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General Ward-Like Relations in Canonical Field Theory*)

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Infinitesimal canonical transformation of a field operator is written in terms of the chronological product of the field operator and the generating current. This relation is then generalized to the infinitesimal transformation of the chronological (and retarded) products of any number of field operators. It is shown that the general relations thus obtained contain identities known as the generalized Ward relations in various forms.

The general relations given in integral form are converted into differential form. The relation to the similar identities obtained by the path integral method is discussed and use of the general relations is suggested.

The argument of a previous paper is, thus, refined and extended in three aspects: (i) constraint variables are treated systematically, (ii) it is pointed out that the general relations hold true for the retarded products, (iii) relation to the path integral method is discussed.

§ 1. Introduction

The generalized Ward relation played a vital role in the past two decades in various fields of physics such as the current algebra supplemented by the PCAC hypothesis,[1] the infra-red problem in quantum electrodynamics,[2] the gauge theory[3] and the dynamical rearrangement theory[4] in particle physics.

Despite extensive developments in techniques in such theories, the origin and the validity of the relations are not entirely clear, and the derivation of the various relations within the conventional field theory is attributed to the property of the chronological ordering and the canonical commutation relations between field operators. On the other hand, the derivation of the relations by the path integral method is general and makes use of the fact that the transformation of integral variables causes no change.[5]

It is the purpose of this article to derive the relations from the first principle in the conventional field theory, hereby showing their validity and unifying the independently developed procedures.

In the next section, we derive the relations between the infinitesimal transformation of the field variables and the generating current associated with it. The derivation is quite general and model-independent. It will be seen that the validity

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of the relations is just the same as that of the canonical equation. Those relations obtained in this section are the generalized Ward relations in integral form.

We can convert them into a differential form by considering a transformation involving a test function. In doing so, it will become essential to know the constraint structure of field variables, which, of course, is model-dependent. This problem will be discussed in § 3 with some elementary examples in § 4.

In order to familiarize ourselves with the derivation of the relations from the general point of view, we adopt quantum electrodynamics and show how the various identities follow, without the requirement of the notion of gauge invariance.

Section 6 is devoted to demonstrate one-to-one correspondence between the general identities and those obtained by the path integral method,[4),5)] Several remarks are given in the last section and use of the general relation is suggested.

§ 2. Derivation of the integral identities

We shall find it useful first to review briefly salient features of the mathematical formalism[6)] of quantum field dynamics that find repeated application in the derivation of the relations.

Let us consider an infinitesimal transformation

$$x_\mu \to x_\mu' = x_\mu + \delta x_\mu\,, \tag{2·1}$$

$$\phi^\alpha(x) \to \phi^{\alpha\prime}(x') = \phi^\alpha(x) + \delta\phi^\alpha(x)\,. \tag{2·2}$$

The change of the action integral is, due to the Euler-Lagrange equation,

$$\begin{aligned}\delta I &= \delta \int_{\sigma_1}^{\sigma_2} d^4x \mathcal{L}(\phi^\alpha(x),\ \partial_\mu\phi^\alpha(x)) \\ &= -\int_{\sigma_1}^{\sigma_2} d^4x \partial_\mu J_\mu(x) = G(\sigma_1) - G(\sigma_2)\,,\end{aligned} \tag{2·3}$$

where

$$\left.\begin{aligned} J_\mu(x) &= -\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\alpha(x)}\delta\phi^\alpha(x) + T_{\mu\nu}(x)\delta x_\nu\,, \\ T_{\mu\nu}(x) &= \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^\alpha(x)}\partial_\nu\phi^\alpha(x) - \delta_{\mu\nu}\mathcal{L}\,. \end{aligned}\right\} \tag{2·4}$$

We have denoted the field variables by ϕ^α, which can be divided into two sets: ϕ^a, called the canonical variables, and ϕ^A, termed the constraint variables. The second set is characterized by the relation

$$\Pi^A \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}^A} = 0\,. \tag{2·5}$$

The Euler-Lagrange equation for the constraint variables degenerates into

$$\frac{\partial \mathcal{L}}{\partial \phi^A} = \partial_i \Pi_i{}^A \tag{2·6}$$

with

$$\Pi_i{}^A \equiv \frac{\partial \mathcal{L}}{\partial \partial_i \phi^A}, \quad i=1,\ 2,\ 3\,. \tag{2·7}$$

The condition that Eq. (2·5) has the Lorentz invariant significance is expressed in the form

$$\delta_{\mu 4} \Pi_i{}^A = \Pi^a S_{i\mu}^{aA}\,, \tag{2·8}$$

where $S_{\mu\nu}$ is the skew-symmetric spin matrix, and Π^a the canonical momentum given by

$$\Pi^a \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a}\,. \tag{2·9}$$

Substituting (2·8) into (2·6), we obtain

$$\frac{\partial \mathcal{L}}{\partial \phi^A} = \partial_i \Pi^a S_{i4}^{aA}\,. \tag{2·10}$$

Under the assumption that (2·10) can be solved for ϕ^A, the constraint variable ϕ^A is given explicitly as a function of the canonical variables ϕ^a and Π^a. Thus, the change of the constraint variable $\delta\phi^A$ is not independent but is calculated from that of the canonical variables ϕ^a and Π^a. The change of Π^a associated with $\delta\phi^a$ can be calculated, but we shall not do so, since for the derivation of the advocated relations, such an information will not be used explicitly.

Returing to Eq. (2·3), we define the generator

$$G(\sigma) = \int d\sigma_\mu J_\mu(x), \tag{2·11}$$

for which the canonical relation holds

$$-i\delta^{\mathrm{L}}\phi^\alpha(x) = [\phi^\alpha(x), G(\sigma)]_{x/\sigma}\,, \tag{2·12}$$

where δ^{L} is the Lie variation defined by

$$\begin{aligned}\delta^{\mathrm{L}}\phi^\alpha(x) &\equiv \phi^{\alpha\prime}(x) - \phi^\alpha(x)\\ &= \delta\phi^\alpha(x) - \delta x_\nu \partial_\nu \phi^\alpha(x).\end{aligned} \tag{2·13}$$

The canonical relation (2·12) is valid regardless of whether the transformation (2·1), (2·2) leaves the action integral invariant.

In order to transform the commutator in (2·12) into the chronological product form, we rewrite it as follows:

$$\begin{aligned}-i\delta^{\mathrm{L}}\phi^\alpha(x) &= \phi^\alpha(x)\{G(\sigma) - G(-\infty)\} + \{G(\infty) - G(\sigma)\}\phi^\alpha(x)\\ &\quad - G(\infty)\phi^\alpha(x) + \phi^\alpha(x) G(-\infty).\end{aligned} \tag{2·14}$$

By the aid of the relation (2·3), we obtain

$$-i\delta^{\mathrm{L}}\phi^{\alpha}(x)=\int d^4x' T\{\phi^{\alpha}(x)\partial_{\mu}{}' J_{\mu}(x')\}$$
$$-\int d^4x'\partial_{\mu}{}' T\{\phi^{\alpha}(x) J_{\mu}(x')\}. \qquad (2\cdot 15)$$

Alternatively, we rewrite (2·12) as

$$-i\delta^{\mathrm{L}}\phi^{\alpha}(x)=[\phi^{\alpha}(x),G(\sigma)-G(-\infty)]+[\phi^{\alpha}(x),G(-\infty)] \qquad (2\cdot 16)$$

to obtain

$$-i\delta^{\mathrm{L}}\phi^{\alpha}(x)=\int d^4x' R\{\phi^{\alpha}(x)\partial_{\mu}{}' J_{\mu}(x')\}$$
$$-\int d^4x'\partial_{\mu}{}' R\{\phi^{\alpha}(x) J_{\mu}(x')\}, \qquad (2\cdot 17)$$

where

$$R\{A(x)B(x')\}\equiv\theta(x_0-x_0')[A(x),B(x')]. \qquad (2\cdot 18)$$

A general definition of the R-product will be given shortly.

The relations (2·15) and (2·17) can be generalized to the case of the chronological (and retarded) product of any number of field operators, in view of the fact that δ^{L} is the infinitesimal operation obeying the product rule and it does not upset the chronological ordering. We thus obtain

$$-i\delta^{\mathrm{L}}T(\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n))$$
$$=\int d^4x' T(\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n)\partial_{\mu}{}' J_{\mu}(x'))$$
$$-\int d^4x'\partial_{\mu}{}' T(\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_n}(x_n) J_{\mu}(x')) \qquad (2\cdot 19)$$

and the identical relation for the R-product.*) The multiple R-product is defined by

$$R(\phi(x)\phi(x_1)\cdots\phi(x_n))$$
$$=\sum_{P(1\cdots n)}[\cdots[\phi(x),\phi(x_1)],\phi(x_2)],\cdots],\phi(x_n)]$$
$$\times\theta(t-t_1)\theta(t_1-t_2)\cdots\theta(t_{n-1}-t_n) \qquad (2\cdot 20)$$

for boson-like operators, and

$$R(\psi(x)\psi(x_1)\cdots\psi(x_n))$$
$$=\sum_{P(1\cdots n)}(-1)^P[\cdots\{[\{\psi(x),\psi(x_1)\},\psi(x_2)]\psi(x_3)\}\cdots\psi(x_n)]$$
$$\times\theta(t-t_1)\theta(t_1-t_2)\cdots\theta(t_{n-1}-t_n) \qquad (2\cdot 21)$$

*) We thank H. Matsumoto for verifying directly Eq. (2·19) written in terms of the R-product.

for fermion-like operators. The summations in $(2\cdot20)$ and $(2\cdot21)$ are taken over all possible permutations among $x_1\cdots x_n$, and

$$(-1)^P=\begin{cases} 1 & \text{for even permutations,} \\ -1 & \text{for odd permutations.}\end{cases}$$

Equation $(2\cdot19)$ is the generalized Ward relation in integral form. The derivation given above clearly shows that it is nothing but the familiar canonical equation $(2\cdot12)$, and how the transformation property of ϕ^α is related to the current generating the transformation.

The two alternative forms given in terms of the T-product and the R-product have their own merit. The T-product form is appropriate in dealing with the S-matrix or the relation between the causal Green's functions, whereas the R-product form is powerful when it is combined with the LSZ expansion of the Heisenberg operator in terms of the asymptotic fields. Note that the operators in the T or R product can be composite objects such as interpolating field operators of a bound state.

We conclude this section by three remarks: (i) Since the relation involves only the T-product or the R-product, it is no longer necessary to express the current J_μ in terms of canonical variables, provided that the Lie variation in the left-hand side is correctly obtained for the constraint variables. (ii) The identity given in $(2\cdot19)$ is valid even when the current J_μ happens to be a fermion-like operator. (iii) Since only the 4th component of the current $J_\mu(x)$ plays the role in the relation, the $\delta\phi^A$ in J_μ can be ignored.

§ 3. The identity in differential form

It is possible to convert $(2\cdot19)$ into a differential form by considering a transformation containing an arbitrary test function $f(x)$. Let us consider the transformation

$$x_\mu\to x_\mu'=x_\mu+f(x)\delta x_\mu\,, \tag{3·1}$$

$$\phi^a(x)\to\phi^{a\prime}(x')=\phi^a(x)+f(x)\delta\phi^a(x). \tag{3·2}$$

The variation of the constraint variables cannot be given in a general form, but can be calculated from $(3\cdot2)$ and $(2\cdot10)$. We shall give such examples later.

On account of the remark (iii) at the end of preceding section, the current associated with $(3\cdot2)$ is effectively given by

$$f(x)J_\mu(x). \tag{3·3}$$

Restricting ourselves to the canonical variables, we therefore obtain

$$-i\sum_{i=1}^{n}f(x_i)T\{\phi(x_1)\cdots\phi(x_{i-1})\delta^{\mathrm{L}}\phi(x_i)\phi(x_{i+1})\cdots\phi(x_n)\}$$

$$= \int d^4x' f(x') T\{\phi(x_1)\cdots\phi(x_n)\partial_\mu' J_\mu(x')\}$$

$$- \int d^4x' f(x') \partial_\mu' T\{\phi(x_1)\cdots\phi(x_n) J_\mu(x')\}. \qquad (3\cdot4)$$

In calculating the right-hand side of $(2\cdot19)$ with the current $(3\cdot3)$, we encounter the derivatives of the test function. However, since f is c-number, the T-symbol has no effect, thereby they cancel out leaving $(3\cdot4)$.

Since $f(x)$ is arbitrary, we arrive at

$$-i\sum_{i=1}^{n}\delta^{(4)}(x-x_i)T\{\phi(x_1)\cdots\phi(x_{i-1})\delta^{\mathrm{L}}\phi(x_i)\phi(x_{i+1})\cdots\phi(x_n)\}$$

$$=T\{\phi(x_1)\cdots\phi(x_n)\partial_\mu J_\mu(x)\}-\partial_\mu T\{\phi(x_1)\cdots\phi(x_n)J_\mu(x)\}. \qquad (3\cdot5)$$

A general formula in differential form involving the constraint variables can be worked out, but it complicates the equations a great deal with no practical advantage.

§ 4. Examples

We shall first illustrate, by taking the free Duffin-Kemmer field,[7] how the constraint variables are handled. The field equation is

$$(\beta_\lambda\partial_\lambda+m)\psi(x)=0 \qquad (4\cdot1)$$

with

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=\delta_{\mu\lambda}\beta_\nu+\delta_{\nu\lambda}\beta_\mu\,. \qquad (4\cdot2)$$

The canonical variables

$$\psi^a(x)\equiv\beta_4{}^2\psi(x) \qquad (4\cdot3)$$

and the constraint variable

$$\psi^A(x)\equiv(1-\beta_4{}^2)\psi(x) \qquad (4\cdot4)$$

are related by

$$\psi^A(x)=-\frac{1}{m}\beta_i\partial_i\psi^a(x). \qquad (4\cdot5)$$

The transformation

$$\delta^{\mathrm{L}}\psi^a(x)=-\varepsilon_\nu(x)\partial_\nu\psi^a(x) \qquad (4\cdot6)$$

induces

$$\delta^{\mathrm{L}}\psi^A(x)=-\frac{1}{m}\beta_i\partial_i\delta^{\mathrm{L}}\psi^a(x)$$

$$=-\varepsilon_\nu(x)\partial_\nu\psi^A(x)+\frac{1}{m}\partial_i\varepsilon_\nu(x)\partial_\nu\beta_i\psi^a(x). \qquad (4\cdot7)$$

Combining (4·6) and (4·7), we have the Lie variation for the total $\psi(x)$

$$\delta^{\mathrm{L}}\psi(x) = -\varepsilon_\nu(x)\partial_\nu\psi(x) + \frac{1}{m}\partial_i\varepsilon_\nu(x)\partial_\nu\beta_i\psi^a(x)$$

$$= -i\int d^4x'\varepsilon_\nu(x')\partial_\mu' T\{\psi(x)T_{\mu\nu}(x')\}. \tag{4·8}$$

Hence, the identity in a differential form is

$$\delta^{(4)}(x-x')\partial_\nu\psi(x) + \frac{1}{m}\partial_i'\delta^{(4)}(x-x')\partial_\nu\beta_i\psi^a(x)$$

$$= i\partial_\mu' T\{\psi(x)T_{\mu\nu}(x')\}. \tag{4·9}$$

This relation can be verified directly if we know the canonical commutation relation

$$[\psi(x), \overline{\psi}(x')]\delta(x_0-x_0')$$

$$= \left\{\beta_4 - \frac{1}{m}(\beta_i\beta_4+\beta_4\beta_i)\partial_i\right\}\delta^{(4)}(x-x') \tag{4·10}$$

and the explicit form

$$T_{\mu\nu}(x) = -\overline{\psi}(x)\beta_\mu\partial_\nu\psi(x). \tag{4·11}$$

It is worth pointing out that in deriving (4·9), the explicit form (4·11) has never been used. It is also interesting to see that due to the relation[7]

$$T\{\psi(x)\overline{\psi}(x')\} = T^*\{\psi(x)\overline{\psi}(x')\} + \frac{i}{m}(1-\beta_4^2)\delta^{(4)}(x-x') \tag{4·12}$$

and (4·11), we can cast (4·9) into the form

$$\delta^{(4)}(x-x')\partial_\nu\psi(x) = i\partial_\mu' T^*\{\psi(x)T_{\mu\nu}(x')\}. \tag{4·13}$$

In other words, the constraint variables can be treated as if they were independent, when the T^*-method is employed. We conjecture that this is the case in interacting fields also, but it is difficult to prove it generally.

The Proca field can be treated in a similar manner, but the calculation is much more complicated.

As the second example of the use of the relation, let us take the spinor-scalar system and derive the Dyson equation as a special case. For the sake of concreteness we adopt the Lagrangian

$$\mathcal{L} = -\overline{\psi}(x)(\gamma\partial+m)\psi(x) - \tfrac{1}{2}\{\partial_\mu\phi(x)\partial_\mu\phi(x) + \mu^2\phi(x)\phi(x)\}$$

$$-g\overline{\psi}(x)\Gamma\psi(x)\phi(x). \tag{4·14}$$

The field equation for $\phi(x)$ is

$$(\Box-\mu^2)\phi(x) = g\overline{\psi}(x)\Gamma\psi(x) \equiv J(x). \tag{4·15}$$

Consider the transformation

$$\phi(x)\to\phi(x)+f(x) \tag{4·16}$$

with an arbitrary c-number function $f(x)$. We shall denote

$$f(x)\equiv\delta^f\phi(x) \tag{4·17}$$

for the obvious reason. The current is

$$J_\mu^f=\partial_\mu\phi(x)f(x). \tag{4·18}$$

Hence, we obtain the identity

$$-i\delta^f T(\cdots)=\int d^4x f(x)T(\Box\phi(x)\cdots)$$
$$-\int d^4x f(x)\partial_\mu T(\partial_\mu\phi(x)\cdots), \tag{4·19}$$

where

$$T(\cdots)=T\{\psi(x_1)\cdots\bar\psi(y_1)\cdots\phi(z_1)\cdots\}. \tag{4·20}$$

Again, the arbitrariness of $f(x)$ yields

$$\partial_\mu T(\partial_\mu\phi(x)\cdots)-T(\Box\phi(x)\cdots)$$
$$=i\sum_i\delta^{(4)}(x-z_i)T(\psi(x_1)\cdots\bar\psi(y_1)\cdots\phi(z_1)\cdots\phi(z_{i-1})\phi(z_{i+1})\cdots)$$
$$=i\frac{\delta}{\delta\phi(x)}T(\cdots), \tag{4·21}$$

where we used the formal stipulation

$$\frac{\delta\phi(z)}{\delta\phi(x)}=\delta^{(4)}(x-z). \tag{4·22}$$

In view of the relation

$$T(\partial_\mu\phi(x)\cdots)=\partial_\mu T(\phi(x)\cdots), \tag{4·23}$$

we can rewrite (4·21) as

$$(\Box-\mu^2)T(\phi(x)\cdots)-T(J(x)\cdots)=i\frac{\delta}{\delta\phi(x)}T(\cdots). \tag{4·24}$$

Upon taking the vacuum expectation value of (4·24), we obtain the Dyson equation. It is straightforward to derive the Dyson equation for the spinor field.

§ 5. The divergence relations in quantum electrodynamics

We shall derive general relations in quantum electrodynamics. For this purpose we adopt the Lagrange multiplier method.[8), 10)] The Lagrangian is given by

$$\mathcal{L} = -\bar{\psi}(\gamma\partial + m_0)\psi - \frac{1}{4}F_{\mu\nu}F_{\mu\nu} + j_\mu A_\mu + B\partial_\mu A_\mu + \frac{\alpha}{2}B^2, \tag{5·1}$$

where B is the Lagrange multiplier and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5·2}$$

$$j_\mu = ie\bar{\psi}\gamma_\mu\psi. \tag{5·3}$$

The electromagnetic potential satisfies the equation

$$\begin{aligned}\partial_\mu F_{\mu\nu}(x) &= (\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)A_\mu(x)\\ &= -j_\nu(x) + \partial_\nu B(x),\end{aligned} \tag{5·4}$$

$$\partial_\mu A_\mu + \alpha B(x) = 0. \tag{5·5}$$

The current $j_\mu(x)$ is conserved, i.e.,

$$\partial_\mu j_\mu(x) = 0. \tag{5·6}$$

Consequently, we obtain

$$\Box B(x) = 0, \tag{5·7}$$

$$\Box\partial_\mu A_\mu(x) = 0. \tag{5·8}$$

The canonical momentum conjugate to A_μ is

$$\Pi_i = \dot{A}_i + \partial_i A_0 = iF_{4i}, \quad i = 1, 2, 3, \tag{5·9}$$

$$\Pi_4 = -iB. \tag{5·10}$$

The canonical commutation relations among A_μ and Π_ν are reduced to

$$[A_i(x), \dot{A}_j(x')] = i\delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{5·11}$$

$$[A_4(x), B(x')] = -\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{5·12}$$

$$[\dot{A}_i(x), B(x')] = i\partial_i\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{5·13}$$

$$\begin{aligned}[A_i(x), B(x')] &= [\dot{A}_i(x), \dot{A}_j(x')]\\ &= [B(x), B(x')] = 0\end{aligned} \tag{5·14}$$

at $x_0 = x_0'$.

Let us now consider the following three transformations:

(A)
$$\begin{aligned}\delta\psi(x) &= ie\Lambda(x)\psi(x) \equiv \delta^A\psi(x),\\ \delta\bar{\psi}(x) &= -ie\Lambda(x)\bar{\psi}(x) \equiv \delta^A\bar{\psi}(x),\\ \delta A_\mu(x) &= \delta B(x) = 0\end{aligned} \tag{5·15}$$

with an arbitrary c-number function $\Lambda(x)$. The current associated with (5·15) is

$$J_\mu{}^A(x) = j_\mu(x)\Lambda(x) \tag{5·16}$$

and

$$\partial_\mu J_\mu{}^A(x) = j_\mu(x)\partial_\mu \Lambda(x). \tag{5·17}$$

Denoting as

$$T(\cdots) = T(\psi(x_1)\cdots\bar{\psi}(y_1)\cdots A_{\mu_1}(z_1)\cdots B(u_1)\cdots), \tag{5·18}$$

we obtain the relation

$$i\delta^A T(\cdots) = \int d^4x'\partial_\mu{}' T(\cdots j_\mu(x'))\Lambda(x'), \tag{5·19}$$

(B)

$$\delta\psi(x) = \delta\bar{\psi}(x) = 0,$$
$$\delta A_\mu(x) = f_\mu(x) \equiv \delta^B A_\mu(x),$$
$$\delta B(x) = 0, \tag{5·20}$$

where $f_\mu(x)$ is an arbitrary c-number vector function. The current is given by

$$J_\mu{}^B(x) = F_{\mu\nu}(x)f_\nu(x) - B(x)f_\mu(x), \tag{5·21}$$

which yields the relation

$$\begin{aligned} i\delta^B T(\cdots) = & \int d^4x' \{f_\nu(x')\partial_\mu{}' T(\cdots F_{\mu\nu}(x')) - f_\mu(x')\partial_\mu{}' T(\cdots B(x')) \\ & - f_\nu(x') T(\cdots \partial_\mu{}' F_{\mu\nu}(x')) + f_\mu(x') T(\cdots\partial_\mu{}' B(x'))\} \\ = & \int d^4x' f_\nu(x')\{T(\cdots j_\nu(x')) + (\Box'\delta_{\mu\nu} - \partial_\mu{}'\partial_\nu{}')T(\cdots A_\mu(x')) \\ & - \partial_\nu{}' T(\cdots B(x'))\}. \end{aligned} \tag{5·22}$$

In the last step, we have used Eq. (5·4) and the commutation relations (5·11) $\sim$(5·14).

(C)

$$\delta\psi(x) = \delta\bar{\psi}(x) = \delta A_\mu(x) = 0,$$
$$\delta B(x) = \lambda(x) \equiv \delta^c B(x). \tag{5·23}$$

Here, $\lambda(x)$ is an arbitrary c-number function. The current

$$J_\mu{}^c(x) = A_\mu(x)\lambda(x) \tag{5·24}$$

yields the identity

$$\begin{aligned} i\delta^c T(\cdots) = & \int d^4x' \lambda(x')\partial_\mu{}' T(\cdots A_\mu(x')) \\ & + \alpha \int d^4x' \lambda(x') T(\cdots B(x')). \end{aligned} \tag{5·25}$$

The local gauge transformation

$$\delta^L \psi(x) = ie\Lambda(x)\psi(x),$$
$$\delta^L \bar{\psi}(x) = -ie\Lambda(x)\bar{\psi}(x),$$

$$\delta^{\mathrm{L}} A_\mu(x) = \partial_\mu \Lambda(x),$$

$$\delta^{\mathrm{L}} B(x) = 0 \tag{5·26}$$

is a combination of (A) and (B) with

$$f_\mu(x) = \partial_\mu \Lambda(x). \tag{5·27}$$

Since the divergence relation holds true regardless of the invariance of the Lagrangian, we *do not* require the condition

$$\Box \Lambda(x) = 0. \tag{5·28}$$

As was mentioned earlier, the identities (5·19), (5·22) and (5·25) are valid for the retarded product also. Since the functions $\Lambda(x)$, $f_\mu(x)$ and $\lambda(x)$ appearing in the above transformations are all arbitrary, we can convert the relations into differential form. They are respectively

$$ie\{\sum_{i=1} \delta^{(4)}(x'-x_i) - \sum_{j=1} \delta^{(4)}(x'-y_j)\} T(\cdots) = -i\partial_\mu' T(\cdots j_\mu(x')), \tag{5·29}$$

$$i\frac{\delta}{\delta A_\nu(x')} T(\cdots) = T(\cdots j_\nu(x')) + (\Box'\delta_{\mu\nu} - \partial_\mu'\partial_\nu') T(\cdots A_\mu(x')) - \partial_\nu' T(\cdots B(x')), \tag{5·30}$$

$$i\frac{\delta}{\delta B(x')} T(\cdots) = \partial_\mu' T(\cdots A_\mu(x')) + \alpha T(\cdots B(x')) \tag{5·31}$$

with the stipulation that

$$\frac{\delta A_\mu(x)}{\delta A_\nu(x')} = \delta_{\mu\nu}\delta^{(4)}(x-x'), \tag{5·32}$$

$$\frac{\delta B(x)}{\delta B(x')} = \delta^{(4)}(x-x'). \tag{5·33}$$

It is easy to show that the combination of (5·30) and (5·29) gives

$$\Box' T(\cdots B(x')) = -e\{\sum_{i=1} \delta^{(4)}(x'-x_i) - \sum_{j=1} \delta^{(4)}(x'-y_j)\} T(\cdots) - i\partial_\nu' \frac{\delta}{\delta A_\nu(x')} T(\cdots), \tag{5·34}$$

from which follows, for example,

$$\Box' T(B(x) B(x')) = 0, \tag{5·35}$$

$$\Box' T(A_\mu(x) B(x')) = -i\partial_\mu' \delta^{(4)}(x-x'), \tag{5·36}$$

$$\Box' T(\psi(x)\bar{\psi}(y) B(x')) = -e(\delta^{(4)}(x'-x) - \delta^{(4)}(x'-y)) T(\psi(x)\bar{\psi}(y)). \tag{5·37}$$

The vacuum expectation value of the last equation is the familiar relation originally proposed by Green.[9]

The pole structure can now be determined by the use of (5·31), (5·35) and (5·36) together with the spectral representation of the Green's functions

$$\langle T(B(x)B(x'))\rangle_0 = i\int d\kappa^2 \rho_0(\kappa^2)\Delta_c(x-x';\kappa^2), \tag{5·38}$$

$$\langle T(A_\mu(x)A_\nu(x'))\rangle_0 = i\int d\kappa^2\rho_1(\kappa^2)\Delta_c(x-x';\kappa^2)\delta_{\mu\nu} - i\int d\kappa^2\rho_2(\kappa^2)\partial_\mu\partial_\nu\Delta_c(x-x';\kappa^2), \tag{5·39}$$

$$\langle T(B(x)A_\mu(x'))\rangle_0 = -\langle T(A_\mu(x)B(x'))\rangle_0 = -i\int d\kappa^2\rho_3(\kappa^2)\partial_\mu\Delta_c(x-x';\kappa^2). \tag{5·40}$$

If we substitute (5·38) and (5·40) into

$$\partial_\mu'\langle T(B(x)A_\mu(x'))\rangle_0 + \alpha\langle T(B(x)B(x'))\rangle_0 = i\delta^{(4)}(x-x'), \tag{5·41}$$

which is obtained from (5·31), we have

$$\kappa^2\rho_3(\kappa^2) + \alpha\rho_0(\kappa^2) = 0, \tag{5·42}$$

$$\int d\kappa^2\rho_3(\kappa^2) = 1. \tag{5·43}$$

On the other hand, Eq. (5·36) gives

$$\kappa^2\rho_3(\kappa^2) = 0, \tag{5·44}$$

which implies on account of (5·42) and (5·43),

$$\rho_3(\kappa^2) = \delta(\kappa^2), \tag{5·45}$$

$$\rho_0(\kappa^2) = 0. \quad (\text{if } \alpha\neq 0) \tag{5·46}$$

Hence,

$$\langle T(B(x)A_\mu(x'))\rangle_0 = -i\partial_\mu D_c(x-x'), \tag{5·47}$$

$$\langle T(B(x)B(x'))\rangle_0 = 0. \quad (\text{if } \alpha\neq 0) \tag{5·48}$$

To determine ρ_1 and ρ_2, we make use of the equation

$$\partial_\mu'\langle T(A_\nu(x)A_\mu(x'))\rangle_0 + \alpha\langle T(A_\nu(x)B(x'))\rangle_0 = 0 \tag{5·49}$$

following from (5·31). The substitution of (5·39) into (5·47) gives

$$\rho_1(\kappa^2) - \kappa^2\rho_2(\kappa^2) = \alpha\delta(\kappa^2), \tag{5·50}$$

$$\int d\kappa^2\rho_2(\kappa^2) = 0. \tag{5·51}$$

Let us assume

$$\rho_1(\kappa^2)=Z_3\delta(\kappa^2-\mu^2)+\delta(\kappa^2), \tag{5·52}$$

where μ^2, which may or may not vanish, is the mass of the electromagnetic field. Equation (5·50) then gives

$$\kappa^2\rho_2(\kappa^2)=-\alpha\delta(\kappa^2)+Z_3\delta(\kappa^2-\mu^2)+\sigma(\kappa^2), \tag{5·53}$$

and consequently

$$\rho_2(\kappa^2)=-K\delta(\kappa^2)-\frac{\alpha}{\kappa^2}\delta(\kappa^2)+\frac{Z_3}{\kappa^2}\delta(\kappa^2-\mu^2)+\frac{1}{\kappa^2}\sigma(\kappa^2), \tag{5·54}$$

where K is a constant to be determined. Due to Eq. (5·51) we obtain

$$K=-\alpha\int d\kappa^2\frac{1}{\kappa^2}\delta(\kappa^2)+Z_3\int d\kappa^2\frac{1}{\kappa^2}\delta(\kappa^2-\mu^2)+\int d\kappa^2\frac{\sigma(\kappa^2)}{\kappa^2}. \tag{5·55}$$

It is convenient to consider the following two cases separately.

(i) $\mu^2=0$. Then, the quantity K becomes

$$K=(Z_3-\alpha)\int d\kappa^2\frac{1}{\kappa^2}\delta(\kappa^2)+\int d\kappa^2\frac{\sigma(\kappa^2)}{\kappa^2}. \tag{5·56}$$

If we choose the gauge parameter

$$\alpha=Z_3, \tag{5·57}$$

the first term in the right-hand side of (5·56) disappears, i.e., the dipole ghost is eliminated. Hence, we have

$$\begin{aligned}\langle T(A_\mu(x)A_\nu(x'))\rangle_0&=i(Z_3\delta_{\mu\nu}+K_0\partial_\mu\partial_\nu)D_c(x-x')\\&\quad+i\int_{\kappa^2\neq 0}d\kappa^2\sigma(\kappa^2)\left(\delta_{\mu\nu}-\frac{1}{\kappa^2}\partial_\mu\partial_\nu\right)\Delta_c(x-x';\kappa^2)\\&=iZ_3\delta_{\mu\nu}D_c(x-x')+i(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\int d\kappa^2\frac{\sigma(\kappa^2)}{\kappa^2}\\&\qquad\times\{\Delta_c(x-x';\kappa^2)-D_c(x-x')\},\end{aligned} \tag{5·58}$$

which agrees, under the condition (5·57), with the expression obtained by Nakanishi.[10] Here,

$$K_0\equiv\int d\kappa^2\frac{\sigma(\kappa^2)}{\kappa^2}. \tag{5·59}$$

Note that

$$\partial_\mu\langle T(A_\mu(x)A_\nu(x'))\rangle_0=iZ_3\partial_\nu D_c(x-x'). \tag{5·60}$$

(ii) $\mu^2 \neq 0$.

In order to eliminate the term $\delta(\kappa^2)/\kappa^2$, we choose

$$\alpha = 0 . \tag{5·61}$$

Denoting K with $\alpha=0$ by K_m, we obtain

$$\begin{aligned}\langle T(A_\mu(x)A_\nu(x'))\rangle_0 &= iZ_3\left(\delta_{\mu\nu}-\frac{1}{\mu^2}\partial_\mu\partial_\nu\right)\varDelta_c(x-x';\mu^2)\\
&\quad + iK_m\partial_\mu\partial_\nu D_c(x-x') + i\int_{\kappa^2\neq 0} d\kappa^2\sigma(\kappa^2)\left(\delta_{\mu\nu}-\frac{1}{\kappa^2}\partial_\mu\partial_\nu\right)\varDelta_c(x-x';\kappa^2)\\
&= i(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\left\{\frac{Z_3}{\mu^2}(\varDelta_c(x-x';\mu^2)-D_c(x-x'))\right.\\
&\qquad\left. + \int d\kappa^2\frac{\sigma(\kappa^2)}{\kappa^2}(\varDelta_c(x-x';\kappa^2)-D_c(x-x'))\right\}\\
&= i(\Box_\mu\delta_\nu-\partial_\mu\partial_\nu)\int d\kappa^2\frac{\rho_1(\kappa^2)}{\kappa^2}(\varDelta_c(x-x';\kappa^2)-D_c(x-x')) .\end{aligned} \tag{5·62}$$

Obviously,

$$\partial_\mu\langle T(A_\mu(x)A_\nu(x'))\rangle_0 = 0 . \tag{5·63}$$

In the above argument, we have chosen the gauge parameter α in such a way that the spectral function does not contain the dipole ghost term $\delta(\kappa^2)/\kappa^2$. With this restriction, both the massless and massive solutions do not contradict the canonical relation and the gauge invariance. The above restriciton enables us to write down the following in-field expansion.[16)]

(i) $\mu^2=0,\quad \alpha=Z_3,$

$$A_\mu(x) = Z_3^{1/2}a_\mu^{\text{in}}(x)+\cdots , \tag{5·64}$$

$$B(x) = -Z_3^{-1/2}\partial_\mu a_\mu^{\text{in}}(x)+\cdots , \tag{5·65}$$

where

$$[a_\mu^{\text{in}}(x), a_\nu^{\text{in}}(x')] = i\delta_{\mu\nu}D(x-x'), \tag{5·66}$$

$$\Box a_\mu^{\text{in}}(x) = 0 , \tag{5·67}$$

(ii) $\mu^2\neq 0,\quad \alpha=0,$

$$A_\mu(x) = Z_3^{1/2}u_\mu^{\text{in}}(x)+\frac{Z_3}{\mu}\partial_\mu b^{\text{in}}(x)+\cdots , \tag{5·68}$$

$$B(x) = -\mu Z_3^{-1/2}(b^{\text{in}}(x)-\chi^{\text{in}}(x))+\cdots , \tag{5·69}$$

where $u_\mu^{\text{in}}(x)$ is the Proca field satisfying

$$\{(\Box-\mu^2)\delta_{\mu\nu}-\partial_\mu\partial_\nu\}u_\nu^{\text{in}}(x)=0\,, \tag{5·70}$$

$$[u_\mu^{\text{in}}(x),u_\nu^{\text{in}}(x')]=i\left(\delta_{\mu\nu}-\frac{1}{\mu^2}\partial_\mu\partial_\nu\right)\Delta(x-x';\mu^2) \tag{5·71}$$

and

$$\Box b^{\text{in}}(x)=\Box\chi^{\text{in}}(x)=0\,, \tag{5·72}$$

$$\begin{aligned}[b^{\text{in}}(x),b^{\text{in}}(x')]&=-[\chi^{\text{in}}(x),\chi^{\text{in}}(x')]\\&=-iD(x-x')\,.\end{aligned} \tag{5·73}$$

It is important to note that the $b^{\text{in}}(x)$ field is of negative norm, whereas the $\chi^{\text{in}}(x)$ field is of positive norm. The field $\chi^{\text{in}}(x)$ is the massless bound state composed of an electron and a positron. This statement can be justified by (5·37) and the singular nature of the electron propagator.*)

§ 6. The relation to identities in the path integral method

The relation obtained in (2·19) is quite different in appearance from that is derived by the path integral method.[4),5),12)] To show their close connection, we first define the generating function

$$W(J)=T\exp\int d^4x\mathcal{L}_s(x), \tag{6·1}$$

where $\mathcal{L}_s(x)$ is given by

$$\mathcal{L}_s(x)\equiv\phi^\alpha(x)J^\alpha(x) \tag{6·2}$$

with a c-number function $J^\alpha(x)$. Since the right-hand side of (6·1) is given in T-product, the operators $\phi^\alpha(x)$ can be treated as if they were c-number.

If we use the symbol

$$\langle F(x)\rangle_J\equiv T(F(x)e^{\int d^4x\mathcal{L}_s(x)}), \tag{6·3}$$

the effect of the variation

$$\phi^\alpha(x)\to\phi^{\alpha\prime}(x)=\phi^\alpha(x)+\delta^L\phi^\alpha(x) \tag{6·4}$$

can be expressed as

$$\delta^LW(J)=\int d^4xJ^\alpha(x)\langle\delta^L\phi^\alpha(x)\rangle_J\,. \tag{6·5}$$

Using the relation (2·15), we obtain the master relation

$$\int d^4xJ^\alpha(x)\langle\delta^L\phi^\alpha(x)\rangle_J=i\int d^4xd^4x'\{J^\alpha(x)\langle\phi^\alpha(x)\partial_\mu'J_\mu(x')\rangle_J$$

*) The singularity of the electron propagator at $x=x'$ is related to the G-I-S term.[11)]

$$-J^{\alpha}(x)\partial_{\mu}'\langle\phi^{\alpha}(x)J_{\mu}(x')\rangle_{J}\}$$
$$=i\int d^4x\{\langle\partial_{\mu}J_{\mu}(x)\rangle_J-\partial_{\mu}\langle J_{\mu}(x)\rangle_J\}. \tag{6·6}$$

Differentiating (6·6) with respect to the c-number function $J^{\alpha}(x)$ n times and putting $J^{\alpha}=0$, we obtain the relation (2·19). We emphasize here that our relations are q-number relations.

The R-product relation is also obtained from the generating function defined by

$$\mathcal{R}(x;J)=W^{-1}(J)\frac{\partial}{\partial J^{\alpha}(x)}W(J). \tag{6·7}$$

We can now see from the above argument that there is a one to one correspondence between the vacuum expectation value of the relations derived above and relations obtained by the path integral method. We do not claim, however, that two methods give the identical result.

§ 7. Discussion

We have expressed the infinitesimal transformation of the T-product (or R-product) of any number of field operators in terms of the T-product (or R-product) of field operators and the generating current. The content of such relations is completely identical to the canonical equation involving equal-time commutator. Hence, the validity of our general relations is identical to the canonical equation.

If we define the generating function of the T-product, the derivation of the general relation resembles that by the path integral method. This enables us to translate relations obtained by the path integral method into the language of the conventional field theory. It should be borne in mind, however, that the derivation of the general relation given here is only formal. A correct derivation would have to employ an appropriate regularization procedure and would bring out the anomalous terms which we have ignored. This point calls for further study.

As we proposed in our earlier paper,[13] the spontaneous breakdown of symmetry can be accommodated in our formalism if we combine our method with the LSZ and the NHZ construction. We have shown in Ref. 13) that the transformation of the Goldstone boson correctly reproduces the transformation property of the Heisenberg operators. We hereby confirmed the idea of the dynamical rearrangement of symmetry originated by Umezawa.

The notion of the dynamical rearrangement of symmetry was further extended by Umezawa and his collaborators with the help of the so-called boson theorem.[4] We can prove the boson theorem within our formalism by extending the discussion given in § 4. More direct and simple proof of the boson theorem is provided, however, by the Yang-Feldman formalism, which we shall not elaborate here any further.

The essence of the boson theorem lies in its capability of treating the macroscopic quantum state. Umezawa and his collaborators have treated successfully the problem of superconductivity,[14] and also magnons,[15] all of which are based on the idea of the dynamical rearrangement and the boson theorem, obtained by the aid of the path integral method.

This paper provides us with a method entirely based on the conventional canonical formalism to deal with the same problems. It is hoped that our method will clarify physical mechanism of the problems thus far hindered somewhat by the mathematical formulation.

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