagger}(t)-2\varkappa_1\tilde{a}(t),\end{aligned} \tag{4·10b}$$

and the basic requirement *A4,* we obtain

$$\partial_t\langle\!\langle 1|a^{\dagger\dagger}(t)\,a(t)=-2(\varkappa_1-\varkappa_2)\langle\!\langle 1|a^{\dagger\dagger}(t)\,a(t)+2\sigma\varkappa_2. \tag{4·11}$$

By applying the thermal vacuum ket-vector $|W(t_0)\rangle\!\rangle$ to (4·11), we have

$$\partial_t n(t-t_0)=-2(\varkappa_1-\varkappa_2)n(t-t_0)+2\sigma\varkappa_2, \tag{4·12}$$

where we defined $n(t-t_0)$ by [c.f. (3·17)]

$$n(t-t_0)=\langle\!\langle 1|a^{\dagger\dagger}(t)\,a(t)|\,W(t_0)\rangle\!\rangle. \tag{4·13}$$

In the limit $t\to\infty$, we obtain from (4·12)

$$n(\infty)=\langle\!\langle 1|a^\dagger a|\,W(\infty)\rangle\!\rangle=\frac{\sigma\varkappa_2}{\varkappa_1-\varkappa_2}. \tag{4·14}$$

Inspecting (4·11) and (4·14), we know that it will be convenient to introduce *positive* quantities $\varkappa$ and $\bar{n}$ by

$$\varkappa=\varkappa_1-\varkappa_2, \tag{4·15a}$$

$$\bar{n}=\frac{\sigma\varkappa_2}{\varkappa_1-\varkappa_2}=(\bar{f}^{-1}-\sigma)^{-1}, \tag{4·15b}$$

where we used (4·9) in the second equality of (4·15). The reason why we can determine that $\varkappa$ and $\bar{n}$ should be positive, is the stability of the system.

Then the Tildian (4·8) reduces to

$$\hat{H}=\varepsilon(a^{\dagger}a-\tilde{a}^{\dagger}\tilde{a})+i\hat{\Pi} \tag{4·16a}$$

with

$$\hat{\Pi}=-\varkappa[(1+2\bar{n}_{\sigma})(a^{\dagger}a+\tilde{a}^{\dagger}\tilde{a})-2(1+\bar{n}_{\sigma})\tilde{a}a-2\bar{n}_{\sigma}\tilde{a}^{\dagger}a^{\dagger}]-2\sigma\varkappa\bar{n}_{\sigma}$$
$$=-\varkappa(\bar{a}^{\alpha}A_{\sigma}^{\alpha\beta}a^{\beta}-\sigma), \tag{4·16b}$$

where

$$\bar{n}_{\sigma}=\sigma\bar{n}. \tag{4·17}$$

We have introduced the thermal doublet notation

$$a^{\alpha}=\begin{pmatrix}a\\ \tilde{a}^{\dagger}\end{pmatrix}, \qquad \bar{a}^{\alpha}=(a^{\dagger},\tilde{a})I_{\sigma} \tag{4·18}$$

with

$$I_{\sigma}=\begin{pmatrix}1, & 0\\ 0, & -\sigma\end{pmatrix}. \tag{4·19}$$

and

$$A_{\sigma}=I_{\bar{\sigma}}C_{\sigma}I_{\sigma}, \tag{4·20}$$

$$C_{\sigma}=C_{\sigma}^{(r)}+C_{\sigma}^{(a)} \tag{4·21}$$

with

$$C_{\sigma}^{(r)}=\begin{pmatrix}1+\bar{n}_{\sigma}, & \bar{n}_{\sigma}\\ 1+\bar{n}_{\sigma}, & \bar{n}_{\sigma}\end{pmatrix}, \quad C_{\sigma}^{(a)}=\begin{pmatrix}\bar{n}_{\sigma}, & \bar{n}_{\sigma}\\ 1+\bar{n}_{\sigma}, & 1+\bar{n}_{\sigma}\end{pmatrix}, \tag{4·22}$$

$$I_{\bar{\sigma}}=I_{\sigma}\tau_{3}, \tag{4·23}$$

$$\tau_{3}=\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}. \tag{4·24}$$

It should be noted that the Tildian Hamiltonian (4·16) has exactly the same form as the unperturbed part of the Tildian Hamiltonian[1),2)] which was obtained by eliminating the reservoir variables. Note, however, that the entire consideration in this paper does not need any reference to the reservoir.

The time derivative of $\Omega(t)$ for the phase-invariant bilinear model is given by

$$d_{t}\Omega(t)=\langle\!\langle 1|(\hat{\Pi}+\hat{\Pi}^{\dagger})|W(t)\rangle\!\rangle \tag{4·25}$$

with (4·16b). By using the thermal state condition properly (see Appendix A for details), we obtain

$$d_t \ln \Omega(t) = \begin{cases} 4\kappa \dfrac{1}{1+2n(t-t_0)}[n(t-t_0)-\bar{n}] & \text{for boson}, \\ 4\kappa \dfrac{1-2n(t-t_0)}{[1-n(t-t_0)]^2+n(t-t_0)^2}[n(t-t_0)-\bar{n}] & \text{for fermion}, \end{cases} \tag{4·26}$$

where $n(t-t_0)$ is defined by (4·13) the explicit form of which is given by

$$n(t) = \bar{n} + [n(0)-\bar{n}]e^{-2\kappa t}. \tag{4·27}$$

If f and $\bar{f}$ are equal to $\exp(-\beta_0\varepsilon)$ and $\exp(-\beta\varepsilon)$, respectively, with $T_0 = 1/\beta_0$ and $T = 1/\beta (k_B = 1)$ being the initial and final temperatures of the system, respectively, we obtain

$$\bar{n} = (e^{\beta\varepsilon} - \sigma)^{-1}, \tag{4·28a}$$

$$n(0) = (e^{\beta_0\varepsilon} - \sigma)^{-1} \tag{4·28b}$$

and

$$0 \le n(t-t_0) < \infty \quad \text{for boson}, \tag{4·29a}$$

$$0 \le n(t-t_0) < \frac{1}{2} \quad \text{for fermion}, \tag{4·29b}$$

as can be seen from (4·23) and (4·24). Then (4·22) tells us that

$$d_t S(t) \geqq 0 \qquad \text{for } T \geqq T_0, \tag{4·30a}$$

$$d_t S(t) < 0 \qquad \text{for } T < T_0. \tag{4·30b}$$

From (4·27) and (4·30), we can see that the phase-invariant bilinear model describes an open system coupling to the particle reservoir with temperature T if the chemical potential of the reservoir remains constant.

§ 5. Two-point Green's function of the semi-free field

The two-point Green's function of the semi-free field is the basic element of the Feynman diagram, and is defined by

$$G^{\alpha\beta}(t,s) = -i\langle\langle 1|T[a^\alpha(t)\bar{a}^\beta(s)]|W_S(t_0)\rangle\rangle, \tag{5·1}$$

where $\langle\langle 1|$ and $|W_S(t_0)\rangle\rangle$ are the thermal vacuum states for the system which satisfy the thermal state conditions (2·4) and (2·7), respectively. We introduced the thermal doublet

$$a^\alpha(t) = \begin{pmatrix} a(t) \\ \tilde{a}^{\dagger\dagger}(t) \end{pmatrix}, \qquad \bar{a}^\alpha(t) = (a^{\dagger\dagger}(t), \tilde{a}(t)) I_\sigma, \tag{5·2}$$

the elements of which are defined by (3·9) and its tilde conjugate. The time evolution of $\hat{S}_0(t)$ in (3·9) is determined by the unperturbed Tildian Hamiltonian $\hat{H}$ which is given by the Tildian Hamiltonian (4·16) of the phase-invariant bilinear model. We use the thermal doublet notations also for the quasi-particle superoperators:

$$\gamma^{\alpha}(t)=\begin{pmatrix}\gamma(t)\\ \tilde{\gamma}^{\dagger}(t)\end{pmatrix}, \quad \bar{\gamma}^{\alpha}(t)=(\gamma^{\dagger}(t), \tilde{\gamma}(t))I_{\sigma}\,. \tag{5·3}$$

The two-point Green's function (5·1) is easily evaluated by rewriting it in terms of the quasi-particle superoperator defined by (3·11) and its tilde conjugate, and by using a Wick-type formula. The result is given by

$$G^{\alpha\beta}(t,s)=[I_{\bar{\sigma}}W(t-t_0)\mathcal{G}(t,s)W^{-1}(s-t_0)I_{\bar{\sigma}}]^{\alpha\beta}\,, \tag{5·4}$$

where

$$\mathcal{G}^{\alpha\beta}(t,s)=-i\langle\!\langle 1|T[\gamma^{\alpha}(t)\bar{\gamma}^{\beta}(s)]|W_s(t_0)\rangle\!\rangle$$
$$=\begin{pmatrix}Z^{1/2}(s-t_0)Z^{-1/2}(t-t_0)G^{r}(t-s)\,, & 0\\ 0 & ,Z^{1/2}(t-t_0)Z^{-1/2}(s-t_0)G^{a}(t-s)\end{pmatrix} \tag{5·5}$$

with

$$G^{r}(t)=-i\theta(t)\exp[-i(\varepsilon-i\kappa)t]\,, \tag{5·6a}$$

$$G^{a}(t)=i\theta(-t)\exp[-i(\varepsilon+i\kappa)t] \tag{5·6b}$$

and

$$W(t)=I_{\sigma}B(t)I_{\sigma} \tag{5·7}$$

with

$$B(t)=Z^{1/2}(t)\begin{pmatrix}1\,, & -f(t)\\ -\sigma\,, & 1\end{pmatrix}, \tag{5·8}$$

where $Z(t)$ and $f(t)$ are related to each other through (3·14) (together with (3·15)) in which $n_{\sigma}(t)$ is given by

$$n_{\sigma}(t)=\sigma n(t) \tag{5·9}$$

with

$$n_{\sigma}(0)=f_{\sigma}/(1-f_{\sigma})\,. \tag{5·10}$$

The definition (3·3) of the quasi-particle superoperator can be written in the thermal doublet notation by using (5·8) as

$$\gamma^{\alpha}(t)=B(t-t_0)^{\alpha\beta}a^{\beta}(t)\,, \tag{5·11a}$$

$$\bar{\gamma}^{\alpha}(t)=\bar{a}^{\beta}(t)B^{-1}(t-t_0)^{\beta\alpha}\,. \tag{5·11b}$$

We have

$$B^{-1}(t)=\tau_3B(t)\tau_3\,. \tag{5·12}$$

The explicit expression of $G^{\alpha\beta}(t,s)$ is given in Appendix B.

When we introduce an interaction, we make use of $\hat{H}$ in this section as the unperturbed Tildian Hamiltonian. In this case the two-point Green's function in (5·1) becomes the internal lines in the Feynman diagrams.

As the quasi-particle superoperator satisfies the following Heisenberg equation of motion in the thermal doublet notation:

$$\partial_t \gamma^{\alpha}(t) = -i\left[\varepsilon\, \delta^{\alpha\beta} - i\kappa \frac{1+\bar{n}_\sigma}{1+n_\sigma(t-t_0)} \tau_3{}^{\alpha\beta}\right] \gamma^{\beta}(t)\,, \tag{5·13}$$

the equation of motion for the quasi-particle two-point Green's function is given by

$$i\partial_t\, \mathcal{G}^{\alpha\beta}(t,s) = \delta(t-s)\delta^{\alpha\beta} + \left[\varepsilon\delta^{\alpha\gamma} - i\kappa \frac{1+\bar{n}_\sigma}{1+n_\sigma(t-t_0)} \tau_3{}^{\alpha\gamma}\right] \mathcal{G}^{\gamma\beta}(t,s)\,. \tag{5·14}$$

It may be worthwhile to note that (5·10) is solved to give

$$\begin{pmatrix} \gamma(t) \\ \tilde{\gamma}^{\,\natural}(t) \end{pmatrix} = e^{-i\varepsilon(t-t_0)} \begin{pmatrix} Z^{1/2}(0) Z^{-1/2}(t-t_0) e^{-\kappa(t-t_0)} & 0 \\ 0\,, & Z^{1/2}(t-t_0) Z^{-1/2}(0) e^{\kappa(t-t_0)} \end{pmatrix} \begin{pmatrix} \gamma \\ \tilde{\gamma}^{\,\natural} \end{pmatrix}. \tag{5·15}$$

Especially in the case where the initial state is of the grand canonical distribution with temperature $T_0 = \beta_0{}^{-1}$, (4·28b), and the final state is of the grand canonical distribution with temperature $T = \beta^{-1}$, (4·28a), the Green's function (5·4) with $\sigma = 1$ (i.e., for boson) reduces to that derived by Schwinger.[19]

§ 6. Discussion

We presented the fundamental framework of NETFD in its most general and compact form. The dissipation effect appears in NETFD through the basic requirements which determine the basic structure of the Tildian Hamiltonian $\hat{H}$. In other words, the previous derivation[1),2)] of $\hat{H}$ by eliminating the reservoir variable (i.e., coarse graining in time axis) has been replaced by the basic requirements. The elimination of partial subsystems in the whole thermal-Liouville space, which we called the second step in Refs. 1) and 2), can be formulated by the projection method given in Ref. 2) under the basic requirements of NETFD.

As has been seen in previous papers,[1),2)] NETFD is based on the Liouville equation formalism. Then, although NETFD seems very much similar to the ordinary quantum field theory without thermal degrees of freedom in *technical* points of view, which makes us easy to handle with non-equilibrium phenomena, the thermal-Liouville space, in which NETFD is constructed, is very much different from the Hilbert space of the ordinary quantum field theory. This is the reason why we can introduce the dissipation within the quantum field theoretical framework without any difficulties.

The symmetry breaking situation can be easily added to NETFD through the basic requirements. The phase change problems both in equilibrium and far from equilibrium can be treated dynamically by NETFD. In the latter case, the physical parameters, like mass etc., in $\hat{H}$ will depend on time because of *the time-dependent renormalization procedure* by which the time dependence of an order parameter and the appearance or disappearance of corresponding Goldstone boson field can be treated self-consistently depending on physical situation. The detailed investigation will be given in a forthcoming paper. (Clearly $\hat{S}(t)$ defined by (3·2) should be regarded as the well-known time-ordered exponential, if $\hat{H}$ is interpreted as a renormalized Tildian Hamiltonian.)

As can be seen by the application of NETFD to the phase-invariant bilinear model in § 4, the basic requirements seem intimately related to open systems. However, the fact that the formalism in this paper does not need any reference to a reservoir seems to suggest that the domain of applicability of NETFD formulated in this paper is much wider. Being optimistic, we expect that NETFD can treat closed systems (such as the universe) in which a suitable boundary condition permits local dissipation. This might provide us with a new foundation in our approach to understand nature.

Although we explained the general structure of NETFD using the superoperators which satisfy the canonical commutation relations, we can include the case where the canonical commutation relations are not satisfied. Some investigation of the latter case [7)~10)] in thermal equilibrium has been done in terms of TFD.[11)~17)] These considerations can be generalized to the nonequilibrium situations and can be formulated in the present compact formalism. For example, the basic structure of the Tildian Hamiltonian $\hat{H}$ for spin system is determined through the basic requirements. This will be given in a forthcoming paper.

It should be noted that in the axiomatic formulation of NETFD we can use the notation $\langle 0|$ and $|0\rangle$ for the thermal vacuum bra and ket vectors, respectively, instead of $\langle\langle 1|$ and $|W(t_0)\rangle\rangle$, because we do not need the explicit structure of the vacuum vectors. The thermal vacuum vectors are specified by the thermal state conditions in the basic requirements.

For reader's convenience, we will mention here the recent (after this paper was submitted) development of NETFD based on what developed in Refs. 1), 2) and this paper. The generating functional methods in NETFD have been formulated.[4)] The semi-free quantum fields in NETFD have been constructed in a way parallel to that of the usual quantum field theory without thermal degrees of freedom, i.e., in terms of orthonormalization relations among wave functions, the sum rule for wave functions, the divisor operator and the particle-antiparticle conjugation (c-conjugation).[20)] Symmetry breaking situations have been investigated dynamically within the framework of NETFD.[21),22)] The theory has been developed in order to handle with the time-dependent renormalization procedure, and to show the mechanism of *the spontaneous creation of dissipation*.[23)~25)] See also Refs. 26)~28).

We close this paper by noting that the linear response theory, proposed by Kubo,[18)] and Prigogine's sub-dynamics[6)] can be re-examined from the viewpoint of NETFD.

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Appendix A

—— *Relations between Thermal Averages* ——

The thermal average $\langle\!\langle 1|\tilde{a}a|W(t)\rangle\!\rangle$ for the semi-free field gives us a relation between $\langle\!\langle 1|a^\dagger a|W(t)\rangle\!\rangle$ and $\langle\!\langle 1|W(t)\rangle\!\rangle$ as

$$\langle\!\langle 1|a^\dagger a|W(t)\rangle\!\rangle=[f^{-1}(t-t_0)-\sigma]^{-1}\langle\!\langle 1|W(t)\rangle\!\rangle\,, \tag{A·1}$$

where we used (3·1), (3·7), (3·8) and thermal state condition (3·10). The relation (3·15) is obtained from (A·1) with (3·3).

The thermal average $\langle\!\langle 1|W^\dagger(t)\tilde{a}a|W(t)\rangle\!\rangle=\langle\!\langle W^\dagger(t)|\tilde{a}a|W(t)\rangle\!\rangle$ for the semi-free field gives us a relation between

$\langle\!\langle W^\dagger(t)|a^\dagger a|W(t)\rangle\!\rangle$ and $\langle\!\langle W^\dagger(t)|W(t)\rangle\!\rangle$ as

$$\langle\!\langle W^\dagger(t)|a^\dagger a|W(t)\rangle\!\rangle=[f^{-2}(t-t_0)-\sigma]^{-1}\langle\!\langle W^\dagger(t)|W(t)\rangle\!\rangle\,, \tag{A·2}$$

where we used (3·1), (3·7), (3·8) and the thermal state condition (3·10a) and its Hermite conjugate. To obtain (4·22), we used (A·2).

Similarly, we can obtain relations between *observable* thermal averages.

Appendix B

—— *Two-Point Green's Function* $G^{\alpha\beta}(t,s)$ ——

Equation (5·4) gives us the expression

$$G^{\alpha\beta}(t,s)=\{I_{\bar{\sigma}}[C_\sigma{}^{(r)}G^r(t-s)-C_\sigma{}^{(a)}G^a(t-s)$$
$$-i\begin{pmatrix}1, & 1\\ 1, & 1\end{pmatrix}\Delta n_\sigma(t-t_0, s-t_0)]I_\sigma\}^{\alpha\beta}\,, \tag{B·1}$$

where $C_\sigma{}^{(r)}$ and $C_\sigma{}^{(a)}$ are defined in (5·28), $G^r(t)$ and $G^a(t)$ are defined in (5·6), and

$$\Delta n_\sigma(t-t_0, s-t_0)=[n_\sigma(0)-\bar{n}_\sigma]e^{-i\varepsilon(t-s)}e^{-\kappa(t+s-2t_0)} \tag{B·2}$$

with the definitions (4·15b) and (5·10).

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