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General Two-Dimensional Theory of Laminated Cylindrical Shells

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A general two-dimensional theory of laminated cylindrical shells is presented. The theory accounts for a desired degree of approximation of the displacements through the thickness, thus accounting for any discontinuities in their derivatives at the interface of laminae. Geometric nonlinearity in the sense of the von Kármán strains is also included. Navier-type solutions of the linear theory are presented for simply supported boundary conditions.

Introduction

LAMINATED cylindrical shells are often modeled as equivalent single-layer shells using classical, i.e., Love-Kirchhoff shell theory in which straight lines normal to the undeformed middle surface remain straight, inextensible, and normal to the deformed middle surface. Consequently, transverse normal strains are assumed to be zero and transverse shear deformations are neglected.[1-3] The classical theory of shells is expected to yield sufficiently accurate results when the lateral dimension-to-thickness ratio s/h is large, the dynamic excitations are within the low-frequency range, and the material anisotropy is not severe. However, application of such theories to layered anisotropic composite shells could lead to as much as 30% or more errors in deflections, stresses, and natural frequencies.[4-6]

As pointed out by Koiter,[7] refinements to Love's first approximation theory of thin elastic shells are meaningless unless the effects of transverse shear and normal stresses are taken into account in a refined theory. The transverse normal stress is, in general, of order h/a (thickness-to-radius) times a bending stress, whereas the transverse shear stresses obtained from equilibrium conditions are of order h/ℓ (thickness-to-length along the side of the panel) times a bending stress. Therefore, for $a/\ell > 10$, the transverse normal stress is negligible compared to the transverse shear stresses.

The effects of transverse shear and normal stresses in shells were considered by Hildebrand et al.,[8] Luré,[9] and Reissner,[10] among others. Exact solutions of the three-dimensional equations and approximate solutions using a piecewise variation of the displacements through the thickness were presented by Srinivas,[11] where significant discrepancies were found between the exact solutions and the classical shell theory solutions.

The present study deals with a generalization of the shear deformation theories of laminated composite shells. The theory is based on the idea that the thickness approximation of the displacement field can be accomplished via a piecewise approximation through each individual lamina. In particular, the use of polynomial expansion with compact support (i.e., finite-element approximation) through the thickness proves to be convenient. This approach was introduced recently for laminated composite plates by Reddy.[12] It is shown that the theory gives very accurate results for deflections, stresses, and natural frequencies.[13] The theory is extended here to laminated composite cylindrical shells.

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Formulation of the Theory

Displacements and Strains

The displacements (u_x, u_θ, u_z) at a point (x, θ, z) (see Fig. 1) in the laminated shell are assumed to be of the form

$$u_x(x,\theta,z,t) = u(x,\theta,t) + U(x,\theta,z,t)$$

$$u_\theta(x,\theta,z,t) = v(x,\theta,t) + V(x,\theta,z,t)$$

$$u_z(x,\theta,z,t) = w(x,\theta,t) + W(x,\theta,z,t) \tag{1}$$

where (u,v,w) are the displacements of a point $(x,\theta,0)$ on the reference surface of the shell at time t, and U, V, and W are yet arbitrary functions that vanish on the reference surface as

$$U(x,\theta,0) = V(x,\theta,0) = W(x,\theta,0) = 0 \tag{2}$$

In developing the governing equations, the von Kármán type of strains are considered,[8] in which strains are assumed to be small, rotations with respect to the shell reference surface are assumed to be moderate, and rotations about normals to the shell reference surface are considered negligible. The nonlinear

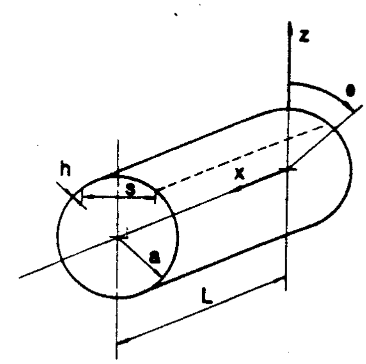

Fig. 1 Shell geometry and coordinate system.

strain-displacement equations in an orthogonal Cartesian coordinate system become

$$e_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\beta_x^2, \quad \beta_x = -\frac{\partial u_z}{\partial x}$$

$$e_{\theta\theta} = \frac{1}{(a+z)}\left(\frac{\partial u_\theta}{\partial \theta} + u_z\right) + \frac{1}{2}\beta_\theta^2$$

$$\beta_\theta = -\frac{1}{(a+z)}\frac{\partial u_z}{\partial \theta}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{x\theta} = \frac{1}{(a+z)}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x} + \beta_x\beta_\theta$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$\gamma_{\theta z} = \frac{1}{(a+z)}\left(\frac{\partial u_z}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial z} \tag{3}$$

where a is the radius of curvature of the shell. Introducing Donnell's approximation,[9] i.e., $z \ll a$, strains $e_{\theta\theta}$, $\gamma_{x\theta}$, and $\gamma_{\theta z}$ can be simplified as

$$e_{\theta\theta} = \frac{1}{a}\left(\frac{\partial u_\theta}{\partial \theta} + u_z\right) + \frac{1}{2}\beta_\theta^2, \quad \beta_\theta = -\frac{1}{a}\frac{\partial u_z}{\partial \theta}$$

$$\gamma_{x\theta} = \frac{1}{a}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x} + \beta_x\beta_\theta$$

$$\gamma_{\theta z} = \frac{1}{a}\left(\frac{\partial u_z}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial z} \tag{4}$$

Substituting for u_x, u_θ, and u_z from Eq. (1) into Eqs. (3) and (4), we obtain

$$e_{xx} = \frac{\partial u}{\partial x} + \frac{\partial U}{\partial z} + \frac{1}{2}\beta_x^2$$

$$e_{\theta\theta} = \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + \frac{\partial V}{\partial \theta} + w + W\right) + \frac{1}{2}\beta_\theta^2$$

$$e_{zz} = \frac{\partial W}{\partial z}$$

$$\gamma_{x\theta} = \frac{\partial v}{\partial x} + \frac{\partial V}{\partial x} + \frac{1}{a}\left(\frac{\partial u}{\partial \theta} + \frac{\partial U}{\partial \theta}\right) + \beta_x\beta_\theta$$

$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial W}{\partial x}$$

$$\gamma_{\theta z} = \frac{1}{a}\left(\frac{\partial w}{\partial \theta} + \frac{\partial W}{\partial \theta} - v - V\right) + \frac{\partial V}{\partial z}$$

$$\beta_x = -\left(\frac{\partial w}{\partial x} + \frac{\partial W}{\partial x}\right)$$

$$\beta_\theta = -\frac{1}{a}\left(\frac{\partial w}{\partial \theta} + \frac{\partial W}{\partial \theta}\right) \tag{5}$$

Variational Formulation

The Hamilton variational principle is used to derive the equations of motion of a cylindrical laminate composed of N constant-thickness orthotropic lamina, whose principal material coordinates are arbitrarily oriented with respect to the laminate coordinates. The principle can be stated, in the absence of body forces and specified tractions, as[16]

$$0 = \int_0^T \Big[\int_V (\sigma_x\delta e_{xx} + \sigma_\theta\delta e_{\theta\theta} + \sigma_z\delta e_{zz} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{\theta z}\delta\gamma_{\theta z} + \sigma_{x\theta}\delta\gamma_{x\theta})\mathrm{d}V - \int_\Omega q\,\delta u_z\,\mathrm{d}\Omega - \int_V \rho(\dot u_x\delta\dot u_x + \dot u_\theta\delta\dot u_\theta + \dot u_z\delta\dot u_z)\mathrm{d}V\Big]\mathrm{d}t \tag{6}$$

where σ_x, σ_θ, σ_z, σ_{xz}, $\sigma_{\theta z}$, etc., are the stresses, q the distributed transverse load, ρ the density, V the total volume of the laminate, Ω the reference surface of the laminate (assumed to be the middle surface of the shell), $(\dot{\ })$ the differentiation with respect to time, and δ the variational symbol.

Substituting the strain-displacements relations [Eq. (5)] into Eq. (6), we obtain

$$\begin{aligned}0 = \int_0^T\Bigg[\int_{-h/2}^{h/2}\int_\Omega\Bigg\{&\sigma_x\left(\frac{\partial\delta u}{\partial x} + \frac{\partial\delta U}{\partial x} + \beta_x\delta\beta_x\right) + \frac{\sigma_\theta}{a}\left(\frac{\partial\delta v}{\partial\theta} + \frac{\partial\delta V}{\partial\theta} + \delta w + \delta W + \beta_\theta\delta\beta_\theta\right)\\ &+ \sigma_z\frac{\partial\delta W}{\partial z} + \sigma_{xz}\left(\frac{\partial\delta w}{\partial x} + \frac{\partial\delta U}{\partial z} + \frac{\partial\delta W}{\partial x}\right)\\ &+ \frac{\sigma_{\theta z}}{a}\left(\frac{\partial\delta w}{\partial\theta} - \delta v + \frac{\partial\delta W}{\partial\theta} - \delta V + a\frac{\partial\delta V}{\partial z}\right)\\ &+ \frac{\sigma_{x\theta}}{a}\left(a\frac{\partial\delta v}{\partial x} + \frac{\partial\delta u}{\partial\theta} + a\frac{\partial\delta V}{\partial x} + \frac{\partial\delta U}{\partial\theta} + a\beta_x\delta\beta_\theta + a\beta_\theta\delta\beta_x\right)\\ &-\rho\Big[(\dot u + \dot U)\delta(\dot u + \dot U) + (\dot v + \dot V)\delta(\dot v + \dot V)\\ &+ (\dot w + \dot W)\delta(\dot w + \dot W)\Big]\Bigg\}\mathrm{d}A\,\mathrm{d}z - \int_\Omega q\,\delta(w + W)\mathrm{d}A\Bigg]\mathrm{d}t\end{aligned} \tag{7}$$

where the following additional approximation, consistent with the Donnel approximation, is used:

$$\int_V f(x,\theta,z)\mathrm{d}z = \int_{r_i}^{r_o}\int_\Omega f r\,\mathrm{d}r\,\mathrm{d}\Omega = \int_{-h/2}^{h/2}\int_\Omega f\cdot\left(1 + \frac{z}{a}\right)\mathrm{d}z\,\mathrm{d}A \approx \int_{-h/2}^{h/2}\int_\Omega f\cdot\mathrm{d}z\,\mathrm{d}A \quad \text{for } z \ll a \tag{8}$$

where r_i and r_o denote the inner and outer radius, respectively, of the cylindrical shell and z is a coordinate measured along the normal to the shell surface with origin at the reference surface.

Approximation through Thickness

In order to reduce the three-dimensional theory to a two-dimensional one, we use a Kantorovich-type approximation,[12,16] where the functions U, V, and W are approximated by

$$U(x,\theta,z,t) = \sum_{j=1}^{n} U^j(x,\theta,t)\phi^j(z)$$

$$V(x,\theta,z,t) = \sum_{j=1}^{n} V^j(x,\theta,t)\phi^j(z)$$

$$W(x,\theta,z,t) = \sum_{j=1}^{m} W^j(x,\theta,t)\psi^j(z) \tag{9}$$

where U^j, V^j, and W^j are undetermined coefficients and $\phi^j(z)$ and $\psi^j(z)$ are any continuous functions that satisfy the conditions [cf Eq. (2)]

$$\phi^j(0) = 0; \quad j = 1,2, \ldots n$$
$$\psi^j(0) = 0; \quad j = 1,2, \ldots m \qquad (10)$$

Governing Equations

To complete the theory, we derive the equations relating the $(3 + 2n + m)$ variables (u, v, w, U^j, V^j, W^j). Substituting Eq. (9) into Eq. (7) and integrating through the thickness, we obtain

$$
\begin{aligned}
0 = \int_0^T \int_\Omega \Bigg\{ \Bigg[& N_x\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + \frac{1}{a}N_\theta\left(\frac{\partial \delta v}{\partial \theta} + \delta w + \frac{1}{a}\frac{\partial w}{\partial \theta}\frac{\partial \delta w}{\partial \theta}\right) + \frac{1}{a}N_{x\theta}\left(a\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial \theta} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial \theta} + \frac{\partial w}{\partial \theta}\frac{\partial \delta w}{\partial x}\right) + Q_{xz}\frac{\partial \delta w}{\partial x} - \frac{1}{a}Q_{\theta z}\left(\frac{\partial \delta w}{\partial \theta} + \delta v\right)\Bigg] \\
& + \sum_{j=1}^{n}\left[M_x^j\frac{\partial \delta U^j}{\partial x} + \frac{1}{a}M_\theta^j\frac{\partial \delta V^j}{\partial \theta} + Q_{xz}^j\delta U^j - \frac{1}{a}M_{\theta z}^j\delta V^j + Q_{\theta z}^j\delta V^j + M_{x\theta}^j\frac{\partial \delta V^j}{\partial x} + \frac{1}{a}M_{x\theta}^j\frac{\partial \delta U^j}{\partial \theta}\right] + \sum_{j=1}^{m}\left[\frac{\hat{M}_\theta^j}{a}\delta W^j + \hat{Q}_z^j\delta W^j + \hat{M}_{xz}^j\frac{\partial \delta W^j}{\partial x} + \frac{\hat{M}_{\theta z}^j}{a}\frac{\partial \delta W^j}{\partial \theta}\right] \\
& + \sum_{j=1}^{m}\Bigg[\hat{M}_x^j\left(\frac{\partial w}{\partial x}\frac{\partial \delta W^j}{\partial x} + \frac{\partial \delta w}{\partial x}\frac{\partial W^j}{\partial x}\right) + \frac{\hat{M}_\theta^j}{a^2}\left(\frac{\partial w}{\partial \theta}\frac{\partial \delta W^j}{\partial \theta} + \frac{\partial \delta w}{\partial \theta}\frac{\partial W^j}{\partial \theta}\right) + \frac{\hat{M}_{x\theta}^j}{a}\left(\frac{\partial w}{\partial x}\frac{\partial \delta W^j}{\partial \theta} + \frac{\partial \delta w}{\partial x}\frac{\partial W^j}{\partial \theta} + \frac{\partial \delta w}{\partial \theta}\frac{\partial W^j}{\partial x} + \frac{\partial w}{\partial \theta}\frac{\partial \delta W^j}{\partial x}\right) \\
& + \sum_{j=1}^{m}\sum_{k=1}^{m}\left[L_x^{jk}\frac{\partial W^j}{\partial x}\frac{\partial \delta W^k}{\partial x} + \frac{1}{a^2}L_\theta^{jk}\frac{\partial W^j}{\partial \theta}\frac{\partial \delta W^k}{\partial \theta} + \frac{1}{a}L_{x\theta}^{jk}\frac{\partial W^j}{\partial x}\frac{\partial \delta W^k}{\partial \theta} + \frac{1}{a}L_{x\theta}^{jk}\frac{\partial \delta W^j}{\partial x}\frac{\partial W^k}{\partial \theta}\right] \\
& - \Bigg[I^o(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) + \sum_{j=1}^{n}I^j(\dot{u}\delta\dot{U}^j + \dot{v}\delta\dot{V}^j + \dot{U}^j\delta\dot{u} + \dot{V}^j\delta\dot{v}) + \sum_{j=1}^{m}\bar{I}^j(\dot{w}\delta\dot{W}^j + \dot{W}^j\delta\dot{w}) + \sum_{j=1}^{n}\sum_{k=1}^{n}I^{jk}(\dot{U}^j\delta\dot{U}^k + \dot{V}^j\delta\dot{V}^k) \\
& + \sum_{j=1}^{m}\sum_{k=1}^{m}\bar{I}^{jk}\dot{W}^j\delta\dot{W}^k\Bigg] - q\delta w\Bigg\}\mathrm{d}A\,\mathrm{d}t \qquad (11)
\end{aligned}
$$

Here N_x, N_θ, etc., denote the stress resultants,

$$(N_x, N_\theta, N_{x\theta}, Q_{xz}, Q_{\theta z}) = \int_{-h/2}^{h/2}(\sigma_x, \sigma_\theta, \sigma_{x\theta}, \sigma_{xz}, \sigma_{\theta z})\mathrm{d}z$$

$$(M_x^j, M_\theta^j, M_{x\theta}^j, M_{\theta z}^j) = \int_{-h/2}^{h/2}(\sigma_x, \sigma_\theta, \sigma_{x\theta}, \sigma_{\theta z})\phi^j\mathrm{d}z$$

$$(Q_{xz}^j, Q_{\theta z}^j) = \int_{-h/2}^{h/2}(\sigma_{xz}, \sigma_{\theta z})\frac{\mathrm{d}\phi^j}{\mathrm{d}z}\mathrm{d}z$$

$$(\hat{M}_x^j, \hat{M}_\theta^j, \hat{M}_{x\theta}^j, \hat{M}_{xz}^j, \hat{M}_{\theta z}^j) = \int_{-h/2}^{h/2}(\sigma_x, \sigma_\theta, \sigma_{x\theta}, \sigma_{xz}, \sigma_{\theta z})\psi^j\mathrm{d}z$$

$$\hat{Q}_z^j = \int_{-h/2}^{h/2}\sigma_z\frac{\mathrm{d}\psi^j}{\mathrm{d}z}\mathrm{d}z$$

$$(L_x^{jk}, L_\theta^{jk}, L_{x\theta}^{jk}) = \int_{-h/2}^{h/2}(\sigma_x, \sigma_\theta, \sigma_{x\theta})\psi^j\psi^k\mathrm{d}z$$

and I^o, I^j, and $\bar{I}^j$ are the inertias,

$$I^o = \int_{-h/2}^{h/2}\rho\mathrm{d}z, \quad (I^j, \bar{I}^j) = \int\rho(\phi^j, \psi^j)\mathrm{d}z$$

$$(I^{jk}, \bar{I}^{jk}) = \int\rho(\phi^j\phi^k, \psi^j\psi^k)\mathrm{d}z \qquad (12)$$

Equations of Motion

The Euler-Lagrange equations of the theory are obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of $\delta u, \delta v, \delta w, \delta U^j, \delta V^j$, and δW^j as

$$N_{x,x} + \frac{1}{a}N_{x\theta,\theta} = I^o\ddot{u} + \sum_{j=1}^{n}I^j\ddot{U}^j \qquad (13a)$$

$$\frac{1}{a}N_{\theta,\theta} + \frac{1}{a}Q_{\theta z} + N_{x\theta,x} = I^o\ddot{v} + \sum_{j=1}^{n}I^j\ddot{V}^j \qquad (13b)$$

$$
\begin{aligned}
& -\frac{1}{a}N_\theta + Q_{xz,x} + \frac{1}{a}Q_{\theta z,\theta} + \left(N_x\frac{\partial w}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}N_\theta\frac{\partial w}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{a}N_{x\theta}\frac{\partial w}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}N_{x\theta}\frac{\partial w}{\partial \theta}\right)_{,x} \\
& + \sum_{j=1}^{m}\left[\left(\hat{M}_x^j\frac{\partial W^j}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}\hat{M}_\theta^j\frac{\partial W^j}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}_{x\theta}^j\frac{\partial W^j}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}_{x\theta}^j\frac{\partial W^j}{\partial \theta}\right)_{,x}\right] \\
& = I\ddot{w} + \sum_{j=1}^{m}\ddot{W}^j\bar{I}^j - q \qquad (13c)
\end{aligned}
$$

$$M_{x,x}^j - Q_{xz}^j + \frac{1}{a}M_{x\theta,\theta}^j = I^j\ddot{u} + \sum_{k=1}^{n}I^{jk}\ddot{U}^k \qquad (13d)$$

$$\frac{1}{a}M_{\theta,\theta}^j + \frac{1}{a}M_{\theta z}^j - Q_{\theta z}^j + M_{x\theta,x}^j = I^j\ddot{v} + \sum_{k=1}^{n}I^{jk}\ddot{V}^k \qquad (13e)$$

$$
\begin{aligned}
& -\frac{1}{a}\hat{M}_\theta^j - \hat{Q}_z^j + \hat{M}_{xz,x}^j + \frac{1}{a}\hat{M}_{\theta z,\theta}^j + \left(\hat{M}_x^j\frac{\partial w}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}\hat{M}_\theta^j\frac{\partial w}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}_{x\theta}^j\frac{\partial w}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}_{x\theta}^j\frac{\partial w}{\partial \theta}\right)_{,x} \\
& + \sum_{k=1}^{m}\left[\left(L_x^{jk}\frac{\partial W^k}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}L_\theta^{jk}\frac{\partial W^k}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{a}L_{x\theta}^{jk}\frac{\partial W^k}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}L_{x\theta}^{jk}\frac{\partial W^k}{\partial \theta}\right)_{,x}\right] = \bar{I}^j\ddot{w} + \sum_{k=1}^{m}\bar{I}^{jk}\ddot{W}^k \qquad (13f)
\end{aligned}
$$

where underscored terms denote the nonlinear terms due to the von Kármán strains.

Boundary Conditions

The virtual work principle gives the following geometric and force boundary conditions for the theory:

Geometric (essential)	Force (natural)
u	$aN_xn_x + N_{x\theta}n_\theta = 0$
v	$aN_{x\theta}n_x + N_\theta n_\theta = 0$
w	$aQ_{xz}n_x + Q_{\theta z}n_\theta = 0$
U^j	$aM_x^jn_x + M_{x\theta}^jn_\theta = 0$
V^j	$aM_{x\theta}^jn_x + M_\theta^jn_\theta = 0$
W^j	$a\hat{M}_{xz}^jn_x + \hat{M}_{\theta z}^jn_\theta = 0$ (14)

where (n_x, n_θ) denote the direction cosines of a unit normal to the boundary of the reference surface Ω.

Further Approximations

The theory can be easily simplified for linear behavior and/or zero normal strain ($e_{zz} = 0$). The term $(1/a)Q_{\theta z}$ in Eq. (13) is neglected in Donnell's quasishallow shell equations[14,15] and it can be neglected here. To be consistent, the term $(1/a)\hat{M}^j_{\theta z}$ should also be neglected simultaneously in this theory.

Consistent with the assumptions made in the derivation of the kinematic equations for the intermediate class of deformations, we can assume that the transverse normal strain is small and neglect the products of the derivatives of the interface transverse displacements,

$$\frac{\partial W^j}{\partial \alpha}\frac{\partial W^j}{\partial \beta} \approx 0 \quad \text{with } \alpha, \beta = x, \theta \tag{15}$$

In this form, we keep a nonlinear coupling between the transverse deflection of the middle surface (w) and the transverse deflections of the interfaces. All remain unchanged but Eq. (13f), which reduces to

$$-\frac{1}{a}\hat{M}^j_\theta - \hat{Q}^j_z + \hat{M}^j_{xz,x} + \frac{1}{a}\hat{M}^j_{\theta z,\theta} + \left(\hat{M}^j_x \frac{\partial w}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}\hat{M}^j_\theta \frac{\partial w}{\partial \theta}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}^j_{x\theta}\frac{\partial w}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}\hat{M}^j_{x\theta}\frac{\partial w}{\partial \theta}\right)_{,x} = \hat{I}^j\ddot{w} + \sum_{k=1}^{m}\hat{I}^{jk}\ddot{W}^k \tag{16}$$

In addition, we can assume that the normal strains in the transverse direction $\partial W^j/\partial x$ are very small and neglect the products $(\partial w/\partial\alpha)(\partial W^j/\partial\beta)$. In this case, the third and sixth of Eqs. (13) reduce to

$$-\frac{1}{a}N_\theta + Q_{xz,x} + \frac{1}{a}Q_{\theta z,\theta} + \left(N_x\frac{\partial w}{\partial x}\right)_{,x} + \left(\frac{1}{a^2}N_\theta\frac{\partial w}{\partial\theta}\right)_{,\theta} + \left(\frac{1}{a}N_{x\theta}\frac{\partial w}{\partial x}\right)_{,\theta} + \left(\frac{1}{a}N_{x\theta}\frac{\partial w}{\partial\theta}\right)_{,x} = I^0\ddot{w} + \sum_{j=1}^{m}\ddot{W}^j\hat{I}^j - q$$

$$-\frac{1}{a}\hat{M}^j_\theta - \hat{Q}^j_z + \hat{M}^j_{xz,x} + \frac{1}{a}\hat{M}^j_{\theta z,\theta} = \hat{I}^j\ddot{w} + \sum_{k=1}^{m}\hat{I}^{jk}\ddot{W}^k \tag{17}$$

Obviously, there is a range of applicability for each of the cases discussed above.

Constitutive Equations

The constitutive equations of an arbitrarily oriented, orthotropic laminae in the laminate coordinate system are

$$\begin{Bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_z\\ \sigma_{xz}\\ \sigma_{\theta z}\\ \sigma_{x\theta}\end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{55} & C_{45} & 0\\ 0 & 0 & 0 & C_{45} & C_{44} & 0\\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}\end{bmatrix} \left[\begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{1}{a}\dfrac{\partial v}{\partial\theta} - \dfrac{w}{a} + \dfrac{1}{2}\left(\dfrac{1}{a}\dfrac{\partial w}{\partial\theta}\right)^2\\ 0\\ \dfrac{\partial w}{\partial x}\\ \dfrac{1}{a}\dfrac{\partial w}{\partial\theta} - \dfrac{v}{a}\\ \dfrac{\partial v}{\partial x} + \dfrac{1}{a}\dfrac{\partial u}{\partial\theta} + \dfrac{\partial w}{\partial x}\dfrac{1}{a}\dfrac{\partial w}{\partial\theta}\end{Bmatrix} + \sum_{j=1}^{n}\begin{Bmatrix}\dfrac{\partial U^j}{\partial x}\phi^j\\ \dfrac{1}{a}\dfrac{\partial V^j}{\partial\theta}\phi^j\\ 0\\ U^j\dfrac{d\phi^j}{dz}\\ V^j\left(\dfrac{d\phi^j}{dz} - \dfrac{1}{a}\phi^j\right)\\ \left(\dfrac{1}{a}\dfrac{\partial U^j}{\partial\theta} + \dfrac{\partial V^j}{\partial x}\right)\phi^j\end{Bmatrix} + \sum_{j=1}^{m}\begin{Bmatrix}\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial x}\psi^j\\ \dfrac{1}{a}\left(W^j + \dfrac{1}{a}\dfrac{\partial w}{\partial\theta}\dfrac{\partial W^j}{\partial\theta}\right)\psi^j\\ W^j\dfrac{d\psi^j}{dz}\\ \dfrac{\partial W^j}{\partial x}\psi^j\\ \dfrac{1}{a}\dfrac{\partial W^j}{\partial\theta}\psi^j\\ \dfrac{1}{a}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial\theta} + \dfrac{\partial w}{\partial\theta}\dfrac{\partial W^j}{\partial x}\right)\psi^j\end{Bmatrix}\right] \tag{18}$$

where C_{ij} denote the elastic constants. Here the nonlinear strains used are those consistent with an intermediate class of deformations and correspond with the simplifications made to arrive at Eq. (16).

Substitution of Eqs. (18) into Eq. (12) gives the following laminate constitutive equations:

$$
\begin{Bmatrix} N_x \\ N_\theta \\ Q_{xz} \\ Q_{\theta z} \\ N_{x\theta} \end{Bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & A_{16} \\
A_{12} & A_{22} & 0 & 0 & A_{26} \\
0 & 0 & A_{55} & A_{54} & 0 \\
0 & 0 & A_{45} & A_{44} & 0 \\
A_{16} & A_{26} & 0 & 0 & A_{66}
\end{bmatrix}
\begin{Bmatrix}
\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\
\dfrac{1}{a}\dfrac{\partial v}{\partial \theta} + \dfrac{w}{a} + \dfrac{1}{2}\left(\dfrac{1}{a}\dfrac{\partial w}{\partial \theta}\right)^2 \\
\dfrac{\partial w}{\partial x} \\
\dfrac{1}{a}\dfrac{\partial w}{\partial \theta} - \dfrac{v}{a} \\
\dfrac{1}{a}\dfrac{\partial u}{\partial \theta} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{1}{a}\dfrac{\partial w}{\partial \theta}
\end{Bmatrix}
$$

$$
+ \sum_{j=1}^{n}
\begin{bmatrix}
B_{11} & B_{12} & 0 & 0 & B_{16} \\
B_{12} & B_{22} & 0 & 0 & B_{26} \\
0 & 0 & B_{55} & B_{54} & 0 \\
0 & 0 & B_{45} & B_{44} & 0 \\
B_{16} & B_{26} & 0 & 0 & B_{66}
\end{bmatrix}^{(j)}
\begin{Bmatrix}
\dfrac{\partial U^j}{\partial x} \\
\dfrac{1}{a}\dfrac{\partial V^j}{\partial \theta} \\
U^j \\
V^j \\
\dfrac{1}{a}\dfrac{\partial U^j}{\partial \theta} + \dfrac{\partial V^j}{\partial x}
\end{Bmatrix}
$$

$$
+ \sum_{j=1}^{m}
\begin{bmatrix}
\hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13} & 0 & 0 & \hat{B}_{16} \\
\hat{B}_{12} & \hat{B}_{22} & \hat{B}_{23} & 0 & 0 & \hat{B}_{26} \\
0 & 0 & 0 & \hat{B}_{55} & \hat{B}_{54} & 0 \\
0 & 0 & 0 & \hat{B}_{45} & \hat{B}_{44} & 0 \\
\hat{B}_{16} & \hat{B}_{26} & \hat{B}_{63} & 0 & 0 & \hat{B}_{66}
\end{bmatrix}^{(j)}
\begin{Bmatrix}
\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial x} \\
\dfrac{1}{a}\left(W^j + \dfrac{1}{a}\dfrac{\partial w}{\partial \theta}\dfrac{\partial W^j}{\partial \theta}\right) \\
W^j \\
\dfrac{\partial W^j}{\partial x} \\
\dfrac{1}{a}\dfrac{\partial W^j}{\partial \theta} \\
\dfrac{1}{a}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial \theta} + \dfrac{\partial w}{\partial \theta}\dfrac{\partial W^j}{\partial x}\right)
\end{Bmatrix}
\tag{19}
$$

$$
\begin{Bmatrix} M_x^j \\ M_\theta^j \\ Q_{xz}^j \\ Q_{\theta z}^j \\ M_{x\theta}^j \end{Bmatrix}
=
\begin{bmatrix}
F_{11} & F_{12} & 0 & 0 & F_{16} \\
F_{12} & F_{22} & 0 & 0 & F_{26} \\
0 & 0 & F_{55} & F_{54} & 0 \\
0 & 0 & F_{45} & F_{44} & 0 \\
F_{16} & F_{26} & 0 & 0 & F_{66}
\end{bmatrix}^{(j)}
\begin{Bmatrix}
\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\
\dfrac{1}{a}\left(\dfrac{\partial v}{\partial \theta} + w + \dfrac{1}{2}\dfrac{\partial w}{\partial \theta}\right) \\
\dfrac{\partial w}{\partial x} \\
\dfrac{1}{a}\left(\dfrac{\partial w}{\partial \theta} - v\right) \\
\dfrac{1}{a}\left(\dfrac{\partial u}{\partial \theta} + a\dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial \theta}\right)
\end{Bmatrix}
$$

$$
+\sum_{k=1}^{n}\begin{bmatrix} D_{11} & D_{12} & 0 & 0 & D_{16} \\ D_{12} & D_{22} & 0 & 0 & D_{26} \\ 0 & 0 & D_{55} & D_{54} & 0 \\ 0 & 0 & D_{45} & D_{44} & 0 \\ D_{16} & D_{26} & 0 & 0 & D_{66} \end{bmatrix}^{(j,k)} \begin{Bmatrix} \dfrac{\partial U^j}{\partial x} \\ \dfrac{1}{a}\dfrac{\partial V^j}{\partial \theta} \\ U^j \\ V^j \\ \dfrac{1}{a}\dfrac{\partial U^j}{\partial \theta}+\dfrac{\partial V^j}{\partial x} \end{Bmatrix}
$$

$$
+\sum_{k=1}^{m}\begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} & \hat{D}_{13} & 0 & 0 & \hat{D}_{16} \\ \hat{D}_{12} & \hat{D}_{22} & \hat{D}_{23} & 0 & 0 & \hat{D}_{26} \\ 0 & 0 & 0 & \hat{D}_{55} & \hat{D}_{54} & 0 \\ 0 & 0 & 0 & \hat{D}_{45} & \hat{D}_{44} & 0 \\ \hat{D}_{16} & \hat{D}_{26} & \hat{D}_{63} & 0 & 0 & \hat{D}_{66} \end{bmatrix}^{(j,k)} \begin{Bmatrix} \dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial x} \\ \dfrac{1}{a}\left(W^j+\dfrac{1}{a}\dfrac{\partial w}{\partial \theta}\dfrac{\partial W^j}{\partial \theta}\right) \\ W^j \\ \dfrac{\partial W^j}{\partial x} \\ \dfrac{1}{a}\dfrac{\partial W^j}{\partial \theta} \\ \dfrac{1}{a}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial \theta}+\dfrac{\partial w}{\partial \theta}\dfrac{\partial W^j}{\partial x}\right) \end{Bmatrix}
\tag{20}
$$

$$
\begin{Bmatrix} \hat{M}_x \\ \hat{M}_\theta \\ \hat{Q}_z \\ \hat{M}_{xz} \\ \hat{M}_{\theta z} \\ \hat{M}_{x\theta} \end{Bmatrix}^{(j)} = \begin{bmatrix} H_{11} & H_{12} & 0 & 0 & H_{16} \\ H_{12} & H_{22} & 0 & 0 & H_{26} \\ H_{31} & H_{32} & 0 & 0 & H_{36} \\ 0 & 0 & H_{55} & H_{54} & 0 \\ 0 & 0 & H_{45} & H_{44} & 0 \\ H_{16} & H_{26} & 0 & 0 & H_{66} \end{bmatrix}^{(j)} \begin{Bmatrix} \dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{1}{a}\left[\dfrac{\partial v}{\partial \theta}+w+\dfrac{1}{2}\dfrac{1}{a}\left(\dfrac{\partial w}{\partial \theta}\right)^2\right] \\ \dfrac{\partial w}{\partial x} \\ \dfrac{1}{a}\left(\dfrac{\partial w}{\partial \theta}-v\right) \\ \dfrac{1}{a}\left(\dfrac{\partial u}{\partial \theta}+a\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial \theta}\right) \end{Bmatrix}
$$

$$
+\sum_{k=1}^{n}\begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\ Q_{12} & Q_{22} & 0 & 0 & Q_{26} \\ Q_{31} & Q_{32} & 0 & 0 & Q_{36} \\ 0 & 0 & Q_{55} & Q_{54} & 0 \\ 0 & 0 & Q_{45} & Q_{44} & 0 \\ Q_{16} & Q_{26} & 0 & 0 & Q_{66} \end{bmatrix}^{(j,k)} \begin{Bmatrix} \dfrac{\partial U^j}{\partial x} \\ \dfrac{1}{a}\dfrac{\partial V^j}{\partial \theta} \\ U^j \\ V^j \\ \dfrac{1}{a}\dfrac{\partial U^j}{\partial \theta}+\dfrac{\partial V^j}{\partial x} \end{Bmatrix}
$$

$$+\sum_{k=1}^{m}\begin{bmatrix}\hat{Q}_{11} & \hat{Q}_{12} & \hat{Q}_{13} & 0 & 0 & \hat{Q}_{16}\\ \hat{Q}_{12} & \hat{Q}_{22} & \hat{Q}_{23} & 0 & 0 & \hat{Q}_{26}\\ 0 & \hat{Q}_{32} & \hat{Q}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \hat{Q}_{55} & \hat{Q}_{54} & 0\\ 0 & 0 & 0 & \hat{Q}_{45} & \hat{Q}_{44} & 0\\ \hat{Q}_{16} & \hat{Q}_{26} & \hat{Q}_{63} & 0 & 0 & \hat{Q}_{66}\end{bmatrix}^{(j,k)}\begin{Bmatrix}\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial x}\\ \dfrac{1}{a}\left(W^j+\dfrac{1}{a}\dfrac{\partial w}{\partial\theta}\dfrac{\partial W^j}{\partial\theta}\right)\\ W^j\\ \dfrac{\partial W^j}{\partial x}\\ \dfrac{1}{a}\dfrac{\partial W^j}{\partial\theta}\\ \dfrac{1}{a}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial W^j}{\partial\theta}+\dfrac{\partial w}{\partial\theta}\dfrac{\partial W^j}{\partial x}\right)\end{Bmatrix} \tag{21}$$

$$M^j_{\theta z}=\bar{H}^j_{45}\left(\frac{\partial w}{\partial x}\right)+\bar{H}^j_{44}\left(\frac{1}{a}\frac{\partial w}{\partial\theta}-\frac{v}{a}\right)+\sum_{k}^{n}\left(\bar{Q}^{jk}_{45}U^k+\bar{Q}^{jk}_{44}V^k\right)+\sum_{k}^{m}\left(\bar{\bar{Q}}^{jk}_{45}\frac{\partial W^k}{\partial x}+\bar{\bar{Q}}^{jk}_{44}\frac{1}{a}\frac{\partial W^k}{\partial\theta}\right) \tag{22}$$

$$\begin{Bmatrix}L_x\\ L_\theta\\ L_{x\theta}\end{Bmatrix}^{(j,k)}=\begin{bmatrix}E_{11} & E_{12} & E_{16}\\ E_{12} & E_{22} & E_{26}\\ E_{16} & E_{26} & E_{66}\end{bmatrix}^{(j,k)}\begin{Bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{1}{a}\left[\dfrac{\partial v}{\partial\theta}+w+\dfrac{1}{2}\dfrac{1}{a}\left(\dfrac{\partial w}{\partial\theta}\right)^2\right]\\ \dfrac{1}{a}\left(\dfrac{\partial u}{\partial\theta}+a\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial\theta}\right)\end{Bmatrix}$$

$$+\sum_{r=1}^{n}\begin{bmatrix}G_{11} & G_{12} & G_{16}\\ G_{12} & G_{22} & G_{26}\\ G_{16} & G_{26} & G_{36}\end{bmatrix}^{(j,k,r)}\begin{Bmatrix}\dfrac{\partial U^r}{\partial x}\\ \dfrac{1}{a}\dfrac{\partial V^r}{\partial\theta}\\ \dfrac{1}{a}\dfrac{\partial U^r}{\partial\theta}+\dfrac{\partial V^r}{\partial x}\end{Bmatrix}$$

$$+\sum_{r=1}^{m}\begin{bmatrix}\hat{G}_{11} & \hat{G}_{12} & \hat{G}_{16} & \hat{G}_{13}\\ \hat{G}_{12} & \hat{G}_{22} & \hat{G}_{26} & \hat{G}_{23}\\ \hat{G}_{61} & \hat{G}_{62} & \hat{G}_{66} & \hat{G}_{63}\end{bmatrix}^{(j,k,r)}\begin{Bmatrix}\dfrac{\partial w}{\partial x}\dfrac{\partial W^r}{\partial x}\\ \dfrac{1}{a}\left(W^r+\dfrac{1}{a}\dfrac{\partial w}{\partial\theta}\dfrac{\partial W^r}{\partial\theta}\right)\\ \dfrac{1}{a}\left(\dfrac{\partial w}{\partial x}\dfrac{\partial W^r}{\partial\theta}+\dfrac{\partial w}{\partial\theta}\dfrac{\partial W^r}{\partial x}\right)\\ W^r\end{Bmatrix} \tag{23}$$

where A, B, F, etc., are the laminate stiffness defined by

$$A_{pq} = \int_{-h/2}^{h/2} C_{pq} \mathrm{d}z \quad (p,q = 1,2,6)$$

$$B^j_{pq} = \int_{-h/2}^{h/2} C_{pq} \phi^j \mathrm{d}z \quad (p,q = 1,2,6)$$

$$(B^j_{55}, B^j_{45}) = \int_{-h/2}^{h/2} (C_{55}, C_{45}) \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \mathrm{d}z$$

$$(B^j_{54}, B^j_{44}) = \int_{-h/2}^{h/2} (C_{54}, C_{44}) \left(\frac{\mathrm{d}\phi^j}{\mathrm{d}z} - \frac{\phi^j}{a} \right) \mathrm{d}z$$

$$\tilde{B}^j_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \mathrm{d}z \quad (p,q = 1,2,4,5,6)$$

$$F^j_{pq} = \int_{-h/2}^{h/2} C_{pq} \phi^j \mathrm{d}z \quad (p,q = 1,2,6)$$

$$F^j_{pq} = \int_{-h/2}^{h/2} C_{pq} \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \mathrm{d}z \quad (p,q = 4,5)$$

$$D^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \phi^j \phi^k \mathrm{d}z \quad (p,q = 1,2,6)$$

$$(D^{jk}_{55}, D^{jk}_{45}) = \int_{-h/2}^{h/2} (C_{55}, C_{45}) \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \frac{\mathrm{d}\phi^k}{\mathrm{d}z} \mathrm{d}z$$

$$(D^{jk}_{54}, D^{jk}_{44}) = \int_{-h/2}^{h/2} (C_{54}, C_{44}) \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \left(\frac{\mathrm{d}\phi^k}{\mathrm{d}z} - \frac{\phi^k}{a} \right) \mathrm{d}z$$

$$\tilde{D}^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \phi^j \psi^k \mathrm{d}z \quad (p,q = 1,2,6)$$

$$(\tilde{D}^{jk}_{13}, \tilde{D}^{jk}_{23}, \tilde{D}^{jk}_{63}) = \int_{-h/2}^{h/2} (C_{13}, C_{23}, C_{63}) \phi^j \frac{\mathrm{d}\psi^k}{\mathrm{d}z} \mathrm{d}z$$

$$\hat{D}^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \frac{\mathrm{d}\phi^j}{\mathrm{d}z} \psi^k \mathrm{d}z \quad (p,q = 4,5)$$

$$H^j_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \mathrm{d}z \quad (p,q = 1,2,4,5,6)$$

$$Q^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \phi^k \mathrm{d}z \quad (p,q = 1,2,6)$$

$$(Q_{55}, Q_{45})^{jk} = \int_{-h/2}^{h/2} (C_{55}, C_{45}) \psi^j \frac{\mathrm{d}\phi^k}{\mathrm{d}z} \mathrm{d}z$$

$$(Q_{54}, Q_{44})^{jk} = \int_{-h/2}^{h/2} (C_{45}, C_{44}) \psi^j \left(\frac{\mathrm{d}\phi^k}{\mathrm{d}z} - \frac{\phi^k}{a} \right) \mathrm{d}z$$

$$\tilde{Q}^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \psi^k \mathrm{d}z \quad (p,q = 1,2,6,4,5)$$

$$(\hat{Q}_{13}, \hat{Q}_{23}, \hat{Q}_{63})^{jk} = \int_{-h/2}^{h/2} (C_{13}, C_{23}, C_{63}) \psi^j \frac{\mathrm{d}\psi^k}{\mathrm{d}z}$$

$$(H_{31}, H_{32}, H_{36})^j = \int_{-h/2}^{h/2} (C_{31}, C_{32}, C_{36}) \frac{\mathrm{d}\psi^j}{\mathrm{d}z} \mathrm{d}z$$

$$(Q_{31}, Q_{32}, Q_{36})^{jk} = \int_{-h/2}^{h/2} (C_{31}, C_{32}, C_{36}) \frac{\mathrm{d}\psi^j}{\mathrm{d}z} \phi^k \mathrm{d}z$$

$$\hat{Q}_{32} = \int_{-h/2}^{h/2} C_{32} \frac{\mathrm{d}\psi^j}{\mathrm{d}z} \psi^k \mathrm{d}z$$

$$\hat{Q}_{33} = \int_{-h/2}^{h/2} C_{33} \frac{\mathrm{d}\psi^j}{\mathrm{d}z} \frac{\mathrm{d}\psi^k}{\mathrm{d}z} \mathrm{d}z$$

$$(\bar{H}^j_{45}, \bar{H}^j_{44}) = \int_{-h/2}^{h/2} (C_{45}, C_{44}) \phi^j \mathrm{d}z$$

$$\bar{Q}^{jk}_{45} = \int_{-h/2}^{h/2} C_{45} \phi^j \frac{\mathrm{d}\phi^k}{\mathrm{d}z} \mathrm{d}z$$

$$\bar{Q}^{jk}_{44} = \int_{-h/2}^{h/2} C_{44} \phi^j \left(\frac{\mathrm{d}\phi^k}{\mathrm{d}z} - \frac{\phi^k}{a} \right) \mathrm{d}z$$

$$(\bar{\bar{Q}}^{jk}_{45}, \bar{\bar{Q}}^{jk}_{44}) = \int_{-h/2}^{h/2} (C_{45}, C_{44}) \phi^j \psi^k \mathrm{d}z$$

$$E^{jk}_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \psi^k \mathrm{d}z \quad (p,q = 1,2,6)$$

$$G^{jkr}_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \psi^k \phi^r \mathrm{d}z \quad (p,q = 1,2,6)$$

$$\hat{G}^{jkr}_{pq} = \int_{-h/2}^{h/2} C_{pq} \psi^j \psi^k \psi^r \mathrm{d}z \quad (p,q = 1,2,6)$$

$$(\hat{G}_{13}, \hat{G}_{23}, \hat{G}_{63})^{jkr} = \int_{-h/2}^{h/2} (C_{13}, C_{23}, C_{63}) \psi^j \psi^k \frac{\mathrm{d}\psi^r}{\mathrm{d}z} \mathrm{d}z \tag{24}$$

Analytical Solution of the Linear Equations

The theory presented so far is general in the sense that the interpolation functions ϕ^j and ψ^j can be chosen arbitrarily as long as they satisfy the conditions in Eq. (10). In order to produce an actual solution, we choose here linear Lagrange polynomials for both ϕ^j and ψ^j. In this particular case, the coefficients U^j, V^j, and W^j are identified as the displacements of each jth interface between layers. In order to be able to obtain an analytical solution and to compare the results with existing solutions of the three-dimensional elasticity theory, we must restrict ourselves to the linear equations obtained by eliminating the underscored terms in Eqs. (13). The solution of equations of even the linear theory is by no means trivial. These equations of motion combined with the constitutive relations are solved exactly for the case of orthotropic, simply supported laminated shells. Using a Navier-type solution method (see Refs. 11 and 16), a set of kinematically admissible solutions is assumed, as follows

$$\begin{aligned}
u(x,\theta,t) &= \sum_m^\infty \sum_n^\infty X_{mn} \cos m\theta \cos \alpha x \, T_{mn}(t) \\
v(x,\theta,t) &= \sum_m^\infty \sum_n^\infty \Gamma_{mn} \sin m\theta \sin \alpha x \, T_{mn}(t) \\
w(x,\theta,t) &= \sum_m^\infty \sum_n^\infty \Pi_{mn} \cos m\theta \sin \alpha x \, T_{mn}(t) \\
U^j(x,\theta,t) &= \sum_m^\infty \sum_n^\infty \gamma^j_{mn} \cos m\theta \cos \alpha x \, T_{mn}(t) \\
V^j(x,\theta,t) &= \sum_m^\infty \sum_n^\infty \Upsilon^j_{mn} \sin m\theta \sin \alpha x \, T_{mn}(t) \\
W^j(x,\theta,t) &= \sum_m^\infty \sum_n^\infty \Omega^j_{mn} \cos m\theta \sin \alpha x \, T_{mn}(t) \\
T_{mn}(t) &= e^{i\omega_{mn}}
\end{aligned} \tag{25}$$

where $\alpha = n\pi/b$ and b is the length of the cylinder.

After substitution into the constitutive equations and equations of motion, we get a system of $3N+3$ equations that relate the $3N+3$ unknowns $\{\xi\} = (X_{mn}, \Gamma_{mn}, \Pi_{mn}, \gamma^j_{mn}, \Upsilon^j_{mn}, \Omega^j_{mn})$, $j = 1, \ldots, N$ as

$$[K]\{\xi\} = \omega^2_{mn}[M]\{\xi\} \tag{26}$$

for each of the modes (m,n). The solution of the eigenvalue problem [Eq. (26)] gives $3N+3$ frequencies for each mode (m,n).

Table 1 Nondimensional frequencies for a three-layer thin laminate

		First frequency		Second frequency		Third frequency	
b/na	m	Exact[a]	GLST[b]	Exact	GLST	Exact	GLST
1	0	0.32461	0.32706	1.8186	1.8074	3.0037	2.9666
	1	0.33631	0.33855	1.7031	1.6939	3.1438	3.1003
	2	0.36737	0.36917	1.4689	1.4629	3.4615	3.4020
	3	0.40447	0.40603	1.2612	1.2571	3.8451	3.7631
	4	0.42507	0.42687	1.1822	1.17873	4.2502	4.1407
2	0	0.28282	0.28354	0.91614	0.91301	1.5441	1.5389
	1	0.30591	0.30649	0.73938	0.73762	1.7807	1.7725
	2	0.30838	0.30897	0.60419	0.60273	2.2071	2.1913
	3	0.21959	0.22066	0.80118	0.79792	2.6748	2.6466
	4	0.20414	0.20527	1.1467	1.1396	3.1559	3.1098
8	0	0.20999	0.20989	0.22904	0.22898	0.51742	0.51708
	1	0.07054	0.07052	0.41957	0.41930	0.84128	0.83941
	2	0.03594	0.03638	0.78928	0.78801	1.3559	1.3476
	3	0.06940	0.06966	1.1956	1.1900	1.9113	1.8887
	4	0.12237	0.12254	1.6058	1.5909	2.4841	2.4354

[a]Exact results from Srinivas.[11] [b]GLST = generalized laminate shell theory (present).

Table 2 Nondimensional frequencies for a three-layer thick laminate

		First frequency		Second frequency		Third frequency	
b/na	m	Exact[a]	GLST[b]	Exact	GLST	Exact	GLST
1	0	0.40438	0.40838	1.6205	1.5064	1.7475	1.9271
	1	0.42140	0.42401	1.5294	1.4333	1.7530	1.9478
	2	0.46495	0.46524	1.3354	1.2710	1.7633	1.9850
	3	0.50904	0.51186	1.1742	1.1233	1.7626	2.0148
	4	0.52631	0.53482	1.1540	1.1003	1.7309	1.9960
2	0	0.31807	0.32042	0.89129	0.87015	1.4316	1.3733
	1	0.35573	0.35501	0.71061	0.69823	1.4242	1.5282
	2	0.33947	0.34297	0.62782	0.61531	1.3608	1.5689
	3	0.28099	0.28233	0.85051	0.84972	1.5134	1.7085
	4	0.33070	0.32851	1.1398	1.1460	1.7260	1.9032
8	0	0.21844	0.21680	0.22955	0.22916	0.54260	0.53637
	1	0.06638	0.06696	0.46207	0.46070	0.86049	0.85054
	2	0.08773	0.08718	0.82383	0.82943	1.3288	1.2941
	3	0.18459	0.18276	1.1547	1.1553	1.6937	1.6713
	4	0.28616	0.28404	1.3692	1.3842	1.9282	1.9686

[a]Exact results from Srinivas.[11] [b]GLST = generalized laminate shell theory (present).

Table 3 Nondimensional frequencies of a two-ply graphite-epoxy cylinder

		First frequency				Second frequency				Third frequency			
		Rotary inertia included		No rotary inertia		Rotary inertia included		No rotary inertia		Rotary inertia included		No rotary inertia	
b/na	m	$e_z \neq 0$	$e_z = 0$	$e_z \neq 0$	$e_z = 0$	$e_z \neq 0$	$e_z = 0$	$e_z \neq 0$	$e_z = 0$	$e_z \neq 0$	$e_z = 0$	$e_z \neq 0$	$e_z = 0$
1	0	0.6370	0.6370	0.6353	0.6353	0.7716	0.7809	0.7725	0.7838	2.2094	2.2102	2.1558	2.1594
	1	0.4153	0.4163	0.4152	0.4170	1.202	1.212	1.195	1.207	2.2208	2.2219	2.1657	2.1698
	2	0.2936	0.2938	0.2937	0.2947	1.818	1.834	1.788	1.804	2.2582	2.2619	2.1978	2.2052
	3	0.2666	0.2660	0.2667	0.2673	2.272	2.268	2.214	2.210	2.5134	2.5444	2.4253	2.4612
	4	0.3109	0.3092	0.3111	0.3115	2.357	2.355	2.287	2.289	3.1302	3.1670	2.9841	3.0258
2	0	0.3185	0.3185	0.3186	0.3186	0.7606	0.7692	0.7610	0.7700	1.1360	1.1360	1.1273	1.1277
	1	0.2133	0.2135	0.2133	0.2136	1.108	1.112	1.104	1.108	1.1678	1.1748	1.1579	1.1667
	2	0.1397	0.1398	0.1397	0.1400	1.211	1.209	1.200	1.199	1.7587	1.7785	1.7314	1.7524
	3	0.1518	0.1510	0.1518	0.1515	1.302	1.300	1.287	1.286	2.4325	2.4602	2.3586	2.3879
	4	0.2313	0.2291	0.2314	0.2303	1.419	1.418	1.399	1.398	3.0777	3.1126	2.9432	2.9816
8	0	0.0797	0.0797	0.0798	0.0977	0.2864	0.2863	0.2863	0.2862	0.7601	0.7684	0.7603	0.7685
	1	0.0439	0.0439	0.0439	0.0439	0.3585	0.3583	0.3582	0.3580	1.0958	1.1078	1.0923	1.1045
	2	0.0454	0.0451	0.0454	0.0451	0.5145	0.5144	0.5134	0.5132	1.7338	1.7530	1.7086	1.7285
	3	0.1147	0.1133	0.1147	0.1136	0.7015	0.7014	0.6984	0.6983	2.4132	2.4404	2.3421	2.3706
	4	0.2133	0.2105	0.2133	0.2115	0.9004	0.9003	0.8937	0.8936	3.0623	3.0968	2.9310	2.9686

[a]Exact results from Srinivas.[11] [b]GLST = generalized laminate shell theory (present).

As an example, a three-ply laminate with orthotropic layers is analyzed. The stiffnesses of the inner layer are assumed to be: $C_{11} = 0.08$, $C_{12} = 0.05$, $C_{13} = 0.07$, $C_{22} = 0.19$, $C_{23} = 0.32$, $C_{33} = 1.0$, $C_{44} = 0.04$, $C_{55} = 0.03$, and $C_{66} = 0.34$; the outer layers are assumed to have stiffnesses 20 times those of the inner layer. The results are presented in terms of a nondimensional parameter λ as

$$\lambda = \omega r_o \left[\frac{\sum_{i=1}^{N} \rho^i (r_{i+1}^2 - r_i^2)}{\sum_{i=1}^{N} C_{33}^i (r_{i+1}^2 - r_i^2)} \right]^{1/2} \tag{27}$$

where r_i is the radius of the ith interface and r_o is the outer radius of the cylinder.

Results for a thin laminate ($r_1 = 0.95 r_o$, $r_2 = 0.955 r_o$, $r_3 = 0.995 r_o$) are presented in Table 1. Similar results for a thick laminate ($r_1 = 0.8 r_o$, $r_2 = 0.82 r_o$, $r_3 = 0.98 r_o$) are presented in Table 2. The exact results using three-dimensional elasticity are taken from Srinivas.[11] In Table 3, results for a two-ply cylindrical shell are presented. The material properties used are those of a graphite-epoxy material ($E_1 = 19.6$ msi, $E_2 = 1.56$ msi, $\nu_{12} = 0.24$, $\nu_{23} = 0.47$, $G_{12} = 0.82$ msi, $G_{23} = 0.523$ msi) and the thickness of each layer is 0.05 r_o. The three lowest frequencies are presented in Table 3 in nondimensional form as before. While neglecting the rotary inertia, the in-plane inertia still needs to be considered for cylindrical shells because the displacements tangential to the reference surface, mainly u_θ, play an important role in the behavior of the shell. This is in contrast to plate theory, where the in-plane inertia are usually neglected along with rotary inertia. Results obtained for zero transverse normal strain are also presented. They were obtained using the reduced stiffness matrix[16] instead of the three-dimensional stiffness matrix. The present results are, in general, in good agreement with those presented by Srinivas.[11]

Conclusions

A general two-dimensional shear deformation theory of laminated cylindrical shells is presented. The theory allows for the inclusion of a desired degree of approximation of the displacements through the thickness. Geometric nonlinearity in the von Kármán sense is also considered. Exact solutions of the linear equations for simply supported cylindrical shells are presented. The results correlate very well with the three-dimensional exact solutions. The validity of Donnell's approximations and the applicability of various simplifications made for the nonlinear equations are to be investigated further. The finite-element models of the theory are to be developed in order to solve cylindrical shells with general boundary conditions, lamination scheme, loading, and geometric nonlinearity.

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