General Wiener-Hopf Operators and Complete Biorthogonal Systems

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1. Introduction. Let A be a linear operator defined on a dense subset of a separable Hilbert space H. Let P be an orthogonal projection onto the closed subspace $R(P)$. The *general Wiener-Hopf operator* associated with A and P is

$$T_P(A) = PA \mid R(P), \tag{1.1}$$

the vertical bar denoting restriction. Shinbrot [3] has developed an inversion formula for such operators assuming A, and therefore $T_P(A)$, has strongly positive real part. This was achieved by embedding A in a family of operators $A(z)$ depending analytically on z and then making use of some basic properties of analytic functions. He thereby obtained an inversion formula for $T_P(A(z))$ which takes the form of a series expansion:

$$\sum_{i=1}^{\infty} (\cdot, \psi_i(\bar{z}))\psi_i(z),$$

where $\bar{z}$ denotes the complex conjugate of z and the $\psi_i(z)$'s are analytic functions of z taking values in $R(P)$ and satisfying certain additional properties. In particular, it was required that

$$(A(z)\psi_i(z), \psi_j(\bar{z})) = \delta_{ij}$$

for z contained in some prescribed complex domain, δ_{ij} denoting the Kronecker delta.

In this paper we broaden the above class of inversion formulas by considering ordered pairs of sequences $\{\psi_i\}$, $\{\omega_i\}$ from $R(P)$ related by

$$(A\psi_i, \omega_j) = \delta_{ij}.$$

Results which extend those of [3] are obtained in §2 without the use of analytic function theory. Our inversion formula takes the analogous form

$$\sum_{i=1}^{\infty} (\cdot, \omega_i)\psi_i .$$

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Using a Gram–Schmidt type construction we prove that there always exists an ordered system $\{\psi_i\}$, $\{\omega_i\}$ such that the above formula equals $T_P(A)^{-1}$.

One of the conditions we require of the system in order to obtain the inversion formula is the spanning property:

$$\text{sp}\{\psi_i\}_{i=1}^n = \text{sp}\{\omega_i\}_{i=1}^n \qquad (n = 1, 2, \cdots).$$

But this condition is most annoying since it is sometimes easy to produce an ordered system $\{\psi_i\}$, $\{\omega_i\}$ satisfying all of the necessary requirements but this one. Moreover the condition would seem unnecessary since without it the inversion formula must still be valid on a dense subset of $R(P)$ (Theorem 2.2 (ii)). It was conjectured to be extraneous by Shinbrot [3, p. 356] within the class of inversion formulas which he considered. In §3 we show that this spanning condition is essential at least within the broader class of inversion formulas which we consider. These formulas come from this larger class in a natural way when solving specific problems.

The proofs used in [3] are in error when the operator A is unbounded, leaving the results in question. But if A is assumed to be bounded the results remain valid. In this paper it will be assumed that A is bounded. The only exception to this is contained in the remark at the end of §2 where one means of generalizing our results to unbounded operators is given. The remark includes an explanation of the error in [3] and a means of correcting it.

In §4 the results of §2 are used to prove a new sufficient condition for a complete biorthogonal system in a separable Hilbert space to be a basis. This result is then used to prove the expansion theorem of Paley and Wiener [1]. Further applications of general Wiener–Hopf operators may be found in [3] and [4]. Some open questions related to our results are listed in §5.

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2. The inversion formula. Let A denote a bounded linear operator on a separable Hilbert space H. An ordered pair of sequences $\{\psi_i\}$, $\{\omega_i\}$ from H is called an *A-biorthogonal system* if

$$(A\psi_i, \omega_j) = \delta_{ij}. \tag{2.1}$$

Given two sequences $\{\psi_i\}$, $\{\omega_i\}$ from H and any integer $n > 0$, we define

(2.2) $\tilde{P}_n$ to be orthogonal projection onto $\text{sp}\{\psi_i\}_{i=1}^n$;

(2.3) P_n to be orthogonal projection onto $\text{sp}\{\omega_i\}_{i=1}^n$,

where "sp" denotes the span of the set in question, *i.e.*, the set of all (finite) linear combinations of elements from that set with scalars from the complex numbers **C**. A new operator is defined by

$$T_n = P_n A \mid R(\tilde{P}_n),$$

$R(\tilde{P}_n)$ denoting the range of $\tilde{P}_n$. Finally for x in H we set

$$S_n x = \sum_{i=1}^{n} (x, \omega_i)\psi_i , \tag{2.4}$$

so that for each n, S_n is a bounded linear operator on H.

Lemma 2.1. *If $\{\psi_i\}$, $\{\omega_i\}$ is any A-biorthogonal system in H, then T_n is invertible for all n. Moreover,*

$$T_n^{-1} = S_n \mid R(P_n). \tag{2.5}$$

Proof. It is apparent that

$$T_n : R(\tilde{P}_n) \to R(P_n)$$

and

$$S_n \mid R(P_n) : R(P_n) \to R(\tilde{P}_n).$$

If $x \in R(\tilde{P}_n)$, we have $x = \sum_{j=1}^{n} a_j\psi_j$ for some $a_j \in \mathbf{C}$. Thus,

$$S_n T_n x = \sum_{i=1}^{n} \left(P_n A \sum_{j=1}^{n} a_j\psi_j , \omega_i\right)\psi_i$$
$$= \sum_{i=1}^{n} a_i\psi_i = x$$

by (2.1) and (2.3). On the other hand, for $y \in R(P_n)$ we have that

$$\begin{aligned}(T_n S_n y, \omega_j) &= \left(P_n A \sum_{i=1}^{n} (y, \omega_i)\psi_i , \omega_j\right) \\ &= (y, \omega_j) \qquad (j = 1, \cdots , n).\end{aligned} \tag{2.6}$$

Since $T_n S_n y \in R(P_n)$, (2.6) shows that

$$T_n S_n y = y,$$

and this completes the proof.

The main objective of this section is to impose sufficient conditions on the operator A and an A-biorthogonal system $\{\psi_i\}$, $\{\omega_i\}$ so that we may pass to the limit in (2.5) in order to obtain an inversion formula for the Wiener–Hopf operator $T_P(A)$ defined by (1.1):

$$T_P(A)^{-1}x = \sum_{i=1}^{\infty} (x, \omega_i)\psi_i .$$

Given an A-biorthogonal system $\{\psi_i\}$, $\{\omega_i\}$ set

$$Sx = \sum_{i=1}^{\infty} (x, \omega_i)\psi_i \tag{2.7}$$

where the domain of S, $D(S)$, is the set of all $x \in H$ for which the right hand side of (2.7) converges. An operator A is said to have *positive real part* if

$$\text{Re}\,(Ax, x) > 0 \tag{2.8}$$

for all non-zero x in H. We abbreviate this Re $A > 0$. Clearly Re $A > 0$ implies Re $T_P(A) > 0$ for any orthogonal projection P. It follows that $T_P(A)$ is one to one and, since $T_P(A)^*$ $(= T_P(A^*)$, $*$ denoting the adjoint) also has positive real part, $T_P(A)$ has dense range. Hence if Re $A > 0$, $T_P(A)^{-1}$ exists and is defined on a dense subset of $R(P)$, but it need not be bounded. The following result extends Theorem 5.1 of [3].

Theorem 2.2. *Let P be an orthogonal projection and suppose* Re $A > 0$. *If $\{\psi_i\}$, $\{\omega_i\}$ is any A-biorthogonal system such that ψ_i, $\omega_i \in R(P)$ for all i, then*:

(i) *if $x \in T_P(A)\text{sp}\{\psi_i\}$, then $x \in D(S)$ and*

$$Sx = T_P(A)^{-1}x; \tag{2.9}$$

(ii) *if* $\overline{\text{sp}\{\psi_i\}} = R(P)$, *then* (2.9) *holds for all x in a dense subset of $R(P)$*;

(iii) $\overline{\text{sp}\{\omega_i\}} = R(P)$ *if and only if* (2.9) *holds for all $x \in D(S) \cap R(P)$*.

Proof of (i). If $y \in \text{sp}\{\psi_i\}$, we have $y = \sum_{j=1}^{n} a_j\psi_j$ for some n and some $a_j \in \mathbf{C}$. Then for $i > n$

$$(T_P(A)y, \omega_i) = \sum_{j=1}^{n} a_j(A\psi_j, \omega_i) = 0$$

since $\omega_i \in R(P)$ and the system $\{\psi_i\}$, $\{\omega_i\}$ is A-biorthogonal. Therefore $T_P(A)y \in D(S)$. Moreover, (2.9) is obtained from the following:

$$ST_P(A)y = \sum_{i=1}^{\infty}\left(PA\sum_{j=1}^{n} a_j\psi_j, \omega_i\right)\psi_i$$

$$= \sum_{i=1}^{n} a_i\psi_i = y.$$

Proof of (ii). By (i) it suffices to prove that $T_P(A)\text{sp}\{\psi_i\}$ is dense in $R(P)$ if $\text{sp}\{\psi_i\}$ is. But if $y \in R(P)$ is orthogonal to $T_P(A)\text{sp}\{\psi_i\}$, then we have

$$(y, T_P(A)\psi_i) = (y, A\psi_i) = (A^*y, \psi_i) = 0$$

for all i. Therefore $T_P(A^*)y = 0$. Now Re $A > 0$ implies Re $T_P(A^*) > 0$ which asserts that $y = 0$.

Proof of (iii). Suppose $\text{sp}\{\omega_i\}$ is dense in $R(P)$ and let x be any element from $D(S) \cap R(P)$. Observe that $S_nx = S_nP_nx$ so that by Lemma 2.1

$$P_nAS_nx = T_nS_nP_nx = P_nx\,.$$

Now for $i > n$

$$P_nA\psi_i = 0$$

via the A-biorthogonality of the system $\{\psi_i\}$, $\{\omega_i\}$. Hence

$$P_n ASx = P_n AS_n x = P_n x. \tag{2.10}$$

Since $\overline{\mathrm{sp}\{\omega_i\}} = R(P)$, (2.9) is obtained by letting n tend to infinity in (2.10).

Suppose, on the other hand, that (2.9) holds for all $x \in D(S) \cap R(P)$. If $x \in R(P)$ is orthogonal to ω_i for all i, then $x \in D(S)$ and $Sx = 0$. Therefore, by (2.9),

$$x = T_P(A)Sx = 0,$$

so that $\mathrm{sp}\{\omega_i\}$ is dense in $R(P)$. This completes the proof.

An operator A is said to have *strongly positive real part* if

$$\mathrm{Re}\,(Ax, x) \geqq \delta\,||x||^2 \tag{2.11}$$

for some $\delta > 0$ and for all $x \in H$. We abbreviate this $\mathrm{Re}\,A \geqq \delta$. If $\mathrm{Re}\,A \geqq \delta$, then $T_P(A)$ inherits this property for any orthogonal projection P. It follows that $T_P(A)$ is not only one to one with dense range, but has closed range as well. Hence if $\mathrm{Re}\,A \geqq \delta$, then $T_P(A)$ possesses a bounded inverse. The following theorem generalizes Corollary 5.2 of [3].

Theorem 2.3. *Let P be an orthogonal projection on H and suppose* $\mathrm{Re}\,A \geqq \delta$. *If $\{\psi_i\}$, $\{\omega_i\}$ is any A-biorthogonal system such that:*

$$\overline{\mathrm{sp}\{\psi_i\}} = R(P);$$

$$\mathrm{sp}\{\psi_i\}_{i=1}^n = \mathrm{sp}\{\omega_i\}_{i=1}^n \qquad (n = 1, 2, \cdots), \tag{2.12}$$

then $T_P(A)$ has a bounded inverse and for every $x \in R(P)$

$$T_P(A)^{-1}x = \sum_{i=1}^{\infty} (x, \omega_i)\psi_i\ .$$

Proof. $T_P(A)$ has a bounded inverse because $\mathrm{Re}\,A \geqq \delta$. Since $\omega_i \in R(P)$ for all i, S is defined and equals zero on the orthogonal complement of $R(P)$. Thus by Theorem 2.2 (ii), the domain of S is dense in H. It follows that we can extend the domain of S to all of H if and only if the S_n's (2.4) are uniformly bounded in norm. Moreover, if $D(S) = H$, then we get the desired inversion formula by Theorem 2.2 (iii). Since $\mathrm{Re}\,A \geqq \delta$, for $x \in H$ we have

$$\begin{aligned} \delta\,||S_n x||^2 &\leqq \mathrm{Re}\,(AS_n x, S_n x) \\ &= \mathrm{Re}\,(\tilde{P}_n AS_n x, S_n x) \\ &\leqq ||\tilde{P}_n AS_n x||\ ||S_n x||. \end{aligned} \tag{2.13}$$

By hypothesis (2.12), $\tilde{P}_n = P_n$ for all n, so that

$$\tilde{P}_n AS_n x = T_n S_n x.$$

But by Lemma 2.1

$$T_nS_nx = T_nS_nP_nx = P_nx,$$

and therefore

$$\tilde{P}_nAS_nx = P_nx.$$

Using this in (2.13), we have that

$$\begin{aligned} ||S_nx|| &\leqq \delta^{-1}\, ||\tilde{P}_nAS_nx|| \\ &= \delta^{-1}\, ||P_nx|| \\ &\leqq \delta^{-1}\, ||x||. \end{aligned} \tag{2.14}$$

This completes the proof.

The theorem below extends Lemma 3.1 of [3]. We prove that the inverse of any general Wiener–Hopf operator arising from an operator with strongly positive real part can always be expanded in terms of an A-biorthogonal system satisfying the hypotheses of Theorem 2.3. In fact the following is true.

Theorem 2.4. *Let P be an orthogonal projection on H and suppose* Re $A > 0$. *Then there exists an A-biorthogonal system $\{\psi_i\}$, $\{\omega_i\}$ such that:*

$$\overline{\text{sp}\{\psi_i\}} = R(P);$$

$$\text{sp}\{\psi_i\}_{i=1}^n = \text{sp}\{\omega_i\}_{i=1}^n \qquad (n = 1, 2, \cdots).$$

Proof. Let $\{\alpha_i\}$ be any complete linearly independent sequence from $R(P)$. In particular, $\{\alpha_i\}$ could be an orthonormal basis for $R(P)$. Define

$$\tau_n = \alpha_n - \sum_{i=1}^{n-1} b_{ni}\alpha_i \;; \tag{2.15}$$

$$\omega_n = \alpha_n - \sum_{i=1}^{n-1} d_{ni}\alpha_i \qquad (n = 2, 3, \cdots), \tag{2.16}$$

(and $\tau_1 = \omega_1 = \alpha_1$) where the coefficients b_{ni} and d_{ni} are determined by requiring that

$$(A\tau_n\,, \alpha_j) = (A^*\omega_n\,, \alpha_j) = 0 \tag{2.17}$$

for $j = 1, \cdots, n - 1$. Rewriting (2.17), we require that for $n > 1$

$$\sum_{i=1}^{n-1} b_{ni}(A\alpha_i\,, \alpha_j) = (A\alpha_n\,, \alpha_j); \tag{2.18}$$

$$\sum_{i=1}^{n-1} d_{ni}(A^*\alpha_i\,, \alpha_j) = (A^*\alpha_n\,, \alpha_j) \tag{2.19}$$

for $j = 1, \cdots, n - 1$. Now the determinant, det $(A\alpha_i\,, \alpha_j)$, is not zero. For if it were zero, there would exist $a_i \in \mathbf{C}$, not all zero, such that

$$\sum_{i=1}^{n-1} a_i(A\alpha_i\,,\alpha_j) = 0$$

for $j = 1, \cdots, n-1$. Then if we set $x = \sum_{i=1}^{n-1} a_i\alpha_i$ and evaluate:

$$(Ax, x) = \sum_{j=1}^{n-1} \bar{a}_j\left[\sum_{i=1}^{n-1} a_i(A\alpha_i\,,\alpha_j)\right] = 0.$$

But this is impossible since Re $A > 0$ and x is not zero. Re $A > 0$ implies Re $A^* > 0$ so that det $(A^*\alpha_i\,,\alpha_j)$ is also not zero. Hence the b_{ni}'s and d_{ni}'s are uniquely determined by the linear systems (2.18) and (2.19). It follows from (2.17) that

$$(A\tau_n\,,\alpha_n) = (A\tau_n\,,\tau_n)$$

which is not zero since $\tau_n \neq 0$ and Re $A > 0$. Finally, set

(2.20) $$\psi_n = (A\tau_n\,,\alpha_n)^{-1}\tau_n \qquad (n = 1, 2, \cdots).$$

It is an easy matter to verify that the ordered system $\{\psi_i\}$, $\{\omega_i\}$ defined by (2.16) and (2.20) is A-biorthogonal and that

$$\text{sp}\{\alpha_i\}_{i=1}^{n} = \text{sp}\{\psi_i\}_{i=1}^{n} = \text{sp}\{\omega_i\}_{i=1}^{n} \qquad (n = 1, 2, \cdots, .$$

Lastly, since $\{\alpha_i\}$ is complete in $R(P)$, we have that

$$\overline{\text{sp}\{\psi_i\}} = R(P).$$

Remark. Suppose A is a densely defined *unbounded* linear operator. One can show that $T_P(A)$ is closed and has a bounded inverse provided PA is closed, $D(AP)$ and $D(A^*P)$ are dense, and Re A, Re $A^* \geqq \delta$ on their respective domains. Now suppose $\{\psi_i\}$, $\{\omega_i\}$ is an A-biorthogonal system ($\psi_i \in D(A)$, $\omega_i \in D(A^*)$) satisfying the hypotheses of Theorem 2.3. The S_n's (2.4) remain uniformly bounded so that the domain of S is a closed set. But does this set contain all of $R(P)$? As in Theorem 2.2 (i), we have

$$T_P(A)\text{sp}\{\psi_i\} \subset D(S).$$

But is $T_P(A)\text{sp}\{\psi_i\}$ dense in $R(P)$ when $\text{sp}\{\psi_i\}$ is? In general the answer is no, but the answer is of course affirmative if $T_P(A)$ is bounded (see the proof of Theorem 2.2 (ii)). This is why the proofs used in [3] are in error when A is unbounded. An incorrect proof was given for the false assertion that if $\text{sp}\{\psi_i\}$ is dense in $R(P)$, then $T_P(A)\text{sp}\{\psi_i\}$ is also.

One way around this difficulty is to simply assume that $T_P(A)\text{sp}\{\psi_i\}$ is itself dense in $R(P)$ and thereby obtain the inversion formula in the unbounded case. But this may be a difficult hypothesis to verify because it requires a knowledge of the range of $T_P(A)$ restricted to a certain given dense set, namely, $\text{sp}\{\psi_i\}$. However, A-biorthogonal systems can always be constructed so that this additional hypothesis is satisfied (provided that $D(AP) \subseteq D(A^*P)$). Start with a sequence $\{\theta_i\} \subset R(AP)$ which is a basis for $\overline{R(AP)}$ and set $\alpha_i = A^{-1}\theta_i$. Then build the ψ_i's and ω_i's from the α_i's as in the proof of Theorem 2.4.

Thus the results of [3] are valid provided the following two additional hypotheses are made: for all $z \in \Omega$ both $PA(z)$ is closed and $T_P(A(z))\text{sp}\{\psi_i(z)\}$ is dense in $R(P)$. This latter condition replaces (3.1) of [3]. The other assumptions made above already hold for the operators considered in [3].

3. The spanning condition. In this section we illustrate the essential character of the spanning condition (2.12) in obtaining the inversion formula (2.7) for $T_P(A)$. Theorem 2.3 states that the inversion formula gives the solution of the Wiener–Hopf equation

$$T_P(A)x = y$$

for all $y \in R(P)$ provided that the bounded linear operator A satisfies:

$$\text{Re } A \geqq \delta, \tag{3.1}$$

and that the ordered system $\{\psi_i\}$, $\{\omega_i\}$ satisfies:

$$(A\psi_i, \omega_j) = \delta_{ij} ; \tag{3.2}$$

$$\overline{\text{sp}\{\psi_i\}} = R(P); \tag{3.3}$$

$$\text{sp}\{\psi_i\}_{i=1}^n = \text{sp}\{\omega_i\}_{i=1}^n \qquad (n = 1, 2, \cdots). \tag{3.4}$$

In applications we can sometimes produce an A-biorthogonal system for which (3.1–3.3) hold. The sequences can arise, for example, as solutions to an eigenvalue problem associated with A (to get $\{\psi_i\}$) and its adjoint (to get $\{\omega_i\}$). Examples of this are given in [4] for positive A. But the spanning condition (3.4) will, in general, not hold for such a system. Theorem 2.2 (ii) says that if (3.1–3.3) hold, the inversion formula gives the Wiener–Hopf inverse on a dense subset of $R(P)$ regardless of whether or not (3.4) holds. But the following result proves that the discrepancy between S and $T_P(A)^{-1}$ can be quite severe without condition (3.4).

Theorem 3.1. *Let P be an orthogonal projection with infinite dimensional range and suppose A satisfies* (3.1). *Then there exists an ordered system satisfying* (3.2, 3.3) *and such that*:

$$Sx = T_P(A)^{-1}x \text{ for } x \text{ in a dense subset of } R(P);$$

$$Sx \neq T_P(A)^{-1}x \text{ for } x \text{ in a dense subset of } R(P).$$

Proof. Since $\text{Re } A \geqq \delta$, $T_P(A)^{-1}$ is a bounded linear operator. Let $\{\alpha_i\}$ be any orthonormal basis for $R(P)$ and set

$$\phi_n = \sum_{i=1}^n 2^{-i}\alpha_i ;$$

$$\psi_n = T_P(A)^{-1}\phi_n ;$$

$$\omega_n = 2^n\alpha_n - 2^{n+1}\alpha_{n+1} \qquad (n = 1, 2, \cdots).$$

From these definitions we see that

$$(A\psi_i , \omega_j) = (T_P(A)\psi_i , \omega_j)$$
$$= (\phi_i , \omega_j) = \delta_{ij} .$$

Moreover, $\text{sp}\{\psi_i\}$ is dense in $R(P)$ since $T_P(A)^{-1}$ is bounded and

$$\text{sp}\{\phi_i\}_{i=1}^n = \text{sp}\{\alpha_i\}_{i=1}^n \qquad (n = 1, 2, \cdots).$$

Therefore, by Theorem 2.2 (ii), $Sx = T_P(A)^{-1}x$ for all x belonging to the dense subset $T_P(A)\text{sp}\{\psi_i\}$ of $R(P)$. For any $y \in R(P)$, set

$$y_k = \sum_{i=1}^{k} (y, \alpha_i)\alpha_i + \sum_{i=k+1}^{\infty} 2^{-i}\alpha_i \qquad (k = 1, 2, \cdots).$$

Notice that $y_k \in R(P)$ and that $y_k \to y$ as $k \to \infty$. Now fix k and observe that if $i > k$, $(y_k , \omega_i) = 0$. Hence $y_k \in D(S)$. It is not difficult to verify that

$$\sum_{i=1}^{\infty} (y_k , \omega_i)\phi_i = y_k - \sum_{i=1}^{\infty} 2^{-i}\alpha_i ,$$

and therefore that

$$T_P(A)Sy_k = \sum_{i=1}^{\infty} (y_k , \omega_i)\phi_i \neq y_k$$

for all k. We conclude that the set of $x \in D(S) \cap R(P)$ for which $Sx \neq T_P(A)^{-1}x$ forms a dense subset of $R(P)$. This completes the proof.

Theorem 2.2 (iii) says that the discrepency between S and $T_P(A)^{-1}$ in the preceding result is due to the fact that $\text{sp}\{\omega_i\}$ isn't dense in $R(P)$. That $\overline{\text{sp}\{\omega_i\}} \neq R(P)$ is easy to see since $\sum_{n=1}^{\infty} 2^{-n}\alpha_n$ is a non-zero element of $R(P)$ which is orthogonal to ω_i for all i. We could remedy this difficulty quite simply, not by requiring condition (3.4), but rather the weaker and very reasonable condition that

$$\overline{\text{sp}\{\omega_i\}} = R(P). \tag{3.5}$$

With this assumption and (3.1–3.3) we know that S and $T_P(A)^{-1}$ agree on all of the dense subset $D(S) \cap R(P)$ of $R(P)$. But condition (3.5) is still not strong enough to yield a viable inversion formula.

Theorem 3.2. *Let P be an orthogonal projection with infinite dimensional range and suppose A satisfies* (3.1). *Then there exists an ordered system* $\{\psi_i\}$, $\{\omega_i\}$ *satisfying* (3.2, 3.3, 3.5) *and such that:*

$Sx = T_P(A)^{-1}x$ for x in a dense subset of $R(P)$;

Sx is undefined for x in a dense subset of $R(P)$.

Proof. Let $\{\alpha_i\}$ be an orthonormal basis for $R(P)$ and set

$$\phi_n = \sum_{j=1}^{n} j^{-2}\alpha_j ;$$

$$\psi_n = T_P(A)^{-1}\phi_n ;$$
$$\omega_n = n^2\alpha_n - (n+1)^2\alpha_{n+1} \qquad (n = 1, 2, \cdots).$$

The ordered system $\{\psi_i\}$, $\{\omega_i\}$ thus defined satisfies (3.2) and (3.3). Moreover, $\mathrm{sp}\{\omega_i\}$ is dense in $R(P)$. For if $x \in R(P)$ is orthogonal to ω_n for all n, then

$$(x, \alpha_{n+1}) = n^2(n+1)^{-2}(x, \alpha_n).$$

This says x must equal zero since

$$||x||^2 = \sum_{n=1}^{\infty} |(x, \alpha_n)|^2 < \infty .$$

Therefore, by Theorem 2.2, $Sx = T_P(A)^{-1}x$ on the dense subset $D(S) \cap R(P)$ of $R(P)$.

We now show that the set of elements in $R(P)$ and in the complement of $D(S)$ is also dense in $R(P)$. Let x be any element in $R(P)$ and set

$$x_k = \sum_{j=1}^{k} (x, \alpha_j)\alpha_j + \sum_{j=k+1}^{\infty} j^{-1}\alpha_j \qquad (k = 1, 2, \cdots).$$

Observe that $x_k \in R(P)$ and that $x_k \to x$ as $k \to \infty$. An easy computation reveals that

$$\sum_{i=1}^{n} (x, \omega_i)\phi_i = \sum_{i=1}^{n} (x, \alpha_i)\alpha_i - (n+1)^2(x, \alpha_{n+1})\phi_n .$$

Therefore, for k fixed,

$$T_P(A)S_n x_k = \sum_{i=1}^{n} (x_k , \alpha_i)\alpha_i - (n+1)^2(x_k , \alpha_{n+1})\phi_n .$$

It follows that for n large enough $(n > k)$

$$||T_P(A)||\ ||S_n x_k|| \geqq (n+1)\ ||\phi_n|| - \left|\left|\sum_{i=1}^{n} (x_k , \alpha_i)\alpha_i\right|\right|$$
$$\geqq (n+1) - ||x_k||.$$

We conclude that $x_k \notin D(S)$ for all k and this gives the desired result.

In general it is no easy matter to determine the domain of S. Therefore, if the inversion formula is to be of practical use, we must insist that the spanning condition (3.4) as well as (3.1–3.3) hold. We thereby obtain an inversion formula which is valid everywhere.

4. Complete biorthogonal systems. A pair of sequences $\{\phi_i\}$, $\{\theta_i\}$ from H is called a *biorthogonal system* if

$$(\phi_i , \theta_j) = \delta_{ij} .$$

Such a system is said to be complete if one of the sequences is complete, that is, if the set of all finite linear combinations of elements from that sequence is dense in H. A biorthogonal system is called a *basis* if one of the sequences is a basis for H, that is, if every element in H can be expressed uniquely as a linear combination (perhaps infinite) of elements from that sequence.

Theorem 4.1. *If* $\{\phi_i\}$, $\{\theta_i\}$ *is a complete biorthogonal system and if there exists a bounded linear operator* B *satisfying* $\operatorname{Re} B \geqq \delta$ *and such that*

$$\operatorname{sp}\{B\phi_i\}_{i=1}^{n} = \operatorname{sp}\{\theta_i\}_{i=1}^{n} \qquad (n = 1, 2, \cdots),$$

then the system is a basis.

Proof. B has a bounded inverse since $\operatorname{Re} B \geqq \delta$. Using the Schwarz inequality it is easy to verify that $\operatorname{Re} B^{-1} \geqq \delta \|B\|^{-2}$. Now set $\psi_i = B\phi_i$, $\omega_i = \theta_i$, and $A = B^{-1}$. Observe that the operator A and the ordered system $\{\psi_i\}$, $\{\omega_i\}$ satisfy the hypotheses of Theorem 2.3 with P equal to the identity. Hence we have a series expansion for A^{-1}:

$$A^{-1}x = \sum_{i=1}^{\infty} (x, \omega_i)\psi_i .$$

But since B has a bounded inverse, this becomes

$$x = \sum_{i=1}^{\infty} (x, \theta_i)\phi_i$$

for *all* x in H. That this representation is unique follows easily from the biorthogonality of $\{\phi_i\}$ to $\{\theta_i\}$.

Recalling the proof of Theorem 2.3, we see that the above proof is based on showing that the finite sums,

$$\sum_{i=1}^{n} (x, \theta_i)\phi_i ,$$

are uniformly bounded in n for each x in H. This is the uniform boundedness condition of S. Banach for the system $\{\phi_i\}$, $\{\theta_i\}$ to be a basis.

It should be pointed out that a biorthogonal system need not be a basis even if both sequences in the system are complete. Moreover, there exist biorthogonal systems in which one sequence is complete while the other is not. Examples of these remarks are contained in the proofs of Theorems 3.2 and 3.1 respectively (consider the systems $\{\phi_i\}$, $\{\omega_i\}$). These facts are known to hold even in a Banach space setting [5]. Also, one can show that if a biorthogonal system is a basis, then both sequences in the system are bases so that every x in H has the unique expansions:

$$x = \sum_{i=1}^{\infty} (x, \theta_i)\phi_i = \sum_{i=1}^{\infty} (x, \phi_i)\theta_i .$$

The most obvious case in which a complete biorthogonal system is a basis occurs when $\phi_i = \theta_i$ for all i so that the system reduces to the usual complete orthonormal sequence. This happens when we take B equal to the identity in Theorem 4.1. If $\{\phi_i\}$ is "close enough" to a complete orthonormal sequence, then there exists a sequence $\{\theta_i\}$ biorthogonal to $\{\phi_i\}$ and such that the system $\{\phi_i\}$, $\{\theta_i\}$ is a basis. There are a number of "near orthogonality" results of this kind which comprise the so called stability theorems of Paley–Wiener type. These results, together with several others which are very abstract, are the only known sufficient conditions for when a complete biorthogonal system is a basis. They can all be proved in a Banach space setting [5]. Although a version of Theorem 4.1 may be stated for Banach spaces, our proof fails to generalize. The difficulty is that a sequence of projections in a Banach space need not be uniformly bounded in norm so that step (2.14) of Theorem 2.3 need not be valid.

Theorem 4.1 can be used to prove the original expansion theorem of Paley and Wiener for Hilbert spaces [1, p. 100]. The author wishes to thank M. Shinbrot for communicating this fact to him.

Corollary 4.2. *(Paley–Wiener). Let $\{\alpha_i\}$ be a complete orthonormal sequence and suppose $\{\phi_i\}$ is a sequence such that for every finite sequence $\{a_i\}$ of complex numbers*

$$||\sum a_i(\alpha_i - \phi_i)||^2 \leqq \theta \sum |a_i|^2, \qquad 0 \leqq \theta < 1. \tag{4.1}$$

Then there exists a sequence $\{\theta_i\}$ biorthogonal to $\{\phi_i\}$ and such that the system $\{\phi_i\}$, $\{\theta_i\}$ is a basis.

Proof. As in Riesz–Nagy [2, p. 209], define an operator K by

$$Kx = \sum_{i=1}^{\infty} (x, \alpha_i)(\alpha_i - \phi_i),$$

wherever the series converges. Taking $a_i = (x, \alpha_i)$ for any x in H, (4.1) implies that Kx is well defined and that $||Kx|| \leqq \theta\, ||x||$, $0 \leqq \theta < 1$. Hence the operator $T = I - K$ is bounded and has a bounded inverse. Notice that

$$Tx = \sum_{i=1}^{\infty} (x, \alpha_i)\phi_i\ ,$$

so that $T\alpha_i = \phi_i$. Now define $B = (TT^*)^{-1}$ and $\theta_i = B\phi_i$. The system $\{\phi_i\}$, $\{\theta_i\}$ is biorthogonal since

$$\begin{aligned}(\phi_i\ , \theta_j) &= (T\alpha_i\ , (TT^*)^{-1}T\alpha_j) \\ &= (\alpha_i\ , \alpha_j) = \delta_{ij}\ .\end{aligned}$$

Moreover, $\{\phi_i\}$ is complete, for if $(x, \phi_i) = 0$ for all i, then $(x, T\alpha_i) = (T^*x, \alpha_i) = 0$. Since $\{\alpha_i\}$ is complete, $T^*x = 0$. Therefore, $x = 0$.

The spanning hypothesis of Theorem 4.1 obviously holds since $\theta_i = B\phi_i$. Also, Re $B \geqq \delta$. In fact,

$$(Bx, x) = ||T^{-1}x||^2 \geqq ||T||^{-2}\, ||x||^2.$$

The system $\{\phi_i\}$, $\{\theta_i\}$ is therefore a basis by Theorem 4.1.

If we ignore the existence part of the expansion theorem of Paley and Wiener, then as remarked earlier, the theorem is a "near orthogonality" result, (4.1) being a sufficient condition for a complete biorthogonal system to be a basis. When viewed in this way, Corollary 4.2 shows that Theorem 4.1 is a natural generalization of the expansion theorem of Paley and Wiener.

There exist an abundance of complete biorthogonal systems which are not necessarily "close" to a given complete orthonormal sequence, but which do satisfy the hypotheses of Theorem 4.1. They are therefore bases.

Theorem 4.3. *Let B be any bounded linear operator such that* Re $B \geqq \delta$. *There exists a complete biorthogonal system* $\{\phi_i\}$, $\{\theta_i\}$ *such that*

$$\operatorname{sp}\{B\phi_i\}_{i=1}^n = \operatorname{sp}\{\theta_i\}_{i=1}^n \qquad (n = 1, 2, \cdots). \tag{4.2}$$

Proof. Re $B \geqq \delta$ implies that B has a bounded inverse and that Re $B^{-1} \geqq \delta \, \|B\|^{-2}$. By Theorem 2.4 with $A = B^{-1}$ and P equal to the identity, we get an A-biorthogonal system $\{\psi_i\}$, $\{\omega_i\}$ such that $\{\psi_i\}$ is complete and

$$\operatorname{sp}\{\psi_i\}_{i=1}^n = \operatorname{sp}\{\omega_i\}_{i=1}^n \qquad (n = 1, 2, \cdots). \tag{4.3}$$

Now set $\phi_i = B^{-1}\psi_i$ and $\theta_i = \omega_i$ and observe that $\{\phi_i\}$, $\{\theta_i\}$ forms a complete biorthogonal system. Moreover (4.2) is simply a restatement of (4.3).

5. Questions. Below are some open questions listed in the order of the section to which they refer.

§2. Can our results be generalized to unbounded operators without the assumption that $T_P(A)\operatorname{sp}\{\psi_i\}$ is dense in $R(P)$?

§3. Is the spanning condition (3.4) necessary to obtain the inversion formula (2.7) within the smaller class of "analytic" formulas considered in [3]? Can the spanning condition be replaced by something weaker which will hold for systems arising from solutions to eigenvalue problems (see [4])?

§4. Can a meaningful condition on a complete biorthogonal system be given in order that there exist an operator B satisfying the hypotheses of Theorem 4.1? Does such a B exist if the system is a basis? Does Theorem 4.1 generalize to Banach spaces?

References

1. R. E. A. C. Paley & N. Wiener, *Fourier transforms in the complex domain,* A. M. S. Colloq. Publ. V. **19**, New York, 1934.
2. F. Riesz & B. S. Nagy, *Functional analysis,* F. Ungar Publ. Co., New York, 1955.
3. M. Shinbrot, *An inversion formula for certain general Wiener–Hopf operators,* Arch. Rational Mech. and Anal. **37** (1970), 344–362.
4. M. Shinbrot, *The solution of some integral equations of Wiener–Hopf type,* Quart. Appl. Math. **28** (1970), 15–36.
5. I. Singer, *Bases in Banach spaces I,* Grundl. d. Math. Wiss. **154**, Springer–Verlag, New York, 1970.

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