General U(N) raising and lowering operators

Mark D. Gould

Published on: 01 Jan 1981 - Journal of Mathematical Physics (American Institute of Physics)

Topics: Operator theory, Spectral theorem, Mathematical Operators and Ladder operator

AIP | Journal of Mathematical Physics

General U(N) raising and lowering operators

M. D. Gould

Citation: Journal of Mathematical Physics **22**, 267 (1981); doi: 10.1063/1.524898
View online: http://dx.doi.org/10.1063/1.524898
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/22/2?ver=pdfcov
Published by the AIP Publishing

General U(*N*) raising and lowering operators

M. D. Gould [a)]
Department of Mathematical Physics, University of Adelaide, G.P.O. Box 498, Adelaide, South Australia, 5001

(Received 21 September 1979; accepted for publication 10 December 1979)

It is the aim of this paper to obtain the general form of U(n) raising and lowering operators. The raising and lowering operators constructed previously by several authors are then compared. The Hermiticity properties of these operators are also investigated. The methods presented extend, with trivial modifications, to the orthogonal groups.

PACS numbers: 02.20.Qs, 02.20.Rt

1. INTRODUCTION

In the 1960's a great deal of interest was generated in extending the angular momentum techniques of Wigner and Racah to the general unitary and orthogonal groups. This has led to the introduction of the group-theoretical concept of operators that lower or raise the highest weights of representations of a subgroup contained in an irreducible representation of the group. Such operators may be regarded as a generalization of the raising and lowering operators $L_{\pm}$ appearing in the theory of angular momenta.

Such operators were first constructed for the unitary groups by Nagel and Moshinsky[1] who applied them to the analysis of many body problems.[2] Subsequently raising and lowering operators were constructed for the orthogonal groups by Pang and Hecht[3] and Wong.[4] Following the definition of Nagel and Moshinsky[1] the lowering (raising) operators shall be polynomials of the group generators that, when acting on a basis vector of an irreducible representation of the group which is of given weight with respect to the subgroup, lower (raise) the weight. Furthermore, they shall, when acting on a basis vector of highest weight of the subgroup transform it into a basis vector of highest weight of a lowered (raised) irreducible representation of the subgroup.

It is important to note that the raising and lowering operators for a subgroup are essentially only defined by their action on a state of highest weight for the subgroup. We see therefore that such operators are not unique. Hence the raising and lowering operators constructed previously for O(n) and U(n) are only one particular solution to the problem.

Recently Bincer[5] obtained raising and lowering operators for the orthogonal and unitary groups using methods based on the characteristic identities satisfied by the infinitesimal generators of the group.[6] These operators of Bincer appear in a compact product form which is useful for manipulations. In subsequent independent work of the author[7] an alternative set of raising and lowering operators for O(n) and U(n) were constructed using techniques similar in content to Bincer's. Our operators, like those of Bincer, may also be written in a compact product form.

Recent work of the author[8] shows how these techniques may be extended to obtain the matrix elements of the group generators. Central in this approach is the concept of "simultaneous shift operator" which shifts the representation labels of U(n) and each of its canonical subgroups in a certain prescribed way. These operators may therefore be regarded as generalizations of the raising and lowering operators discussed in this paper. A general procedure for constructing raising and lowering operators for a general semi-simple Lie group is discussed in Ref. 7.

It is the aim of the present paper to investigate the connection between the various raising and lowering operators. We shall obtain the general form for a raising (resp. lowering) operator for U(n). It shall be shown that the raising and lowering operators constructed in Refs. 1 and 5 are identical. By contrast the operators constructed in Ref. 7 are shown to be different. The behavior of these raising and lowering operators under Hermitian conjugation is also investigated. It shall be shown that the raising and lowering operators constructed in Ref. 7 are unique with respect to the property of being Hermitian conjugates of one another.

The techniques employed in this paper are similar in content to Bincer's except for our use of the U(n) contragredient identity. This enables raising operators for U(n) (which are absent in the work of Bincer) to be constructed in analogy with the lowering operators.

Although we shall only discuss the unitary group it is clear that the arguments extend to O(n) with little modification. Also it is of interest to extend these results to the noncompact groups O(n,1) and U(n,1). Patera[9] has shown that the Nagel–Moshinsky operators are also a suitable choice for U(n,1) while Wong[10] has shown that his operators for O(n) extend to O(n,1). Wong and Yeh[11] have also recently investigated the extension of Bincer's operators to O(n,1). One may follow through their derivation to conclude that the raising and lowering operators constructed in Ref. 7 also extend to O(n,1) and U(n,1) as does any general raising (resp. lowering) operator for O(n) and U(n) (defined in the sense of Nagel and Moshinsky).

2. THE CHARACTERISTIC IDENTITIES

The generators $a^i_{\ j}$ of the Lie group U(n) may be assembled into a square matrix a which, on an irreducible representation of the group (finite or infinite dimensional) with highest weight $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, satisfies the polynomial identity[6]

$$\prod_{r=1}^{n} (a - \lambda_r - n + r) = 0 .$$

This polynomial identity may be written in a representation

[a)] Present address: School of Physical Sciences, The Flinders University of South Australia, South Australia, 5042.

 0022-2488/81/020267-04$1.00

independent way as

$$\prod_{r=1}^{n}(a-\alpha_r)=0\,, \tag{1}$$

where the operators α_r lie in an algebraic extension of the center of the enveloping algebra.[12] (Note that any symmetric combination of the α_r necessarily lies in the center Z. In particular the coefficients of the identity (1) are central elements.) The eigenvalues of these operators on any representation admitting an infinitesimal character χ_λ (or equivalently on any irreducible representation with highest weight λ) are given by

$$\chi_\lambda(\alpha_r)=\lambda_r+n-r\,.$$

Associated with the matrix a is its "contragredient" $\bar{a}$ with entries given by

$$\bar{a}_j{}^i=-a^i_j\,.$$

The matrix $\bar{a}$ satisfies the polynomial identity

$$\prod_{r=1}^{n}(\bar{a}-\bar{\alpha}_r)=0\,, \tag{2}$$

where the roots $\bar{\alpha}_r$ are related to the α_r by $\bar{\alpha}_r=n-1-\alpha_r$.

By virtue of the identities (1) and (2) one may construct projection operators

$$P[r]=\prod_{l\neq r}\left(\frac{a-\alpha_l}{\alpha_r-\alpha_l}\right),$$

$$\bar{P}[r]=\prod_{l\neq r}\left(\frac{\bar{a}-\bar{\alpha}_l}{\bar{\alpha}_r-\bar{\alpha}_l}\right),$$

which enables arbitrary functions of the matrices a and $\bar{a}$ to be defined by setting

$$P(a)=\sum_{r=1}^{n}P(\alpha_r)P[r]\,,$$

$$P(\bar{a})=\sum_{r=1}^{n}P(\bar{\alpha}_r)\bar{P}[r]\,.$$

The projection operators $P[r]$ and $\bar{P}[r]$ are well defined elements of an extension of the enveloping algebra although they need not be defined on representations where the eigenvalues of some α_r and α_k $(r\neq k)$ coincide. This however cannot occur on finite dimensional representations [nor on unitary representations of the noncompact groups $U(p,q)$] and hence, for the applications we have in mind, the projectors $P[r]$ and $\bar{P}[r]$ are always well defined.

If ψ (resp. $\psi^\dagger$) is a vector (resp. contragredient vector) operator of $U(n)$ then we may resolve ψ and $\psi^\dagger$ into shift vectors[6]

$$\psi=\sum_{r=1}^{n}\psi[r]\,,\quad \psi^\dagger=\sum_{r=1}^{n}\psi^\dagger[r]\,,$$

which alter the $U(n)$ representation labels according to

$$\alpha_k\psi[r]=\psi[r](\alpha_k+\delta_{kr})\,,$$

$$\alpha_k\psi^\dagger[r]=\psi^\dagger[r](\alpha_k-\delta_{kr})\,.$$

(Note that this shift property also extends to infinite dimensional representations.) Such shift operators may be constructed by applying the projectors $P[r]$ and $\bar{P}[r]$:

$$\psi[r]=P[r]\psi=\psi\bar{P}[r]\,,$$

$$\psi^\dagger[r]=\bar{P}[r]\psi^\dagger=\psi^\dagger P[r]=(\psi[r])^\dagger\,.$$

It was shown in Ref. 7 (see also Ref. 13) that if v_0 is an arbitrary maximal weight state of $U(n)$ with highest weight $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ then

$$\begin{aligned}&P[r]^i{}_j v_0=0\,,\quad \text{for } j>r \text{ and } i \text{ arbitrary}\,,\\ &\bar{P}[r]_j{}^i v_0=0\,,\quad \text{for } j<r \text{ and } i \text{ arbitrary}\,.\end{aligned} \tag{3}$$

As a consequence of (3) we see that on the maximal weight vector v_0 the shift vectors $\psi[r]$ and $\psi^\dagger[r]$ must satisfy

$$\begin{aligned}&\psi[r]^i v_0=0\,,\quad \text{for } i<r\,,\\ &\psi^\dagger[r]_i v_0=0\,,\quad \text{for } i>r\,.\end{aligned} \tag{4}$$

Using the definition of vector operator equation (4) then implies that $\psi[r]^r v_0$ and $\psi^\dagger[r]_r v_0$ are maximal weight states of weight $\lambda+\Delta_r$ and $\lambda-\Delta_r$, respectively.

3. GENERAL RAISING AND LOWERING OPERATORS

It is our aim here to determine the general form for $U(n)$ raising and lowering operators and to compare the operators constructed in Refs. 1, 5, and 7.

Throughout we shall let ψ denote the $U(n)$ vector operator with components $\psi^i=a^i{}_{n+1}$ $(i=1,\ldots,n)$ whose contragredient has components $\psi^\dagger{}_i=a^{n+1}{}_i$. The operators $\psi[r]^r$ and $\psi^\dagger[r]_r$ will shift highest weight vectors of $U(n)$ in a finite dimensional irreducible representation of $U(n+1)$. These are the $U(n)$ raising and lowering operators constructed in Ref. 7. For convenience we denote them by $\psi^r{}_n$ and $\psi^{\dagger r}{}_n$, respectively. Finally we denote a maximal weight state of $U(n)$ [i.e., a semi-maximal state of $U(n+1)$] by the pattern $\left|\begin{smallmatrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{smallmatrix}\right\rangle$. Here $\lambda_{i,n+1}$ and $\lambda_{i,n}$, as usual, refer to highest weights of finite dimensional irreducible representations of $U(n+1)$ and $U(n)$, respectively.

Suppose now that $R^r{}_n$ and $L^r{}_n$ are arbitrary raising and lowering operators of $U(n)$ effecting the shifts

$$\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle\rightarrow\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}+\delta_{ir}\end{matrix}\right\rangle,$$

$$\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle\rightarrow\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}-\delta_{ir}\end{matrix}\right\rangle,$$

respectively. According to Nagel and Moshinsky such operators are of the form

$$R^r{}_n=h(a)^r{}_j a^j{}_{n+1}\,, \tag{5}$$

$$L^r{}_n=a^{n+1}{}_j h(a)^j{}_r\,, \tag{6}$$

for a suitable polynomial $h(x)$.

Resolving $\psi^\dagger{}_i=a^{n+1}{}_i$ into its distinct shift components allows us to write Eq. (6) in the form

$$L^r{}_n=\sum_{l=1}^{n}\psi^\dagger[l]_r h(\alpha_l)\,. \tag{7}$$

Now acting on the state $\left|\begin{smallmatrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{smallmatrix}\right\rangle$ the operators $\psi^\dagger[l]_r$, $l<r$, vanish by virtue of Eq. (4). Hence, acting on the state $\left|\begin{smallmatrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{smallmatrix}\right\rangle$, Eq. (7) reduces to

$$L^r{}_n\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle=\sum_{l\geqslant r}\psi^\dagger[l]_r h(\alpha_l)\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle. \tag{8}$$

However each operator $\psi^\dagger[l]_r$ (for $l>r$) effects the shifts $\lambda_{i,n}\rightarrow\lambda_{i,n}-\delta_{il}$. Hence in order to obtain the required shifts we must have $h(\alpha_l) = 0$ for $l > r$. Hence $h(x)$ must be divisible by the polynominal $\Pi_{l>r}(x - \alpha_l)$. In the limiting case where $h(x) = \Pi_{l>r}(x - \alpha_l)$ we obtain the operators constructed by Bincer.[5] We have shown then that a general lowering operator is of the form

$$\psi^r{}_n\beta + \sum_{l<r}\psi^\dagger[l]_r\beta_l,$$

where β and β_l are invariant multiplies of the identity [or equivalently of the form $a^{n+1}{}_j h(a)^j{}_r$ where $h(x)$ is divisible by $\Pi_{l>r}(x - \alpha_l)$].

Note also that the Nagel–Moshinsky operators $L^r{}_n$ are of the form $a^{n+1}{}_j h(a)^j{}_r$ for a suitable polynomial $h(x)$ which, according to the remarks above, is divisible by the polynomial $\Pi_{l>r}(x - \alpha_l)$. However it is well known that the Nagel–Moshinsky operators $L^r{}_n$ are homogeneous of degree $n - r + 1$ in the group generators from which it follows that $h(x)$ is of degree $n - r$ which is precisely the degree of $\Pi_{l>r}(x - \alpha_l)$. Accordingly we must have

$$h(x) = c\prod_{l>r}(x - \alpha_l),$$

where c is a constant dependent on the roots α_l.

From Eq. (8) acting on the state $\left|\begin{smallmatrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{smallmatrix}\right\rangle$ the Nagel–Moshinsky operators $L^r{}_n$ reduce to

$$L^r{}_n\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle = \psi^{\dagger r}{}_n c\prod_{l>r}(\alpha_r - \alpha_l)\left|\begin{matrix}\lambda_{i,n+1}\\ \lambda_{i,n}\end{matrix}\right\rangle.$$

By comparing the normalization of our lowering operators $\psi^{\dagger r}{}_n$ with the normalization of the $L^r{}_n$ the constant c may be determined. By this means we obtain

$$c = \prod_{l>r}\left(\frac{\alpha_r - \alpha_l - 1}{\alpha_r - \alpha_l}\right).$$

Hence the Nagel–Moshinsky lowering operators may be written

$$L^r{}_n = a^{n+1}{}_j g(a)^j{}_r, \tag{9}$$

where $g(x)$ is the polynomial

$$g(x) = \prod_{l>r}(x - \alpha_l)\left(\frac{\alpha_r - \alpha_l - 1}{\alpha_r - \alpha_l}\right).$$

These are essentially the operators constructed by Bincer[5] (up to multiplication by an invariant multiple of the identity).

Equation (9) is just one representation of the Nagel–Moshinsky operators. Expanding the polynomial $g(x)$ into powers of x we may write

$$g(x) = \sum_{k=0}^{n-r} X^{n-r-k} S_k \prod_{l>r}\left(\frac{\alpha_r - \alpha_l - 1}{\alpha_r - \alpha_l}\right),$$

where S_k is a polynomial in the $\alpha_l (l > r)$. By replacing the α_r with Lie algebra elements $\epsilon_r = a^r{}_r + n - r$ we obtain the Nagel–Moshinsky operators in their original form.[1]

Using Eq. (4) one may show, by the same techniques, that a general raising operator [see Eq. (5)] is of the form

$$R^r{}_n = h(a)^r{}_j a^j{}_{n+1},$$

where $h(x)$ is divisible by the polynomial $\Pi_{l<r}(x - \alpha_l)$. In terms of the matrix $\bar{a}$ a general raising operator may more usefully be written

$$R^r{}_n = a^j{}_{n+1}\, g(\bar{a})_j{}^r,$$

where $g(x)$ is necessarily divisible by $\Pi_{l<r}(x - \bar{\alpha}_l)$. The limiting case where $g(x) = \Pi_{l<r}(x - \bar{\alpha}_l)$ gives the raising analogue of Bincer's lowering operators (although these operators do not appear in the work of Bincer). The raising operators of Nagel and Moshinsky may be writen

$$\begin{aligned} R^r{}_n &= \Pi_{l<r}\left(\frac{\alpha_r - \alpha_l - 1}{\alpha_r - \alpha_l}\right)(a - \alpha_l)^r{}_i a^i{}_{n+1} \\ &= a^i{}_{n+1}\prod_{l<r}(\bar{a} - \bar{\alpha}_l)_i{}^r\left(\frac{\bar{\alpha}_r - \bar{\alpha}_l - 1}{\bar{\alpha}_r - \bar{\alpha}_l}\right). \end{aligned} \tag{10}$$

4. HERMITICITY PROPERTIES

It was shown by Nagel and Moshinsky that their raising and lowering operators are not Hermitian conjugates [as one may show by comparing Eqs. (9) and (10)]. It is natural then to determine under what conditions the Hermitian conjugate of a raising operator is a lowering operator (and vice versa). We answer this by showing that our raising and lowering operators are unique with respect to the property of being Hermitian conjugates.

In the last section it was shown that a general lowering operator may be written in the form

$$L^r{}_n = \psi^{\dagger r}{}_n\beta + \sum_{l<r}\psi^\dagger[l]_r\beta_l, \tag{11}$$

where β and β_l are constants dependent on the roots α_l. Similarly a general raising operator may be written in the form

$$R^r{}_n = \gamma\psi^r{}_n + \sum_{l>r}\gamma_l\psi[l]^r. \tag{12}$$

Comparing Eqs. (11) and (12) we see that $(L^r{}_n)^\dagger$ cannot possibly be a raising operator unless $\beta_l = 0$ for $l < r$; i.e., unless $L^r{}_n$ is an invariant multiple of $\psi^{\dagger r}{}_n$. (An analogous statement holds for $R^r{}_n$.) Accordingly we see that the raising and lowering operators $\psi^r{}_n$ and $\psi^{\dagger r}{}_n$ (constructed in Ref. 7) are unique with respect to the property of being Hermitian conjugates.

ACKNOWLEDGMENTS

It is my pleasure to thank Professor H.S. Green for his interest and encouragement. The author also acknowledges the financial support of a Commonwealth Post-Graduate Research Award.

[1] J.G. Nagel and M. Moshinsky, J. Math. Phys. **6**, 682 (1965); Rev. Mex. Fis. **14**, 29 (1965).

[2] M. Moshinsky, *Many-body problems and Other Selected Topics in Theoretical Physics* (Gordon and Breach, New York, 1966).

[3] J.C. Pang and K.T. Hecht, J. Math. Phys. **8**, 1233 (1967).

[4] M.K.F. Wong, J. Math. Phys. **8**, 1899 (1967).

[5] A.M. Bincer, J. Math. Phys. **18**, 1870 (1977); **19**, 1173 (1978); **19**, 1179 (1978).

[6] H.S. Green, J. Math. Phys. **12**, 2106 (1971); A.J. Bracken and H.S. Green, J. Math. Phys. **12**, 2099 (1971).

[7] M.D. Gould, "An infinitesimal approach to semi-simple Lie Groups and raising and lowering operators of O(N) and U(N)," to appear in J. Math. Phys.

[8]M.D. Gould, "On the matrix elements of the U(N) generators," submitted for publication in J. Math. Phys.
[9]J. Patera, J. Math. Phys. **14**, 279 (1973).
[10]M.K.F. Wong, J. Math. Phys. **15**, 25 (1974); M.K.F. Wong and H.Y. Yeh, J. Math. Phys. **16**, 800 (1975).
[11]M.K.F. Wong and Hsin-Yang Yeh, J. Math. Phys. **4**, 629 (1979).
[12]D.M. O'Brien, A. Cant, and A.L. Carey, Ann. Inst. Henri Poincaré A **26**, 405 (1977); M.D. Gould, Ann. Inst. Henri Poincaré A **24**, 51 (1978).
[13]M.D. Gould, J. Aust. Math. Soc. B **20**, 401 (1978).