

Generalised Calogero-Moser Models and Universal Lax Pair Operators

A. J. BORDNER, E. CORRIGAN^{*,*)} and Ryu SASAKI

*Yukawa Institute for Theoretical Physics, Kyoto University
Kyoto 606-8502, Japan*

**Department of Mathematical Sciences, University of Durham
South Road, Durham DH1-3LE, United Kingdom*

(Received May 13, 1999)

Calogero-Moser models can be generalised for all of the finite reflection groups. These include models based on non-crystallographic root systems, that is the root systems of the finite reflection groups, H_3 , H_4 , and the dihedral group $I_2(m)$, besides the well-known ones based on crystallographic root systems, namely those associated with Lie algebras. Universal Lax pair operators for all of the generalised Calogero-Moser models and for any choices of the potentials are constructed as linear combinations of the reflection operators. The consistency conditions are reduced to functional equations for the coefficient functions of the reflection operators in the Lax pair. There are only four types of such functional equations corresponding to the two-dimensional sub-root systems, A_2 , B_2 , G_2 , and $I_2(m)$. The root type and the minimal type Lax pairs, derived in our previous papers, are given as the simplest representations. The spectral parameter dependence plays an important role in the Lax pair operators, which bear a strong resemblance to the Dunkl operators, a powerful tool for solving quantum Calogero-Moser models.

§1. Introduction

Generalised Calogero-Moser models are integrable many-particle dynamical systems based on finite reflection groups. Finite reflection groups include the dihedral groups $I_2(m)$ and H_3 and H_4 together with the Weyl groups of the root systems associated with Lie algebras, called crystallographic root systems. Integrability of classical Calogero-Moser models based on the crystallographic root systems^{1), 2)} is shown in terms of Lax pairs. The root and the minimal type Lax pairs derived in our previous papers³⁾ provide a universal framework for these Calogero-Moser models, including those based on exceptional root systems and the twisted models. On the other hand, a theory of classical integrability for the models based on non-crystallographic root systems has been virtually non-existent. This is in sharp contrast with the quantum counterpart. Dunkl operators, which are useful for solving quantum Calogero-Moser models, were first explicitly constructed for the models based on the dihedral groups.⁴⁾

In this paper we present a Lax pair in an operator form for generalised Calogero-Moser models, which applies universally to the models based on non-crystallographic root systems as well as those based on crystallographic ones. This Lax pair, as

^{*)} Address after October 1 1999: Department of Mathematics, University of York, Heslington, York YO10 5DD, United Kingdom.

expected, bears a strong resemblance to the Dunkl operators^{4), 5)} and the reflection operators play a central role. The spectral parameter dependence is also essential, in contradistinction to the auxiliary role in the conventional formalism.²⁾ When suitable representation spaces are chosen, the universal Lax pair reproduces the root type and the minimal type Lax pairs for the models based on the crystallographic root systems. This provides another clue that the quantum and classical integrability of the generalised Calogero-Moser models are closely connected. We hope that this Lax pair operator formalism will cross-fertilise the fruitful subject of the quantum and classical Calogero-Moser systems and related ones such as Toda systems.

For the general background and the motivations of this paper and the physical applications of the Calogero-Moser models with various potentials to lower-dimensional physics, ranging from solid state to particle physics and supersymmetric gauge theories, we refer to our previous papers³⁾ and references therein.

This paper is organised as follows. In §2 we summarise the basic concepts of the finite reflection groups in order to set the stage and to introduce appropriate notation. In §3 the generalised Calogero-Moser models are defined with various choices of root systems and potentials. Section 4 is the main body of the paper. The Lax operators L and M are defined as a linear combination of the reflection operators \hat{s}_ρ for all the roots ρ . The coefficient functions depend on the dynamical variables and on the spectral parameter in a rather symmetrical way. Consistency of the Lax pair can be proved quite easily. The Lax equation is at most quadratic in the reflection operators, $\hat{s}_\rho\hat{s}_\sigma$, which becomes an identity operator for $\rho = \sigma$, $\hat{s}_\rho^2 = 1$, and a two-dimensional rotation operator for $\rho \neq \sigma$. The linear (\hat{s}_ρ^1) and the constant (\hat{s}_ρ^0) parts give the canonical equations of motion of the generalised Calogero-Moser models. The quadratic part $\hat{s}_\rho\hat{s}_\sigma, \rho \neq \sigma$, imposes the consistency conditions, which are decomposed into those corresponding to two-dimensional sub-root systems containing ρ and σ . As shown in Table I, there are only four types of two-dimensional root systems, A_2 , B_2 , G_2 , and $I_2(5)$ for all of the Coxeter groups except for the dihedral group $I_2(m)$ which can have many for some values of m . Thus the functions appearing in the Lax pair operator (except for those for $I_2(m)$) need to satisfy at most two functional equations. The solutions are derived in the Appendix. In §5 various possible representations of the Lax pair operator are discussed. The minimal type and the root type Lax pairs are derived as two simplest examples in §6. Various sum rules utilised in previous papers are derived as restrictions of the general functional equations derived in §4. Some comments and discussion are given in §7. Details of the derivation of the solutions are relegated to the Appendix. Two types of solutions, the untwisted and twisted solutions, are derived. The consistency conditions of all of the untwisted solutions are attributed to one functional identity, (A-12), of the Weierstrass σ function. Twisted solutions are obtained as proper linear combinations of the untwisted ones.

§2. Root systems and finite reflection groups

We now review some facts about root systems and their reflection groups in order to introduce notation.⁶⁾ We consider only reflections in Euclidean space.

A root system Δ of rank r is a set of vectors in \mathbf{R}^r which is invariant under reflections in the hyperplane perpendicular to each vector in Δ . In other words,

$$s_\alpha(\beta) \in \Delta, \quad \forall \alpha, \beta \in \Delta, \tag{2.1}$$

in which

$$s_\alpha(\beta) = \beta - 2(\alpha \cdot \beta / |\alpha|^2)\alpha. \tag{2.2}$$

Dual roots are defined by $\alpha^\vee = 2\alpha/|\alpha|^2$, in terms of which

$$s_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha. \tag{2.3}$$

The orbit of $\beta \in \Delta$ is the set of root vectors resulting from the action of the reflections on it $\{s_\alpha(\beta), \alpha \in \Delta\}$. The set of positive roots Δ_+ may be defined in terms of a vector $V \in \mathbf{R}^r$, with $V \cdot \alpha \neq 0, \forall \alpha \in \Delta$, as those roots $\alpha \in \Delta$ such that $\alpha \cdot V > 0$. A unique set of r simple roots Π is defined such that they span the root space and the coefficients $\{a_j\}$ in $\beta = \sum_{j=1}^r a_j \alpha_j$ for $\beta \in \Delta_+, \{\alpha_j \in \Pi, j = 1, \dots, r\}$ are all positive.

The set of reflections $\{s_\alpha, \alpha \in \Delta\}$ generates a group, known as a Coxeter group. It is generated by products of s_α with $\alpha \in \Pi$ subject to the relations

$$(s_\alpha s_\beta)^{m(\alpha, \beta)} = 1, \quad \alpha, \beta \in \Pi. \tag{2.4}$$

The set of positive integers $m(\alpha, \beta)$ uniquely specify the Coxeter group with $m(\alpha, \alpha) = 1, \forall \alpha \in \Pi$. For example, for two-dimensional crystallographic root systems A_2, B_2 , and G_2 , the integer $m(\alpha, \beta)$ is 3, 4, and 6, respectively. Thus $s_\alpha s_\beta$ is a two-dimensional rotation by an angle $\pm 2\pi/3, \pm\pi/2$, and $\pm\pi/3$, respectively. This fact will be used in later sections. We consider here only those Coxeter groups with a finite number of roots in Euclidean space, called the finite reflection groups.

The root systems for finite reflection groups may be divided into two types: crystallographic and non-crystallographic root systems. Crystallographic root systems satisfy the additional condition

$$\alpha^\vee \cdot \beta \in \mathbf{Z}, \quad \forall \alpha, \beta \in \Delta. \tag{2.5}$$

These root systems are associated with simple Lie algebras: $\{A_r, r \geq 1\}, \{B_r, r \geq 2\}, \{C_r, r \geq 2\}, \{D_r, r \geq 4\}, E_6, E_7, E_8, F_4$, and G_2 and $\{BC_r, r \geq 2\}$. The Coxeter groups for these root systems are called Weyl groups. The remaining non-crystallographic root systems are H_3, H_4 , and the dihedral group of order $2m, \{I_2(m), m \geq 4\}$.

Weyl chambers are defined as the open subsets of \mathbf{R}^r that result from removing the reflection hyperplanes $H_\alpha, \alpha \in \Delta, H_\alpha \equiv \{q \in \mathbf{R}^r, \alpha^\vee \cdot q = 0\}$. The action of the reflection group on the Weyl chambers is transitive and free, i.e., any Weyl chamber may be obtained from another by the action of an element of the reflection group and this element is unique. The principal Weyl chamber is defined as the one whose points q satisfy $q \cdot \alpha > 0, \forall \alpha \in \Delta_+$. For crystallographic root systems this implies that all points in the principal Weyl chamber have positive Dynkin labels.

Definitions and properties of the crystallographic root systems may be found in many references, see for example.⁷⁾ The concept of weights may be defined for these root systems. For crystallographic root systems, any positive root is a sum of simple roots in Π with positive integer coefficients. The set of weights Σ , which lie on a lattice, is the set of vectors such that if $\Lambda \in \Sigma$ then $\alpha^\vee \cdot \Lambda$ is an integer for any $\alpha \in \Delta$. Fundamental weights $\Lambda^{(j)}$ are vectors which form a dual basis to the corresponding dual simple roots α_j^\vee , i.e., $\alpha_j^\vee \cdot \Lambda^{(k)} = \delta_{jk}$, $j, k = 1, \dots, r$. Any weight $\Psi \in \Sigma$ may be expressed as a sum of fundamental weights with integer coefficients, $\Psi = \sum_{j=1}^r a_j \Lambda^{(j)}$. The coefficients $\{a_j, j = 1, \dots, r\}$ are called the Dynkin labels of Ψ .

We now briefly describe the non-crystallographic root systems. The dihedral group of order $2m$, $I_2(m)$, is the group of orthogonal transformations that preserve a regular m -sided polygon in two dimensions. It consists of m rotations (through multiples of $2\pi/m$) and m reflections. The angle between adjacent roots is π/m and a possible basis for the roots, if all are chosen to have the same length $|\alpha_j|^2 = 1$, is

$$\alpha_j = (\cos(j\pi/m), \sin(j\pi/m)), \quad j = 1, \dots, 2m. \quad (2.6)$$

For odd m all of the roots are in the same orbit of the reflection group but for even m there are two orbits, one consisting of the α_j with odd j and the other with even j . This then allows two different coupling constants and potential functions for the $I_2(m)$ Calogero-Moser model for even m .

The reflection group of type H_4 is the symmetry group of a regular 120-sided solid, with dodecahedral faces, in \mathbf{R}^4 . It is a group of order 14400. The group of type H_3 is a subgroup of H_4 and is the symmetry group of the icosahedron (with 20 faces) in \mathbf{R}^3 . It is a group of order 120. Define

$$a \equiv \cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}, \quad b \equiv \cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}. \quad (2.7)$$

Then a choice of simple roots for H_4 is the following:

$$\begin{aligned} \alpha_1 &= \left(a, -\frac{1}{2}, b, 0 \right), & \alpha_2 &= \left(-a, \frac{1}{2}, b, 0 \right), \\ \alpha_3 &= \left(\frac{1}{2}, b, -a, 0 \right), & \alpha_4 &= \left(-\frac{1}{2}, -a, 0, b \right). \end{aligned} \quad (2.8)$$

The full set of roots of H_4 in this basis may be obtained from $(1, 0, 0, 0)$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and $(a, \frac{1}{2}, b, 0)$ by even permutations and arbitrary sign changes of coordinates. These 120 roots form a single orbit. A subset of (2.8), $\{\alpha_1, \alpha_2, \alpha_3\}$ is a choice of simple roots for the H_3 root system. In this basis, the full set of roots for H_3 results from even permutations and arbitrary sign changes of $(1, 0, 0)$ and $(a, \frac{1}{2}, b)$. These 30 roots also form a single orbit.

§3. Generalised Calogero-Moser models

A generalised Calogero-Moser model is a Hamiltonian system associated with a root system Δ of rank r . Quantum versions of these models are also integrable,

at least for certain choices of Δ and potential function.⁸⁾ The dynamical variables are the coordinates $\{q^j\}$ and their canonically conjugate momenta $\{p_j\}$, with the Poisson brackets

$$\{q^j, p_k\} = \delta_k^j, \quad \{q^j, q^k\} = \{p_j, p_k\} = 0, \quad j, k = 1, \dots, r. \quad (3.1)$$

These will be denoted by vectors in \mathbf{R}^r

$$q = (q^1, \dots, q^r), \quad p = (p_1, \dots, p_r). \quad (3.2)$$

The Hamiltonian for the Calogero-Moser model is

$$\mathcal{H} = \frac{1}{2}p^2 + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} V_{|\alpha|}(\alpha \cdot q), \quad (3.3)$$

in which the real coupling constants $g_{|\alpha|}$ and potential functions $V_{|\alpha|}$ are defined on orbits of the corresponding finite reflection group, i.e., they are identical for roots in the same orbit. This then ensures that the Hamiltonian is invariant under reflections of the phase space variables about a hyperplane perpendicular to any root

$$q \rightarrow s_\alpha(q), \quad p \rightarrow s_\alpha(p), \quad \forall \alpha \in \Delta \quad (3.4)$$

with s_α defined by (2.3).

The Lax pair operator that we will construct in later sections will apply for the following potentials ($g_{|\alpha|} = g$ for all roots in simply laced models and $g_{|\alpha|} = g_L$ for long roots and $g_{|\alpha|} = g_S$ for short roots in non-simply laced models):

1. *Untwisted elliptic potential* This applies to all of the crystallographic root systems and the potential function is

$$V_{|\alpha|}(\alpha \cdot q) = \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), \quad \text{for all roots,} \quad (3.5)$$

in which \wp is the Weierstrass \wp function with a pair of primitive periods $\{2\omega_1, 2\omega_3\}$. Throughout this paper we adopt the convention that the Weierstrass \wp , ζ , and σ functions have the above standard periods, unless otherwise stated.

2. *Twisted elliptic potential* This applies to all of the non-simply laced root systems. Except for the G_2 model, the potential functions are

$$V_{|\alpha|}(\alpha \cdot q) = \begin{cases} \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), & \text{for long roots,} \\ \wp(\alpha \cdot q | \{\omega_1, 2\omega_3\}), & \text{for short roots.} \end{cases} \quad (3.6)$$

That is, the potential for short roots has one half of the standard period in one direction, which we choose to be ω_1 . For the G_2 model,

$$V_{|\alpha|}(\alpha \cdot q) = \begin{cases} \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), & \text{for long roots,} \\ \wp(\alpha \cdot q | \{\frac{2\omega_1}{3}, 2\omega_3\}), & \text{for short roots.} \end{cases} \quad (3.7)$$

In this case the potential for short roots has one third of the standard period in one direction, which we choose to be ω_1 . The cases of BC_r system and the extended B_r , C_r , and BC_r systems will be discussed separately in later sections.

3. *Trigonometric and hyperbolic potentials* This applies to all crystallographic systems and the potential functions are

$$V_{|\alpha|}(\alpha \cdot q) = \begin{cases} a^2/\sin^2 a(\alpha \cdot q), \\ a^2/\sinh^2 a(\alpha \cdot q), \end{cases} \quad \text{for all roots,} \quad (3.8)$$

in which a is an arbitrary real constant.

4. *Rational potential* This applies to all of the generalised Calogero-Moser models including those based on the dihedral group $I_2(m)$, H_3 , and H_4 and the potential function is

$$V_{|\alpha|}(\alpha \cdot q) = \frac{1}{(\alpha \cdot q)^2}, \quad \text{for all roots.} \quad (3.9)$$

These models are also integrable if a confining harmonic potential

$$\frac{1}{2}\omega^2 q^2 \quad (3.10)$$

is added to the Hamiltonian.

Some remarks are in order. (i) For all of the root systems and for any choice of the potential, the generalised Calogero-Moser model has a hard repulsive potential $\sim \frac{1}{(\alpha \cdot q)^2}$ near the reflection hyperplane $H_\alpha = \{q \in \mathbf{R}^r, \alpha^\vee \cdot q = 0\}$. This repulsion potential is classically insurmountable. Thus the motion is always confined within one Weyl chamber. In other words, the spatial ordering of the particles is unchanged during the time evolution. This simplifying feature is a basic cornerstone of the solvability. The coupling constants $g_{|\alpha|}^2$ (with a scale $1/|\alpha|^2$) are measures of the strength of the repulsive potentials. (ii) The trigonometric, hyperbolic, and the rational potentials are obtained from the elliptic potential as one or both periods tend to infinity. The Lax pairs for these degenerate potentials can be obtained from the one for the elliptic potential by taking the corresponding limit. Thus we do not write down the Lax pairs for degenerate potential cases except for the models based on the non-crystallographic root systems, for which only the rational potentials are integrable. (iii) For all of the Lax pairs based on any root systems and any choice of the potential, except for the rational potential with the harmonic confining potential (3.10), one can introduce an additional complex parameter ξ (*spectral parameter*), which appears in the equation for the spectral curve.^{9),10)} (iv) Independent conserved quantities $\text{Tr}(L^k)$ to be obtained from a Lax equation $\dot{L} = [L, M]$ occur at such $k = 1 + \text{exponent}$ of the corresponding crystallographic root systems. For the non-crystallographic root systems, they arise at $k = 2, m$ for the dihedral group $I_2(m)$, $k = 2, 6, 10$ for H_3 and $k = 2, 12, 20, 30$ for H_4 . These are the degrees at which Coxeter invariant polynomials exist.⁶⁾

§4. Lax pair and functional equations

Here we construct a Lax pair for the generalised Calogero-Moser models in an operator form acting on an as yet unspecified vector space and derive the necessary

and sufficient conditions for the consistency of the Lax equations. The Lax pair (4.5) contains operators as well as functions $x_{|\rho|}(u, w)$, $y_{|\rho|}(u, w)$, which are related to the chosen potential (3.5)–(3.9). The consistency of the Lax pair requires that $x_{|\rho|}(u, w)$ satisfy certain functional equations (4.27) and (A.7), which are closely related with those required for the commutativity of Dunkl operators.^{4),5)} Verification that the solutions (4.34), (4.37), (4.38), (4.41), (4.44) and (4.45) satisfy the functional equations will be presented in the Appendix.

The operators appearing in the Lax pair for a generalised Calogero-Moser model associated with a root system Δ are naturally the reflection operators $\{\hat{s}_\alpha, \alpha \in \Delta\}$ of the root system. They act on a set of \mathbf{R}^r vectors $\Gamma = \{\mu^{(k)} \in \mathbf{R}^r, k = 1, \dots\}$, which is closed under the action of the reflection group. The totality of the vectors in Γ forms the representation space \mathbf{V} . A general construction of the representation space, and some explicit cases will be presented in the subsequent two sections. Another set of operators $\{\hat{H}_j, j = 1, \dots, r\}$ is necessary. If \hat{H}_j acts on a vector $\mu^{(k)} \in \Gamma$, the j -th component is returned:

$$\hat{H}_j \mu^{(k)} = \mu_j^{(k)} \mu^{(k)}.$$

These form the following operator algebra:

$$[\hat{H}_j, \hat{H}_k] = 0, \tag{4.1}$$

$$[\hat{H}_j, \hat{s}_\alpha] = \alpha_j (\alpha^\vee \cdot \hat{H}) \hat{s}_\alpha, \tag{4.2}$$

$$\hat{s}_\alpha \hat{s}_\beta \hat{s}_\alpha = \hat{s}_{s_\alpha(\beta)}, \tag{4.3}$$

$$(\hat{s}_\alpha \hat{s}_\beta)^{m(\alpha, \beta)} = 1. \tag{4.4}$$

The first relation (4.1) implies that the operators $\{\hat{H}_j, j = 1, \dots, r\}$ form an abelian subalgebra and relations (4.3) and (4.4) are just those for the finite reflection group associated with the root system Δ . The set of integers $m(\alpha, \beta)$ are those appearing in the Coxeter relations (2.4) which characterise the reflection group.

Next we describe the Lax pair and the corresponding Hamiltonian for the generalised Calogero-Moser model for the root system Δ . The Lax operators are

$$\begin{aligned} L &= p \cdot \hat{H} + X, & X &= i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho^\vee \cdot \hat{H}) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) \hat{s}_\rho, \\ M &= i \sum_{\rho \in \Delta_+} g_{|\rho|} y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) \hat{s}_\rho. \end{aligned} \tag{4.5}$$

The function $y_{|\rho|}$ is defined by

$$y_{|\rho|}(u, w) \equiv \frac{\partial}{\partial u} x_{|\rho|}(u, w). \tag{4.6}$$

The variable u takes care of the dynamical variable dependence and w is for the spectral parameter dependence. Furthermore, it is required that $x_{|\rho|}(u, w)$ is odd:

$$x_{|\rho|}(-u, -w) = -x_{|\rho|}(u, w) \tag{4.7}$$

so that L and M are independent of the choice of positive roots Δ_+ . This also implies that the sums in (4.5) may be extended to a sum over all roots if an additional factor of $1/2$ is included in front of the sums since the summands are even under $\rho \rightarrow -\rho$. The function $x_{|\rho|}(u, w)$ is assumed to have a simple pole at $u = 0$ with unit residue

$$\lim_{u \rightarrow 0} u x_{|\rho|}(u, w) = 1. \tag{4.8}$$

This condition is related with the unit strength of the repulsive potential near the reflection hyperplane mentioned earlier.

Before proving the consistency of this Lax pair we calculate the Hamiltonian and find the corresponding equations of motion. The Hamiltonian for the theory is defined in terms of a representation of the operator L of (4.5) by

$$\mathcal{H} = \frac{1}{2C_R} \text{Tr}(L^2), \tag{4.9}$$

where the constant C_R , which depends on the representation, is defined by

$$\text{Tr}(\hat{H}_j \hat{H}_k) = C_R \delta_{jk}. \tag{4.10}$$

The resulting Hamiltonian is then

$$\mathcal{H} = \frac{1}{2} p^2 + \sum_{\rho \in \Delta} \frac{g_{|\rho|}^2}{|\rho|^2} V_{|\rho|}(\rho \cdot q) + C, \tag{4.11}$$

in which C is independent of the dynamical variables q and p and therefore unimportant for the classical theory. The potential functions are given in (3.5)–(3.9). The function $x_{|\rho|}$ and the potential function $V_{|\rho|}$ are related (except for the confining harmonic potential (3.10), which has to be added separately) simply as

$$x_{|\rho|}(u, w)x_{|\rho|}(-u, w) = -V_{|\rho|}(u) + C_{|\rho|}(w), \tag{4.12}$$

i.e., the product gives a sum of a function of only u and a function of only w . It is easy to show that all of the functions $x_{|\rho|}(u, w)$ which lead to a consistent Lax equation, (4.34), (4.37), (4.38), (4.41), (4.44) and (4.45), satisfy this property.

The equations of motion following from this Hamiltonian are

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} = p_j, \tag{4.13}$$

$$\begin{aligned} \dot{p}_j &= -\frac{\partial \mathcal{H}}{\partial q_j} \\ &= -\frac{\partial}{\partial q_j} \left[\sum_{\rho \in \Delta} \frac{g_{|\rho|}^2}{|\rho|^2} V_{|\rho|}(\rho \cdot q) \right] \\ &= \sum_{\rho \in \Delta} \frac{g_{|\rho|}^2}{|\rho|^2} \rho_j \left[y_{|\rho|}(\rho \cdot q, w)x_{|\rho|}(-\rho \cdot q, w) - x_{|\rho|}(\rho \cdot q, w)y_{|\rho|}(-\rho \cdot q, w) \right]. \end{aligned} \tag{4.14}$$

Because of (4.12) the last expression in (4.14) is independent of w .

The Lax equation

$$\dot{L} = [L, M] \tag{4.15}$$

may be divided into three parts as

$$\frac{d}{dt}X = [p \cdot \hat{H}, M], \tag{4.16}$$

$$\frac{d}{dt}(p \cdot \hat{H}) = [X, M]_{\text{diagonal}}, \tag{4.17}$$

$$0 = [X, M]_{\text{off-diagonal}}. \tag{4.18}$$

We next prove, in turn, that each of these equations is consistent with the equations of motion (4.13).

The left-hand side of (4.16) is

$$\frac{d}{dt}X = i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho^\vee \cdot \hat{H}) y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) (\rho \cdot \dot{q}) \hat{s}_\rho \tag{4.19}$$

and the right-hand side is

$$\begin{aligned} [p \cdot \hat{H}, M] &= \left[p \cdot \hat{H}, i \sum_{\rho \in \Delta_+} g_{|\rho|} y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) \hat{s}_\rho \right], \\ &= i \sum_{\rho \in \Delta_+} g_{|\rho|} y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) [p \cdot \hat{H}, \hat{s}_\rho], \\ &= i \sum_{\rho \in \Delta_+} g_{|\rho|} y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) (\rho^\vee \cdot \hat{H}) (\rho \cdot p) \hat{s}_\rho, \\ &= \frac{d}{dt}X. \end{aligned} \tag{4.20}$$

The third line follows from the commutation relation (4.2) and the last line follows from the equation of motion $\dot{q} = p$.

The left-hand side of (4.17), after using the equations of motion (4.13), is

$$\frac{d}{dt}(p \cdot \hat{H}) = \sum_{\rho \in \Delta} \frac{g_{|\rho|}^2}{|\rho|^2} (\rho \cdot \hat{H}) \left[y_{|\rho|}(\rho \cdot q, w) x_{|\rho|}(-\rho \cdot q, w) - x_{|\rho|}(\rho \cdot q, w) y_{|\rho|}(-\rho \cdot q, w) \right]. \tag{4.21}$$

The summand $x_{|\rho|}(u, w) y_{|\rho|}(-u, w) - x_{|\rho|}(-u, w) y_{|\rho|}(u, w)$ is independent of the parameter w . The commutator $[X, M]$ reads

$$\begin{aligned} [X, M] &= - \left[\sum_{\rho \in \Delta_+} g_{|\rho|} (\rho^\vee \cdot \hat{H}) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) \hat{s}_\rho, \sum_{\sigma \in \Delta_+} g_{|\sigma|} y_{|\sigma|}(\sigma \cdot q, (\sigma^\vee \cdot \hat{H})\xi) \hat{s}_\sigma \right], \\ &= - \sum_{\rho, \sigma \in \Delta_+} g_{|\rho|} g_{|\sigma|} \left[(\rho^\vee \cdot \hat{H}) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) y_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \hat{H})\xi) \hat{s}_\rho \hat{s}_\sigma \right. \\ &\quad \left. - y_{|\sigma|}(\sigma \cdot q, (\sigma^\vee \cdot \hat{H})\xi) (s_\sigma(\rho)^\vee \cdot \hat{H}) x_{|\rho|}(\rho \cdot q, (s_\sigma(\rho)^\vee \cdot \hat{H})\xi) \hat{s}_\sigma \hat{s}_\rho \right]. \end{aligned} \tag{4.22}$$

Since $\hat{s}_\rho \hat{s}_\sigma$ and $\hat{s}_\sigma \hat{s}_\rho$ are rotations (except for $\rho = \sigma$, $\hat{s}_\rho^2 = 1$) they do not leave any real vectors in the rotation plane invariant. Thus $[X, M]$ is decomposed into the diagonal ($\rho = \sigma$) and the off-diagonal ($\rho \neq \sigma$) parts. The diagonal part gives the equation of motion

$$[X, M]_{\text{diagonal}} = \sum_{\rho \in \Delta_+} g_{|\rho|}^2 (\rho^\vee \cdot \hat{H}) \left[y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) x_{|\rho|}(-\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) - x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) y_{|\rho|}(-\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) \right], \tag{4.23}$$

$$= \frac{d}{dt}(p \cdot \hat{H}). \tag{4.24}$$

Finally, (4.18) will lead to the functional equations which must be satisfied by the functions $x_{|\rho|}(u, w)$ in the Lax pair. Relabeling the roots in the second summation of (4.22) gives the general operator form for the functional equation

$$\begin{aligned} 0 &= [X, M]_{\text{off-diagonal}} \\ &= \sum_{\rho \neq \sigma \in \Delta_+} g_{|\rho|} g_{|\sigma|} \left[(\rho^\vee \cdot \hat{H}) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) y_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \hat{H})\xi) - (s_\rho(\sigma)^\vee \cdot \hat{H}) y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \hat{H})\xi) x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \hat{H})\xi) \right] \hat{s}_\rho \hat{s}_\sigma. \end{aligned} \tag{4.25}$$

This shows that the consistency condition involving all of the roots is decomposed into a sum of two-dimensional ones corresponding to a fixed rotation $\hat{R}_\psi \equiv \hat{s}_\rho \hat{s}_\sigma$ in each two-dimensional plane. Since the coefficient of $\hat{R}_\psi \equiv \hat{s}_\rho \hat{s}_\sigma$ in this equation must separately vanish, this may be decomposed into the functional equations

$$\begin{aligned} 0 &= \sum_{\rho \neq \sigma \in \Phi_+, R_\psi = s_\rho s_\sigma} g_{|\rho|} g_{|\sigma|} \left[(\rho^\vee \cdot \mu) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) y_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) - (s_\rho(\sigma)^\vee \cdot \mu) y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) \right], \end{aligned} \tag{4.26}$$

in which μ is a generic vector in Γ . This equation must be satisfied for a fixed rotation $R_\psi = s_\rho s_\sigma$ and all roots appearing in it are in the two-dimensional sub-root system $\Phi = \{\kappa, \kappa \in (\Delta \cap \text{span}(\rho, \sigma))\}$ with positive roots $\Phi_+ \equiv \Phi \cap \Delta_+$. The only possible two-dimensional root systems Φ are $A_1 \times A_1$, A_2 , B_2 , G_2 , and $I_2(m)$. Table I shows the two-dimensional sub-root systems appearing in the root systems of finite reflection groups. The $A_1 \times A_1$ root system has been omitted since its corresponding functional equation is trivially satisfied for any function and therefore does not give any constraint on the functions $x_{|\rho|}(u, w)$. It should be stressed that the functional equations are determined by the two-dimensional sub-root systems only and not by where they are embedded in the entire root system. Thus, as seen from Table I, each

function $x_{|\rho|}$ must satisfy at most two functional equations, except for the models based on the dihedral groups.

The functional equation in (4.26) may be further simplified to

$$0 = \mu \cdot R_{\psi/2} \frac{\partial}{\partial q} f(q, \mu), \quad (4.27)$$

in which

$$f(q, \mu) = \sum_{\rho \neq \sigma \in \Phi_+, R_\psi = s_\rho s_\sigma} \frac{(\sigma^\vee \cdot \rho)}{|\sigma^\vee \cdot \rho|} x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) \times g_{|\rho|} g_{|\sigma|} \times x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) \quad (4.28)$$

and $R_{\psi/2}$ is a rotation by an angle $\psi/2$ with $R_\psi = s_\rho s_\sigma$. It has the following action on any pair of roots $\rho, \sigma \in \Phi_+$ which satisfy $R_\psi = s_\rho s_\sigma$

$$R_{\psi/2} \rho = -\frac{|\rho|}{|\sigma|} \frac{(\sigma^\vee \cdot \rho)}{|\sigma^\vee \cdot \rho|} s_\rho(\sigma),$$

$$R_{\psi/2} \sigma = \frac{|\sigma|}{|\rho|} \frac{(\sigma^\vee \cdot \rho)}{|\sigma^\vee \cdot \rho|} \rho. \quad (4.29)$$

Using these relations, the simplified form of the functional equation (4.27) may be proven. First substitute the definition of $f(q, \mu)$ in (4.28) into (4.27) to obtain

$$0 = \sum_{\rho \neq \sigma \in \Phi_+, r = s_\rho s_\sigma} g_{|\rho|} g_{|\sigma|} \frac{(\sigma^\vee \cdot \rho)}{|\sigma^\vee \cdot \rho|} \times \left[(\mu \cdot R_{\psi/2} \rho) y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) + (\mu \cdot R_{\psi/2} \sigma) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) y_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) \right]. \quad (4.30)$$

Since the coupling constants $g_{|\alpha|}$ are arbitrary and are constants on the orbits they may be rescaled as $g_{|\alpha|} \rightarrow 2g_{|\alpha|}/|\alpha|$ for all $\alpha \in \Phi$. Performing this rescaling of the coupling constants and using the action of $R_{\psi/2}$ in (4.29), the previous equation becomes

$$0 = \sum_{\rho \neq \sigma \in \Phi_+, r = s_\rho s_\sigma} g_{|\rho|} g_{|\sigma|} \frac{2}{|\rho||\sigma|} \times \left[\frac{|\rho|}{|\sigma|} (-s_\rho(\sigma) \cdot \mu) y_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) + \frac{|\sigma|}{|\rho|} (\rho \cdot \mu) x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) y_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) \right]. \quad (4.31)$$

Table I. Two-dimensional sub-root systems. $A_1 \times A_1$ is not included. †: k divides m .

Root System	Sub-root Systems
$A_r, r > 1$	A_2
$B_r, r \geq 2$	A_2, B_2
$C_r, r \geq 2$	A_2, B_2
$D_r, r > 3$	A_2
$BC_r, r \geq 2$	A_2, B_2
E_6, E_7, E_8	A_2
F_4	A_2, B_2
G_2	A_2, G_2
$I_2(m)$	$I_2(k)^\dagger$
H_3	$A_2, I_2(5)$
H_4	$A_2, I_2(5)$

This is the same as the earlier form of the functional equation in (4.26), after using the definition $\alpha^\vee = 2\alpha/|\alpha|^2$.

We next present the functional equations for the two-dimensional root systems, A_2 , B_2 , G_2 , and $I_2(m)$, and state their solutions. The proofs that the solutions satisfy the relevant functional equations are contained in the Appendix.

We first consider the functional equation for the A_2 root system with simple roots α and β . We choose $R_{2\pi/3} = s_\alpha s_\beta$ and the functional equation (4.25) becomes

$$0 = \mu \cdot R_{\pi/3} \frac{\partial}{\partial q} f_{A_2}(q, \mu) \tag{4.32}$$

in which $f_{A_2}(q, \mu)$ is defined by

$$\begin{aligned} f_{A_2}(q, \mu) = & x((\alpha + \beta) \cdot q, (\alpha + \beta)^\vee \cdot \mu) x(\alpha \cdot q, -\beta^\vee \cdot \mu) \\ & + x(\beta \cdot q, \beta^\vee \cdot \mu) x((\alpha + \beta) \cdot q, \alpha^\vee \cdot \mu) \\ & - x(\alpha \cdot q, \alpha^\vee \cdot \mu) x(\beta \cdot q, (\alpha + \beta)^\vee \cdot \mu). \end{aligned} \tag{4.33}$$

The subscripts on the function x is omitted since all roots are in the same orbit and hence only one function appears. Also the spectral parameter ξ is absorbed into μ by redefinition: $\mu \rightarrow \mu/\xi$. We look for solutions to this functional equation, as well as the ones for other root systems, which are valid for arbitrary vectors q and μ . Therefore these solutions are valid in any representations. In certain representations, such as the minimal and the root type representations discussed in §6, there is a larger class of solutions. The functional equation arising from the rotation $R_{-2\pi/3}$ is equivalent to that given above. A simple solution to (4.32) satisfying the residue condition (4.8) is

$$x(u, w) = \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \tag{4.34}$$

in which b is an arbitrary complex parameter. The potential function in the resulting Hamiltonian (3.3), $V(u) = \wp(u)$, is the Weierstrass elliptic function with the set of primitive periods $2\omega_1$ and $2\omega_3$. The limits as one or both periods diverge yield the potential functions $a^2/\sin^2(au)$ and $a^2/\sinh^2(au)$ or $1/u^2$, respectively.³⁾

Functional equation (4.26) for the B_2 root system may be written in terms of the short and long simple roots, α and β , respectively. We choose $R_{\pi/2} = s_\alpha s_\beta$ in (4.25) and the resulting functional equation is

$$0 = \mu \cdot R_{\pi/4} \frac{\partial}{\partial q} f_{B_2}(q, \mu) \tag{4.35}$$

in which

$$\begin{aligned} f_{B_2}(q, \mu) = & -x_S(\alpha \cdot q, \alpha^\vee \cdot \mu) x_L(\beta \cdot q, (2\alpha + \beta)^\vee \cdot \mu) \\ & + x_S((\alpha + \beta) \cdot q, (\alpha + \beta)^\vee \cdot \mu) x_L((2\alpha + \beta) \cdot q, -\beta^\vee \cdot \mu) \\ & + x_S(\alpha \cdot q, -(\alpha + \beta)^\vee \cdot \mu) x_L((2\alpha + \beta) \cdot q, (2\alpha + \beta)^\vee \cdot \mu) \\ & + x_S((\alpha + \beta) \cdot q, \alpha^\vee \cdot \mu) x_L(\beta \cdot q, \beta^\vee \cdot \mu). \end{aligned} \tag{4.36}$$

In this case there are two sets of elliptic function solutions for the long and short root functions, x_L and x_S , respectively. The solutions along with the corresponding potential functions are

$$\begin{aligned} x_L(u, w) = x_S(u, w) &= \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \\ V_L(u) = V_S(u) &= \wp(u), \end{aligned} \tag{4.37}$$

for the untwisted solution and

$$\begin{aligned} x_L(u, w) &= \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \\ x_S(u, w) &= \frac{\sigma(w/2 - u|\{\omega_1, 2\omega_3\})}{\sigma(w/2|\{\omega_1, 2\omega_3\})\sigma(u|\{\omega_1, 2\omega_3\})} \exp\left[\left(b + \frac{e_1}{2}\right) w u\right], \\ &= \frac{x_L(u, w/2)x_L(u + \omega_1, w/2)}{x_L(\omega_1, w/2)}, \\ V_L(u) &= \wp(u), \quad V_S(u) = \wp(u|\{\omega_1, 2\omega_3\}), \end{aligned} \tag{4.38}$$

for the twisted solution. Here $e_1 \equiv \wp(\omega_1)$ and b is an arbitrary complex constant.

Next we consider the functional equation (4.26) for the G_2 root system with short and long simple roots α and β , respectively. We choose $R_{\pi/3} = s_\alpha s_\beta$ in (4.26) and the functional equation is

$$0 = \mu \cdot R_{\pi/6} \frac{\partial}{\partial q} f_{G_2}(q, \mu), \tag{4.39}$$

in which

$$\begin{aligned} f_{G_2}(q, \mu) &= -x_S(\alpha \cdot q, \alpha^\vee \cdot \mu) x_L(\beta \cdot q, (3\alpha + \beta)^\vee \cdot \mu) \\ &\quad + x_S(\alpha \cdot q, -(2\alpha + \beta)^\vee \cdot \mu) x_L((3\alpha + \beta) \cdot q, (3\alpha + \beta)^\vee \cdot \mu) \\ &\quad + x_S((2\alpha + \beta) \cdot q, (2\alpha + \beta)^\vee \cdot \mu) x_L((3\alpha + \beta) \cdot q, -(3\alpha + 2\beta)^\vee \cdot \mu) \\ &\quad + x_S((2\alpha + \beta) \cdot q, -(\alpha + \beta)^\vee \cdot \mu) x_L((3\alpha + 2\beta) \cdot q, (3\alpha + 2\beta)^\vee \cdot \mu) \\ &\quad + x_S((\alpha + \beta) \cdot q, (\alpha + \beta)^\vee \cdot \mu) x_L((3\alpha + 2\beta) \cdot q, -\beta^\vee \cdot \mu) \\ &\quad + x_S((\alpha + \beta) \cdot q, \alpha^\vee \cdot \mu) x_L(\beta \cdot q, \beta^\vee \cdot \mu). \end{aligned} \tag{4.40}$$

As before, x_S and x_L are the functions for short and long roots, respectively. The elliptic function solutions to (4.39) along with the corresponding potential functions $V_S(u)$ and $V_L(u)$ are the untwisted ones in (4.37) and the following twisted ones

$$\begin{aligned} x_L(u, w) &= \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \\ x_S(u, w) &= \frac{\sigma(w/3 - u|\{2\omega_1/3, 2\omega_3\})}{\sigma(w/3|\{2\omega_1/3, 2\omega_3\})\sigma(u|\{2\omega_1/3, 2\omega_3\})} \exp\left[\left(b + \frac{2}{3}\wp(2\omega_1/3)\right) w u\right] \\ &= \frac{x_L(u, w/3)x_L(u + 2\omega_1/3, w/3)x_L(u + 4\omega_1/3, w/3)}{x_L(2\omega_1/3, w/3)x_L(4\omega_1/3, w/3)} \exp[b w u], \\ V_L(u) &= \wp(u), \quad V_S(u) = \wp(u|\{2\omega_1/3, 2\omega_3\}). \end{aligned} \tag{4.41}$$

Here again, b is an arbitrary complex parameter. The short and long root functions also satisfy the A_2 functional equation (4.32) separately, as expected.

Finally, we consider the functional equation (4.26) for the $I_2(m)$ root system. The $2m$ roots are labelled in order of increasing angle by $\{\alpha_1, \dots, \alpha_{2m}\}$ with $\Delta_+ = \{\alpha_j, j = 1, \dots, m\}$ and $\alpha_{j+m} = -\alpha_j$ for $j = 1, \dots, m$. For example, choosing all roots to have the same length $|\alpha_j| = 1$, a possible basis is given in (2.6). Unlike the crystallographic root systems, there is, in general, more than one functional equation. The functional equation (4.26) corresponding to $R_\psi = s_\rho s_\sigma$ with $\psi = 2\pi N/m$, $N = 1, \dots, [m/2]$ is

$$0 = \mu \cdot R_{N\pi/m} \frac{\partial}{\partial q} f_{I_2(m)}^N(q, \mu) \tag{4.42}$$

with

$$f_{I_2(m)}^N(q, \mu) = \sum_{j=1}^m g_{|j|} g_{|j+N|} x_{|j+N|}(\alpha_{j+N} \cdot q, \alpha_{j+N}^\vee \mu) x_{|j|}(\alpha_j \cdot q, -\alpha_{j+2N}^\vee \mu). \tag{4.43}$$

In contrast to the previous cases, the coupling constants are included in the functional equation since they do not factor out, in general. Because of the many functional equations (4.42) to be satisfied, only rational solutions are allowed in this case. For odd m , $I_2(m)$ roots are all in a single orbit and only one coupling constant and function are possible. The solution to (4.42) in this case is

$$x(u, w) = \left(\frac{1}{u} - \frac{1}{w}\right) \exp[b u w]. \tag{4.44}$$

However, $I_2(m)$ for even m has two orbits: one the set $\{\alpha_j\}$ with odd j and the other with even j . It is possible to have two independent coupling constants and functions in this case. The corresponding functions are denoted x_O and x_E , respectively and the solution of (4.42) is

$$\begin{aligned} x_O(u, w) &= \left(\frac{1}{u} - \frac{1}{w}\right) \exp[b u w], \\ x_E(u, w) &= \left(\frac{1}{u} - \frac{c}{w}\right) \exp[b u w], \end{aligned} \tag{4.45}$$

in which b and c are arbitrary complex constants.

For the other non-crystallographic root systems H_3 and H_4 , the two-dimensional sub-root systems are A_2 and $I_2(5)$. Thus the solution (4.44) satisfies all of the functional equations for the consistency of the Lax pair.

At the end of this section, let us show the Lax pair operator formulation for the rational potential with a confining harmonic force. This applies, as before, to all of the root systems including the non-crystallographic ones. This is a simple generalisation of the Lax pairs given in Ref. 2), which were constructed for the vector representations of the classical root systems, A_r, B_r, C_r, D_r , and BC_r .

Let us start from the Lax operator for the rational potential without spectral parameter:

$$L = p \cdot \hat{H} + X, \quad X = i \sum_{\rho \in \Delta_+} g_{|\rho|} (\rho^\vee \cdot \hat{H}) x(\rho \cdot q) \hat{s}_\rho, \tag{4.46}$$

$$M = i \sum_{\rho \in \Delta_+} g_{|\rho|} y(\rho \cdot q) \hat{s}_\rho,$$

$$x(\rho \cdot q) = \frac{1}{\rho \cdot q}, \quad y(\rho \cdot q) = -\frac{1}{(\rho \cdot q)^2} = -x(\rho \cdot q)^2. \quad (4.47)$$

It corresponds to the following Hamiltonian and the equations of motion:

$$\text{Tr}(L^2) \propto \mathcal{H}_r = \frac{1}{2} p^2 + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} \frac{1}{(\alpha \cdot q)^2}, \quad (4.48)$$

$$\dot{L} = [L, M] \Leftrightarrow \dot{q} = p, \quad \dot{p} = 2 \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} \frac{\alpha}{(\alpha \cdot q)^3}. \quad (4.49)$$

If a confining harmonic potential is added to the Hamiltonian

$$\mathcal{H}_\omega = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} \frac{1}{(\alpha \cdot q)^2}, \quad (4.50)$$

the equations of motion read

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q + 2 \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} \frac{\alpha}{(\alpha \cdot q)^3}. \quad (4.51)$$

It is elementary to see that the above equations can be written in an operator form

$$\dot{L} = [L, M] - \omega^2 Q, \quad \dot{Q} = p \cdot \hat{H} = L - X, \quad (4.52)$$

in which L and M are the same as (4.46) and Q is defined by

$$Q = q \cdot \hat{H}. \quad (4.53)$$

It is easy to verify that

$$[Q, M] = -X. \quad (4.54)$$

This property is valid only for the rational potential without spectral parameter.

Next let us introduce a pair of non-hermitian operators L^\pm by

$$L^\pm = L \pm i\omega Q. \quad (4.55)$$

Their time evolution equations read

$$\begin{aligned} \dot{L}^\pm &= \dot{L} \pm i\omega \dot{Q} \\ &= [L, M] - \omega^2 Q \pm i\omega(L - X) \\ &= [L^\pm \mp i\omega Q, M] - \omega^2 Q \pm i\omega(L - X) \\ &= [L^\pm, M] - \omega^2 Q \pm i\omega(L^\pm \mp i\omega Q) \\ &= [L^\pm, M] \pm i\omega L^\pm. \end{aligned} \quad (4.56)$$

If we define hermitian operators \mathcal{L}_1 and \mathcal{L}_2

$$\mathcal{L}_1 = L^+ L^-, \quad \mathcal{L}_2 = L^- L^+, \quad (4.57)$$

they satisfy Lax-like equations

$$\dot{\mathcal{L}}_j = [\mathcal{L}_j, M], \quad j = 1, 2. \tag{4.58}$$

From these we can construct conserved quantities

$$\text{Tr}(\mathcal{L}_1^k) = \text{Tr}(\mathcal{L}_2^k), \quad k = 1, 2, \dots, \tag{4.59}$$

as before. It is elementary to check that the first conserved quantities give the Hamiltonian (4.50)

$$\text{Tr}(\mathcal{L}_1) = \text{Tr}(\mathcal{L}_2) \propto \mathcal{H}_\omega. \tag{4.60}$$

This then completes the derivation of the Lax pairs for all of the generalised Calogero-Moser models.

§5. Representations of the Lax operators

We next consider representations of the operator algebra (4.1)–(4.4) for operators $\{\hat{H}_j, j = 1, \dots, r\}$ and $\{\hat{s}_\alpha, \alpha \in \Delta\}$ that appear in the Calogero-Moser Lax pair (4.5). In general, the representation of an algebra consists of a vector space \mathbf{V} and a mapping from elements of the algebra to $GL(\mathbf{V})$, e.g., $\hat{s}_\alpha \rightarrow \mathcal{R}(\hat{s}_\alpha)$. When a basis is chosen for \mathbf{V} then $\mathcal{R}(\hat{s}_\alpha)$ becomes a matrix in $GL(d)$, where d is the dimension of \mathbf{V} . First define a monomial $P_\Omega(t) = \prod_{j=1}^N (v^{(j)} \cdot t)$ in an auxiliary vector variable $t \in \mathbf{R}^r$ associated with a set Ω of vectors $\{v^{(j)} \in \mathbf{R}^r, j = 1, \dots, N\}$. The basis vectors of \mathbf{V} , for a representation \mathcal{R}_Ω of the algebra (4.1)–(4.4), are monomials resulting from the orbit of $P_\Omega(t)$ under the reflection group, i.e., $P_\Omega(s_\alpha(t))$ with $\alpha \in \Delta$.*) All of the monomials in \mathbf{V} therefore have the same degree N . A similar representation for only the reflection generators \hat{s}_α with Ω a sub-root system of Δ was introduced by MacDonald for irreducible representations of the Weyl groups.^{11), 12)} The action of the operators on the basis vectors in \mathbf{V} is

$$\begin{aligned} \hat{s}_\alpha \prod_{j=1}^N (v^{(j)} \cdot t) &= \prod_{j=1}^N (s_\alpha(v^{(j)}) \cdot t), \\ \hat{H}_l \prod_{j=1}^N (v^{(j)} \cdot t) &= \left[\sum_{j=1}^N v_l^{(j)} \right] \prod_{j=1}^N (v^{(j)} \cdot t). \end{aligned} \tag{5.1}$$

Note that the representation matrices $\{\mathcal{R}(\hat{H}_l), l = 1, \dots, r\}$ are all diagonal in this basis.

All of the previous versions of Lax pairs for the Calogero-Moser models, namely the minimal and root type Lax pairs, result from (4.5) in a representation \mathcal{R}_Ω with Ω a single vector, i.e., (5.1) with $N = 1$.³⁾ In the case of $N = 1$, the vectors in \mathbf{V} , upon which the representation matrices act, may be denoted in a simple manner as $|\mu\rangle$ with

$$|\mu\rangle \equiv (\mu \cdot t). \tag{5.2}$$

) More generally, the vector space for the representation is $\text{Sym}(W^)$, the symmetric algebra on the r -dimensional dual vector space W^* .

For the minimal type Lax pair, the representation has $|\mu\rangle \in \mathbf{V}$ with μ being a weight in the same Weyl orbit as the highest weight of a minimal representation of the corresponding Lie algebra. For the root type Lax pair, the vectors μ in \mathbf{V} are labelled by all of the roots for the simply laced root systems and either the short or long roots for the non-simply laced root systems. All of the μ which label the basis vectors of \mathbf{V} are then in a single Weyl orbit, as required. It will be demonstrated in §6 that the Lax pairs in these representations agree with those given in Ref. 3).

There is a simple relation between the geometry of the root system and the dimension of the \mathcal{R}_Ω representations for Ω a single vector. The vectors μ labelling the $|\mu\rangle \in \mathbf{V}$ are generated by the orbit of a single vector μ_0 under the action of the finite reflection group. The vector μ_0 may also be assumed, without loss of generality, to be in the principal Weyl chamber or its boundary. The dimension of the representation depends only on which reflection hyperplanes, if any, contain the vector μ_0 . Let Φ_0 be the set of indices j such that $\Phi_0 \equiv \{j, \mu_0 \cdot \alpha_j = 0, \alpha_j \in \Pi\}$. For crystallographic root systems Φ_0 are the Dynkin labels of μ_0 which are equal to zero. Then $s_{\alpha_j}(\mu_0) = \mu_0$ for $j \in \Phi_0$ or the isotropy group I_0 of μ_0 is generated by reflections about simple roots $\alpha_j \in \Pi$ for which $j \in \Phi_0$. Therefore the number of elements in the orbit of μ_0 under the reflection group W , and hence the dimension of the representation, is $D = |W|/|I_0|$. Since the isotropy group is the direct product of the reflection groups corresponding to the connected parts of the Coxeter diagram for the original root system, after deleting those vertices corresponding to the indices j not contained in I_0 , $|I_0|$ may be computed as the product of the orders of the reflection groups of the corresponding Coxeter sub-diagrams.

As a first example, we calculate the dimensions of the minimal and root type representations of the algebra in (4.1)–(4.4) and hence the Lax pairs for A_5 . There are three inequivalent minimal representations with $\mu_0 = \Lambda^{(1)}, \Lambda^{(2)},$ or $\Lambda^{(3)}$. For $\mu_0 = \Lambda^{(1)}$, the vector representation, the isotropy group $I_0 = W_{A_4}$ so $D = |W_{A_5}|/|W_{A_4}| = 6!/5! = 6$. For the antisymmetric product of two vector representations, $\mu_0 = \Lambda^{(2)}$, $I_0 = W_{A_1} \times W_{A_3}$ so we obtain $D = 6!/(2!4!) = 15$. For the antisymmetric product of three vector representations, $\mu_0 = \Lambda^{(3)}$, $I_0 = W_{A_2} \times W_{A_2}$ so we obtain $D = 6!/(3!3!) = 40$. Finally, for the root representation, choose $\mu_0 = \Lambda^{(1)} + \Lambda^{(5)}$, which is the highest root. Then $I_0 = W_{A_3}$ and $D = |W_{A_5}|/|W_{A_3}| = 6!/4! = 30$. As an aside, we note that the highest dimensional representation of the Lax pair for A_5 has $\mu_0 = a_1\Lambda^{(1)} + a_2\Lambda^{(2)} + a_3\Lambda^{(3)} + a_4\Lambda^{(4)} + a_5\Lambda^{(5)}$ with all non-vanishing coefficients $a_j, j = 1, \dots, 5$. Since the dimension of the representation depends only on which Dynkin labels are non-zero, these coefficients a_j may be chosen to be 1, without loss of generality. Then the isotropy group is trivial and we obtain $D = 6!$, which has not been previously described. In fact, even among the representations R_Ω with $N = 1$, most of the matrices for the Lax pairs had not been described before.

As a second example, let us evaluate the dimensions of some lower-dimensional Lax pairs for the E_8 model. Let us take $\mu_0 = \Lambda^{(1)}$ and $\mu_0 = \Lambda^{(7)}$ corresponding to the two end points of the two longer forks of the E_8 Dynkin diagram. For $\mu_0 = \Lambda^{(1)}$ the isotropy group $I_0 = W_{E_7}$ and $D = |W_{E_8}|/|W_{E_7}| = 240$. This gives the root type Lax pair. For $\mu_0 = \Lambda^{(7)}$ the isotropy group $I_0 = W_{D_7}$ and $D = |W_{E_8}|/|W_{D_7}| = 2160$. This gives the second smallest Lax pair for the E_8 model and its weights are a part

of the **3875** representation.

For the non-crystallographic root systems, the root type Lax pairs with dimensions $2m$ for $I_2(m)$, 30 for H_3 and 120 for H_4 give the smallest dimensional Lax pair matrices.

§6. Minimal and root type Lax pairs

As examples of representations of the Lax operators, we consider those representations that yield the minimal and root type Lax pairs previously reported in Ref. 3). The functional equations associated with these representations are also derived by restricting those given in §4. These are the same as those given in our previous papers, which we denote as sum rules. In this section we consider only the crystallographic root systems.

Minimal type Lax pairs are associated with minimal representations of the Lie algebras. All of the fundamental representations of A_r , the spinor representation of B_r , the vector representation of C_r , and the vector, spinor and anti-spinor representations of D_r , the **27** and **$\overline{27}$** of E_6 and the **56** of E_7 are the minimal representations. All weights in a minimal representation are in a single Weyl orbit. The vectors $|\mu\rangle \in \mathbf{V}$ of the representation are $\{|\mu\rangle, \mu \in \Sigma_{\min}\}$, in which Σ_{\min} is the set of weights of the minimal representation. They are characterised by the condition

$$\rho^\vee \cdot \mu = \{0, \pm 1\}, \quad \rho \in \Delta, \quad \mu \in \Sigma_{\min}. \tag{6.1}$$

On the other hand the vectors $|\mu\rangle \in \mathbf{V}$ of the root type Lax pairs are $\{|\alpha\rangle, \alpha \in \Delta\}$ for a simply laced root system Δ . The set of basis vectors of the two possible root type representations for a Lax pair associated with a non-simply laced root system are $\{|\alpha\rangle, \alpha \in \Delta_S\}$, the set of short roots, and $\{|\alpha\rangle, \alpha \in \Delta_L\}$, the set of long roots. Let us collectively denote by Δ_R the set of basis vectors of various root type representations:

$$\Delta_R = \begin{cases} \Delta, & \text{for simply laced models,} & \text{all roots,} \\ \Delta_L, & \text{for non-simply laced models,} & \text{long roots,} \\ \Delta_S, & \text{for non-simply laced models,} & \text{short roots.} \end{cases}$$

The roots are characterised by the condition

$$\rho^\vee \cdot \alpha = \{0, \pm 1, \pm 2\}, \quad \rho, \alpha \in \Delta, \tag{6.2}$$

except for the G_2 case in which ± 3 are also possible.

The fact that the eigenvalues of the operator $\rho^\vee \cdot \hat{H}$ are restricted to these values (6.1) and (6.2) simplifies the representation of the Lax pair operator (4.5) drastically. Especially at the zero eigenvalue of $\rho^\vee \cdot \hat{H}$ the functions $x_{|\rho|}$ and $y_{|\rho|}$ take the following simple forms:

$$\lim_{w \rightarrow 0} w x_{|\rho|}(u, w\xi) = \begin{cases} -1/\xi, & \text{untwisted,} \\ -2/\xi, & \text{twisted short roots,} \\ -3/\xi, & \text{twisted short roots, } G_2, \end{cases} \tag{6.3}$$

$$\lim_{w \rightarrow 0} y_{|\rho|}(u, w) = -V_{|\rho|}(u) + D_{|\rho|}, \tag{6.4}$$

in which $D_{|\rho|}$ are constants possibly depending on the orbits, that is, D_L and D_S for the non-simply laced cases.

With the basis vectors of \mathbf{V} , upon which the representation matrices act, labelled by a single vector, as in (5.2), the matrix elements of the minimal and root type Lax pairs are

$$L|\nu\rangle = \sum_{\mu \in \Sigma_{\min}} \mathcal{R}(L)_{\mu\nu}|\mu\rangle, \quad M|\nu\rangle = \sum_{\mu \in \Sigma_{\min}} \mathcal{R}(M)_{\mu\nu}|\mu\rangle, \quad \mu, \nu \in \Sigma_{\min}, \tag{6.5}$$

$$L|\beta\rangle = \sum_{\alpha \in \Delta_R} \mathcal{R}(L)_{\alpha\beta}|\alpha\rangle, \quad M|\beta\rangle = \sum_{\alpha \in \Delta_R} \mathcal{R}(M)_{\alpha\beta}|\alpha\rangle, \quad \alpha, \beta \in \Delta_R, \tag{6.6}$$

where L and M are the Lax operators defined in (4.5). Hereafter we adopt simplified notation $L_{\mu\nu}$ ($L_{\alpha\beta}$) and $M_{\mu\nu}$ ($M_{\alpha\beta}$) for the matrix elements $\mathcal{R}(L)_{\mu\nu}$ ($\mathcal{R}(L)_{\alpha\beta}$) and $\mathcal{R}(M)_{\mu\nu}$ ($\mathcal{R}(M)_{\alpha\beta}$). For an arbitrary function $f(u)$ we have the following matrix element for the minimal type

$$f(\rho^\vee \cdot \hat{H}) \hat{s}_\rho |\nu\rangle = \sum_{\mu \in \Sigma_{\min}} \left\{ f(-1) \delta_{\mu-\nu, -\rho} + f(0) \delta_{\mu, \nu} \delta_{\rho, \mu, 0} + f(1) \delta_{\mu-\nu, \rho} \right\} |\mu\rangle, \tag{6.7}$$

and for the root type (except for the G_2 case)

$$f(\rho^\vee \cdot \hat{H}) \hat{s}_\rho |\beta\rangle = \sum_{\alpha \in \Delta_R} \left\{ f(-2) \delta_{\alpha-\beta, -2\rho} + f(-1) \delta_{\alpha-\beta, -\rho} + f(0) \delta_{\alpha, \beta} \delta_{\rho, \alpha, 0} \right. \\ \left. + f(1) \delta_{\alpha-\beta, \rho} + f(2) \delta_{\alpha-\beta, 2\rho} \right\} |\alpha\rangle. \tag{6.8}$$

The G_2 case can be handled in a similar way.

By combining (6.3), (6.4) and (6.7), (6.8) it is straightforward to derive the matrix representations of the minimal and root type Lax pairs. For the minimal Lax pair we obtain

$$L_{\mu\nu} = p \cdot \mu \delta_{\mu, \nu} + i \sum_{\rho \in \Delta} g_{|\rho|} x_{|\rho|}(\rho \cdot q, \xi) \delta_{\mu-\nu, \rho} + A_m \delta_{\mu, \nu}, \tag{6.9}$$

$$M_{\mu\nu} = i \left[\sum_{\rho \in \Delta, \rho^\vee \cdot \mu = 1} g_{|\rho|} V_{|\rho|}(\rho \cdot q) \right] \delta_{\mu, \nu} + i \sum_{\rho \in \Delta} g_{|\rho|} y_{|\rho|}(\rho \cdot q, \xi) \delta_{\mu-\nu, \rho} + B_m \delta_{\mu, \nu}, \tag{6.10}$$

in which A_m is a constant and B_m contains the dynamical variables q . Both have no effect on the Lax equation and can thus be omitted. They are

$$A_m = \frac{i}{2} (g_L N_L^m + g_S N_S^m), \quad \text{for untwisted,} \quad \frac{i}{2} (g_L N_L^m + 2g_S N_S^m), \quad \text{for twisted,}$$

$$B_m = \frac{i}{2} (g_L N_L^m D_L + g_S N_S^m D_S) - \frac{i}{2} \sum_{\rho \in \Delta} g_{|\rho|} V_{|\rho|}(\rho \cdot q),$$

in which N_L^m (N_S^m) is the number of long (short) roots $\alpha \in \Delta$ such that $\alpha \cdot \mu = 0$ for a given $\mu \in \Sigma_{\min}$. The integer N_L^m (N_S^m) is well-defined since all vectors labelling

the basis elements in V for the representation are in a single Weyl orbit and hence any other vector, say $\nu = w(\mu)$ for w an element of the Weyl group. This implies that $\alpha \cdot \nu = 0$ if and only if $w(\alpha) \cdot \mu = 0$ and so N_L^m is the same for any choice of $\mu \in \Sigma_{\min}$. When the A_m and B_m terms are dropped, the above Lax pair (6·9) and (6·10) has the same form as that given in our previous paper.³⁾

For the root type Lax pair we obtain

$$L_{\alpha\beta} = p \cdot \alpha \delta_{\alpha,\beta} + i \sum_{\rho \in \Delta} g_{|\rho|} \left[x_{|\rho|}(\rho \cdot q, \xi) \delta_{\alpha-\beta,\rho} + 2 x_{|\rho|}(\rho \cdot q, 2\xi) \delta_{\alpha-\beta,2\rho} \right] + A_r \delta_{\alpha,\beta}, \tag{6·11}$$

$$M_{\alpha\beta} = i \left[g_{|\alpha|} V_{|\alpha|}(\alpha \cdot q) + \sum_{\Delta \ni \rho = \alpha - \sigma, \sigma \in \Delta} g_{|\rho|} V_{|\rho|}(\rho \cdot q) \right] \delta_{\alpha,\beta} + i \sum_{\rho \in \Delta} g_{|\rho|} \left[y_{|\rho|}(\rho \cdot q, \xi) \delta_{\alpha-\beta,\rho} + y_{|\rho|}(\rho \cdot q, 2\xi) \delta_{\alpha-\beta,2\rho} \right] + B_r \delta_{\alpha,\beta}. \tag{6·12}$$

As in the minimal case A_r is a constant and B_r contains the dynamical variables q . Both have no effect on the Lax equation and can be omitted. They are

$$A_r = \frac{i}{2} (g_L N_L^r + g_S N_S^r), \quad \text{for untwisted,} \quad \frac{i}{2} (g_L N_L^r + 2g_S N_S^r), \quad \text{for twisted,}$$

$$B_r = \frac{i}{2} (g_L N_L^r D_L + g_S N_S^r D_S) - \frac{i}{2} \sum_{\rho \in \Delta} g_{|\rho|} V_{|\rho|}(\rho \cdot q),$$

in which N_L^r (N_S^r) is the number of long (short) roots $\alpha \in \Delta$ such that $\alpha \cdot \beta = 0$ for a given $\beta \in \Delta$ (Δ_L, Δ_S). When the A_r and B_r terms are dropped and with the following identification:

$$x_L(u, \xi) \equiv x(u, \xi), \quad x_S(u, \xi) \equiv x(u, \xi) \quad \text{or} \quad x^{(1/2)}(u, \xi),$$

$$x_L(u, 2\xi) \equiv x_d(u, \xi), \quad x_S(u, 2\xi) \equiv x_d(u, \xi) \quad \text{or} \quad x_d^{(1/2)}(u, \xi), \tag{6·13}$$

the above Lax pair (6·11) and (6·12) has the same form as that given in our previous paper.³⁾

The restrictions on the eigenvalues of the operator (6·1) and (6·2), simplify the functional equations, too. Let us examine the A_2 functional equation (4·32) by adopting the variables u, v, ξ_1 , and ξ_2 defined in (A·8). We impose a condition $\xi_1 = \xi = -\xi_2$, so that the restriction of the minimal representation (6·1) is satisfied. The limit formulas (6·3) and (6·4) yield the so called first sum rule of Ref. 3)

$$x(u, \xi) y(v, \xi) - y(u, \xi) x(v, \xi) + x(u + v, \xi) [V(v) - V(u)] = 0. \tag{6·14}$$

Here the suffix $|\rho|$ is omitted since all the roots participating in the A_2 functional equation belong to the same orbit. Likewise, let us impose a condition $\xi_1 = 2\xi, \xi_2 = -\xi$, so that the restriction of the root type representation (6·2) is satisfied. The limit formulas (6·3) and (6·4) yield the so called second sum rule of Ref. 3)

$$0 = x(u + v, \xi) y(u, \xi) - y(u + v, \xi) x(u, \xi) + 2 [x(u, 2\xi) y(v, \xi) - y(-v, \xi) x(u + v, 2\xi)] + x(-v, \xi) y(u + v, 2\xi) - y(u, 2\xi) x(v, \xi). \tag{6·15}$$

One may also verify that all of the functional equations in Ref. 3) which must be satisfied by the functions $x_{|\rho|}(u, w)$ appear as various restrictions of the operator equation in (4.27), although we will not derive them here. Since the functional equations for the minimal and root type representations are restricted, their solution spaces are larger than that for a generic representation. For example, these functional equations have the same solutions as the general solutions given in §4, except that the exponential factor is changed from $\exp[b w u]$ to $\exp[b(w) u]$, in which an arbitrary function $b(w)$ need not be linear in w .

§7. Comments and discussion

Firstly, let us comment on the relation between our work and the paper on elliptic Dunkl operators by Buchstaber, Felder, and Veselov.⁵⁾ From the commutativity of Dunkl operators they derived a functional equation (9) of Ref. 5), which was closely related to our equations $f_{A_2} = 0, f_{B_2} = 0, f_{G_2} = 0$. But their equation did not contain the spectral parameter dependence explicitly. They obtained what would amount to our untwisted solution for the $f_{A_2} = 0$ functional equation. This gives a clue that the classical and quantum integrability of the generalised Calogero-Moser models are closely connected. Secondly, some remarks about the Calogero-Moser models based on $B_r, C_r,$ and BC_r root systems. The short roots of $B_r,$ the long roots of $C_r,$ and the long and short roots of $BC_r,$ when restricted to any two-dimensional plane, form only an $A_1 \times A_1$ sub-root system. This means that their short root function $x_S(u, w)$ (and/or $x_L(u, w)$) in these models are required to satisfy the B_2 functional equation only but not the A_2 one. Thus the solution space is larger than that of the other models. This in turn allows more potential function terms (one more for $B_r,$ and C_r and two more for BC_r) with independent coupling constant(s) in the Hamiltonian without breaking integrability. We call these models extended twisted models. The root type and minimal type Lax pairs for the extended models are given in Refs. 3) and 13). The Lax pair operators for the extended twisted models can be constructed in a similar way as is given in this paper. Thirdly, a few words about the structure of the functions $x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi)$ and their functional equations (4.26) and (4.27), and the self-duality of the two-dimensional crystallographic root systems. The coefficients in the functional equation (4.26), i.e., $\rho^\vee \cdot \mu$ and $s_\rho(\sigma)^\vee \cdot \mu,$ etc., come from the second argument of the function $x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi)$. Namely, they are co-roots. On the other hand, the gradient operator, $\partial/\partial q,$ in (4.27) supplies the coefficient from the first argument of the function $x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi),$ that is the roots. The operator $R_{\pm\pi/4}$ in the B_2 case and $R_{\pm\pi/6}$ in the G_2 case rotates a short root into a long root position and vice versa. In other words, the rotation operator $R_{\psi/2}$ in (4.27) performs the necessary conversion from the roots to the co-roots. This is possible because of the well-known self-duality of the two-dimensional crystallographic root systems, $A_2, B_2,$ and $G_2,$ under $\alpha \leftrightarrow \alpha^\vee = 2\alpha/\alpha^2$. As a final remark, let us comment on the integrability of the generalised Calogero-Moser models. As is well known, the existence of the independent involutive conserved quantities as many as the degrees of freedom is necessary and sufficient for the Liouville integrability. To the best of our knowledge, the involution of the conserved quantities for the models based on

the exceptional root systems and the non-crystallographic root systems as well as all the twisted models is yet to be demonstrated.

Acknowledgements

We thank P. Sutcliffe for bringing Ref. 5) to our attention. We thank S. P. Khastgir, K. Takasaki and A. P. Veselov for useful discussions. This work is partially supported by the Grant-in-Aid from the Ministry of Education, Science, Sports and Culture, priority area (#707) ‘‘Supersymmetry and unified theory of elementary particles’’. A. J. B. is supported by the Japan Society for the Promotion of Science and the National Science Foundation under grant No. 9703595. E. C. thanks the UK Engineering and Physical Sciences Research Council for travel support under grant No. GR/K 79437.

Appendix A

— Solutions to Functional Equations —

In this appendix we demonstrate that the solutions (4.34), (4.37), (4.38), (4.41), (4.44) and (4.45) given in §4 satisfy the functional equations for the consistency of the Lax pair

$$0 = \mu \cdot R_{\psi/2} \frac{\partial}{\partial q} f(q, \mu). \tag{A.1}$$

We do not give a proof that these solutions are the most general ones. First let us remark on a symmetry of the solutions of (A.1) such that if $x_{|\rho|}(u, w)$ is a solution then

$$x_{|\rho|}^{\text{new}}(u, w) \equiv x_{|\rho|}(u, w) \exp[b u w], \quad \forall b \in \mathbf{C}, \tag{A.2}$$

is also a solution. Therefore, it is necessary to prove (A.7) only for one representative function $x_{|\rho|}(u, w)$ among those related by the symmetry (A.2). Assume that the function $x_{|\rho|}(u, w)$ satisfies (A.1) with $f(q, \mu)$ defined by (4.28). The corresponding expression in (4.28) for the new solution $x_{|\rho|}^{\text{new}}(u, w) \equiv x_{|\rho|}(u, w) \exp[b u w]$ will be denoted as $f(q, \mu)^{\text{new}}$. Then

$$f(q, \mu)^{\text{new}} = \sum_{\rho \neq \sigma \in \Phi_+, R_\psi = s_\rho s_\sigma} g_{|\rho|} g_{|\sigma|} \left\{ x_{|\rho|}(\rho \cdot q, (\rho^\vee \cdot \mu)\xi) x_{|\sigma|}(\sigma \cdot q, (s_\rho(\sigma)^\vee \cdot \mu)\xi) \right. \\ \left. \times \exp [b \xi ((\rho \cdot q)(\rho^\vee \cdot \mu) + (\sigma \cdot q)(s_\rho(\sigma)^\vee \cdot \mu))] \right\}. \tag{A.3}$$

The exponent is proportional to

$$q \cdot \mu - q \cdot s_\sigma(s_\rho(\mu)) \equiv q \cdot \mu - q \cdot R_{-\psi} \mu.$$

Two pairs of roots (ρ, σ) and (ρ', σ') which appear in different terms of (A.3) are related by $s_\rho s_\sigma = s_{\rho'} s_{\sigma'}$, which implies also

$$s_\sigma s_\rho = s_{\sigma'} s_{\rho'} = R_{-\psi}. \tag{A.4}$$

Therefore the exponential factor in (A.3) is common to all terms and may be factored out of the sum. The functional equation (A.1) for $f(q, \mu)^{\text{new}}$ then is

$$\begin{aligned} & \mu \cdot R_{\psi/2} \frac{\partial}{\partial q} f(q, \mu)^{\text{new}} \\ &= \mu \cdot R_{\psi/2} \frac{\partial}{\partial q} (f(q, \mu) \exp [b\xi(q \cdot \mu - q \cdot R_{-\psi}\mu)]) \\ &= b\xi f(q, \mu) \exp [b\xi(q \cdot \mu - q \cdot R_{-\psi}\mu)] (\mu \cdot R_{\psi/2}\mu - \mu \cdot R_{-\psi/2}\mu) \\ &= 0. \end{aligned} \tag{A.5}$$

Here the orthogonality of rotation operators

$$\mu \cdot R_{\psi/2}\mu - \mu \cdot R_{-\psi/2}\mu = 0, \quad R_{-\psi/2} = R_{\psi/2}^T \tag{A.6}$$

is used. This then demonstrates that the transformed function $x_{|\rho|}(u, w)^{\text{new}}$ satisfies the functional equation if the original function $x_{|\rho|}(u, w)$ does.

Next we show in turn that all of the solutions given in §4 satisfy

$$f_G(q, \mu) = 0, \quad G = A_2, B_2, G_2, \tag{A.7}$$

which is a sufficient condition for the solutions of (A.1). Note that this is a necessary and sufficient condition for the commutativity of the corresponding Dunkl operators.⁵⁾

A.1. A_2 solution (4.34)

Defining the variables

$$\begin{aligned} u &= \alpha \cdot q, & v &= \beta \cdot q, \\ \xi_1 &= \alpha^\vee \cdot \mu, & \xi_2 &= \beta^\vee \cdot \mu, \end{aligned} \tag{A.8}$$

$f_{A_2}(q, \mu)$ becomes

$$f_{A_2}(q, \mu) = x(u + v, \xi_1 + \xi_2) x(u, -\xi_2) + x(v, \xi_2) x(u + v, \xi_1) - x(u, \xi_1) x(v, \xi_1 + \xi_2). \tag{A.9}$$

By substituting the solution

$$x(u, w) = \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \tag{A.10}$$

it reads, after extracting a common denominator,

$$\begin{aligned} f_{A_2}(u, v, \xi_1, \xi_2) &= \left[\sigma(\xi_1 + \xi_2 - u - v) \sigma(\xi_2 + u) \sigma(v) \sigma(\xi_1) \right. \\ &+ \sigma(\xi_2 - v) \sigma(\xi_1 - u - v) \sigma(u) \sigma(\xi_1 + \xi_2) \\ &\left. - \sigma(\xi_1 - u) \sigma(\xi_1 + \xi_2 - v) \sigma(u + v) \sigma(\xi_2) \right] \\ & / [\sigma(u) \sigma(v) \sigma(u + v) \sigma(\xi_1) \sigma(\xi_2) \sigma(\xi_1 + \xi_2)]. \end{aligned} \tag{A.11}$$

The expression in the numerator may be shown to vanish, by using the following identity (see page 153 of Ref. 14) and Eq. (5.1) of Ref. 15))

$$\begin{aligned} 0 &= \sigma(z - u_1) \sigma(z + u_1) \sigma(u_2 - u_3) \sigma(u_2 + u_3) \\ &\quad + \sigma(z - u_2) \sigma(z + u_2) \sigma(u_3 - u_1) \sigma(u_3 + u_1) \\ &\quad + \sigma(z - u_3) \sigma(z + u_3) \sigma(u_1 - u_2) \sigma(u_1 + u_2) \end{aligned} \quad (\text{A}\cdot 12)$$

with the choice of variables

$$\begin{aligned} z &= \xi_2 + \frac{1}{2}(\xi_1 - v), & u_1 &= u - \frac{1}{2}(\xi_1 - v), \\ u_2 &= \frac{1}{2}(\xi_1 + v), & u_3 &= \frac{1}{2}(\xi_1 - v). \end{aligned} \quad (\text{A}\cdot 13)$$

This implies that $f_{A_2}(q, \mu) = 0$ for the function in (A.10) and after including the symmetry of solutions (A.2), the more general function in (4.34) also gives $f_{A_2}(q, \mu) = 0$ and therefore the A_2 functional equation is satisfied.

A.2. B_2 untwisted solution (4.37)

Next we turn to the functional equation arising from the B_2 sub-root system. This, as well as the functional equation associated with the G_2 root system, admits two types of solutions, the untwisted and the twisted ones. By using the same definitions of the variables u , v , ξ_1 , and ξ_2 as in (A.8), except for the fact that α and β are the short and long simple roots of B_2 , respectively, the expression for $f_{B_2}(q, \mu)$ becomes

$$\begin{aligned} f_{B_2}(u, v, \xi_1, \xi_2) &= -x_S(u, \xi_1) x_L(v, \xi_1 + \xi_2) \\ &\quad + x_S(u + v, \xi_1 + 2\xi_2) x_L(2u + v, -\xi_2) \\ &\quad + x_S(u, -\xi_1 - 2\xi_2) x_L(2u + v, \xi_1 + \xi_2) \\ &\quad + x_S(u + v, \xi_1) x_L(v, \xi_2). \end{aligned} \quad (\text{A}\cdot 14)$$

Substituting in the untwisted solutions

$$x_L(u, w) = x_S(u, w) = \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \quad (\text{A}\cdot 15)$$

this becomes

$$\begin{aligned} f_{B_2}(u, v, \xi_1, \xi_2) &= \left[-\sigma(\xi_1 - u) \sigma(\xi_1 + \xi_2 - v) \sigma(u + v) \sigma(2u + v) \sigma(\xi_2) \sigma(\xi_1 + 2\xi_2) \right. \\ &\quad + \sigma(\xi_1 + 2\xi_2 - u - v) \sigma(\xi_2 + 2u + v) \sigma(u) \sigma(v) \sigma(\xi_1) \sigma(\xi_1 + \xi_2) \\ &\quad + \sigma(\xi_1 + 2\xi_2 + u) \sigma(\xi_1 + \xi_2 - 2u - v) \sigma(v) \sigma(u + v) \sigma(\xi_1) \sigma(\xi_2) \\ &\quad \left. + \sigma(\xi_1 - u - v) \sigma(\xi_2 - v) \sigma(u) \sigma(2u + v) \sigma(\xi_1 + \xi_2) \sigma(\xi_1 + 2\xi_2) \right] \\ &\quad / [\sigma(u) \sigma(v) \sigma(2u + v) \sigma(\xi_1) \sigma(\xi_2) \sigma(\xi_1 + \xi_2) \sigma(\xi_1 + 2\xi_2)]. \end{aligned} \quad (\text{A}\cdot 16)$$

Let us denote the numerator by $g_{B_2}(u, v, \xi_1, \xi_2)$. Gathering terms, one obtains

$$g_{B_2}(u, v, \xi_1, \xi_2) = \sigma(2u + v) \sigma(\xi_1 + 2\xi_2) \left[-\sigma(\xi_1 - u) \sigma(\xi_1 + \xi_2 - v) \sigma(u + v) \sigma(\xi_2) \right.$$

$$\begin{aligned}
 & +\sigma(\xi_1 - u - v) \sigma(\xi_2 - v) \sigma(u) \sigma(\xi_1 + \xi_2) \Big] \\
 & +\sigma(v) \sigma(\xi_1) \Big[\sigma(\xi_1 + 2\xi_2 - u - v) \sigma(\xi_2 + 2u + v) \sigma(u) \sigma(\xi_1 + \xi_2) \\
 & +\sigma(\xi_1 + 2\xi_2 + u) \sigma(\xi_1 + \xi_2 - 2u - v) \sigma(u + v) \sigma(\xi_2) \Big]. \quad (\text{A}\cdot 17)
 \end{aligned}$$

Using the identity (A.12) for the expression in the first set of brackets with the choice of variables

$$\begin{aligned}
 z &= -v + \frac{1}{2}(-u + \xi_1 + \xi_2), & u_1 &= \frac{1}{2}(u - \xi_1 + \xi_2), \\
 u_2 &= \frac{1}{2}(u + \xi_1 + \xi_2), & u_3 &= \frac{1}{2}(-u + \xi_1 + \xi_2), \quad (\text{A}\cdot 18)
 \end{aligned}$$

and for the expression in the second set of brackets with the variables

$$\begin{aligned}
 z &= \frac{1}{2}(u + \xi_1 + 3\xi_2), & u_1 &= \frac{1}{2}(3u + 2v - \xi_1 - \xi_2), \\
 u_2 &= \frac{1}{2}(u + \xi_1 + \xi_2), & u_3 &= \frac{1}{2}(-u + \xi_1 + \xi_2), \quad (\text{A}\cdot 19)
 \end{aligned}$$

gives

$$\begin{aligned}
 g_{B_2}(u, v, \xi_1, \xi_2) &= -\sigma(2u + v) \sigma(\xi_1 + 2\xi_2) \sigma(v) \sigma(\xi_1 + \xi_2 - u - v) \sigma(\xi_1) \sigma(\xi_2 + u) \\
 & +\sigma(v) \sigma(\xi_1) \sigma(\xi_2 + u) \sigma(\xi_1 + 2\xi_2) \sigma(\xi_1 + \xi_2 - u - v) \sigma(2u + v) \\
 & = 0. \quad (\text{A}\cdot 20)
 \end{aligned}$$

Including the possible symmetry transformations (A.2) of the function (A.15) this then proves that $f_{B_2}(q, \mu) = 0$ for the general elliptic untwisted solution of (4.37).

The untwisted solution (4.37) of the G_2 functional equation (4.39) may be proven again using only the σ function identity (A.12). Since the method of proof is essentially the same as for the B_2 functional equation we omit the details of the proof.

A.3. B_2 twisted solution (4.38)

We next demonstrate that the twisted solution in (4.38) satisfies the B_2 functional equation (4.35). First we define a particular untwisted solution $x_0(u, w)$ of (4.37) by assigning a special value of the constant $b = \eta_1/\omega_1$

$$x_S(u, w) = x_L(u, w) = x_0(u, w) \equiv \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[(\eta_1/\omega_1) w u], \quad (\text{A}\cdot 21)$$

where η_1 is defined in terms of the Weierstrass ζ function as $\eta_1 \equiv \zeta(\omega_1)$. The value of b is chosen so that $x_0(u, w)$ is periodic in the ω_1 direction

$$x_0(u + 2\omega_1, w) = x_0(u, w). \quad (\text{A}\cdot 22)$$

Adding a constant vector to q does not affect the equation $f_{B_2}(q, \mu) = 0$ so q is shifted as

$$q \rightarrow q + \frac{2\omega_1 \Lambda^{(\alpha)}}{|\alpha|^2}, \quad (\text{A}\cdot 23)$$

in which $\Lambda^{(\alpha)}$ is the fundamental weight dual to the simple short root α , i.e.,

$$\alpha^\vee \cdot \Lambda^{(\alpha)} = 1, \quad \beta^\vee \cdot \Lambda^{(\alpha)} = 0. \tag{A.24}$$

Then the equation $f_{B_2}(q, \mu) = 0$ reads

$$\begin{aligned} 0 = & -x_0(\alpha \cdot q + \omega_1, \alpha^\vee \cdot \mu) x_0(\beta \cdot q, (2\alpha + \beta)^\vee \cdot \mu) \\ & + x_0((\alpha + \beta) \cdot q + \omega_1, (\alpha + \beta)^\vee \cdot \mu) x_0((2\alpha + \beta) \cdot q, -\beta^\vee \cdot \mu) \\ & + x_0(\alpha \cdot q + \omega_1, -(\alpha + \beta)^\vee \cdot \mu) x_0((2\alpha + \beta) \cdot q, (2\alpha + \beta)^\vee \cdot \mu) \\ & + x_0((\alpha + \beta) \cdot q + \omega_1, \alpha^\vee \cdot \mu) x_0(\beta \cdot q, \beta^\vee \cdot \mu), \end{aligned} \tag{A.25}$$

where the periodicity of $x_0(u, w)$, (A.22), is used. This simply means that

$$\begin{aligned} x_L(u, w) &= \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \\ x_S(u, w) &= \frac{\sigma(w - u - \omega_1)}{\sigma(w)\sigma(u + \omega_1)} \exp[\eta_1 w + b w u], \end{aligned} \tag{A.26}$$

satisfy the equation $f_{B_2}(q, \mu) = 0$. Since $x_L(u, w)$ here is the same function as in (4.37), any linear combinations of $x_S(u, w)$ from the untwisted solution in (4.37) and in (A.26) also satisfy the B_2 functional equation. Requiring that the linear combination should satisfy the A_2 functional equation and that it has a simple pole with unit residue at $u = 0$, we obtain

$$x_S^0(u, w) = \left[\frac{\sigma(w - u)}{\sigma(w)\sigma(u)} + \frac{\sigma(w - u - \omega_1)}{\sigma(w)\sigma(u + \omega_1)} \exp[\eta_1 w] \right] \exp[b w u]. \tag{A.27}$$

This expression for $x_S^0(u, w)$ has the following monodromies in u and w :

$$\begin{aligned} x_S^0(u + \omega_1, w) &= x_S^0(u, w) \exp [(-\eta_1 + b \omega_1)w], \\ x_S^0(u + 2\omega_3, w) &= x_S^0(u, w) \exp [2(-\eta_3 + b \omega_3)w], \\ x_S^0(u, w + 2\omega_1) &= x_S^0(u, w) \exp [2(-\eta_1 + b \omega_1)u], \\ x_S^0(u, w + 4\omega_3) &= x_S^0(u, w) \exp [4(-\eta_3 + b \omega_3)u]. \end{aligned} \tag{A.28}$$

It also has the following poles and zeros in the fundamental regions of u and w : simple poles at $u = 0$ and $w = 0$ with residues 1 and -2 , respectively, and a zero at $u = w/2$. It may also be shown that the twisted solution to the B_2 functional equation $x_S(u, w)$ in (4.38) has the same monodromies and poles. This implies that the ratio $x_S^0(u, w)/x_S(u, w)$ is an elliptic function in both u and w and has no poles and therefore is a constant. Since the residues are the same at all the poles, the ratio is equal to one and $x_S^0(u, w)$ is, in fact, the same as the twisted solution in (4.38).

A.4. G_2 twisted solution (4.41)

The proof that the twisted solutions (4.41) to the G_2 functional equation (4.39) follow from the untwisted solutions proceeds in a similar manner. We start from the

particular untwisted solution $x_S(u, w) = x_L(u, w) = x_0(u, w)$ satisfying $f_{G_2}(q, \mu) = 0$, with the periodicity (A.22), and shift q as

$$q \rightarrow q + \frac{4\omega_1 \Lambda^{(\alpha)}}{3|\alpha|^2}, \tag{A.29}$$

in which $\Lambda^{(\alpha)}$ is the fundamental weight dual to the simple short root α . We obtain

$$\begin{aligned} 0 = & x_0\left(\alpha \cdot q + \frac{2\omega_1}{3}, \alpha^\vee \cdot \mu\right) x_0\left(\beta \cdot q, (3\alpha + \beta)^\vee \cdot \mu\right) \\ & + x_0\left(\alpha \cdot q + \frac{2\omega_1}{3}, -(2\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + \beta) \cdot q, (3\alpha + \beta)^\vee \cdot \mu\right) \\ & + x_0\left((2\alpha + \beta) \cdot q + \frac{4\omega_1}{3}, (2\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + \beta) \cdot q, -(3\alpha + 2\beta)^\vee \cdot \mu\right) \\ & + x_0\left((2\alpha + \beta) \cdot q + \frac{4\omega_1}{3}, -(\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + 2\beta) \cdot q, (3\alpha + 2\beta)^\vee \cdot \mu\right) \\ & + x_0\left((\alpha + \beta) \cdot q + \frac{2\omega_1}{3}, (\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + 2\beta) \cdot q, -\beta^\vee \cdot \mu\right) \\ & + x_0\left((\alpha + \beta) \cdot q + \frac{2\omega_1}{3}, \alpha^\vee \cdot \mu\right) x_0\left(\beta \cdot q, \beta^\vee \cdot \mu\right). \end{aligned} \tag{A.30}$$

Shifting q again by the same amount, or in one step

$$q \rightarrow q + \frac{8\omega_1 \Lambda^{(\alpha)}}{3|\alpha|^2}, \tag{A.31}$$

gives the following equation

$$\begin{aligned} 0 = & x_0\left(\alpha \cdot q + \frac{4\omega_1}{3}, \alpha^\vee \cdot \mu\right) x_0\left(\beta \cdot q, (3\alpha + \beta)^\vee \cdot \mu\right) \\ & + x_0\left(\alpha \cdot q + \frac{4\omega_1}{3}, -(2\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + \beta) \cdot q, (3\alpha + \beta)^\vee \cdot \mu\right) \\ & + x_0\left((2\alpha + \beta) \cdot q + \frac{2\omega_1}{3}, (2\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + \beta) \cdot q, -(3\alpha + 2\beta)^\vee \cdot \mu\right) \\ & + x_0\left((2\alpha + \beta) \cdot q + \frac{2\omega_1}{3}, -(\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + 2\beta) \cdot q, (3\alpha + 2\beta)^\vee \cdot \mu\right) \\ & + x_0\left((\alpha + \beta) \cdot q + \frac{4\omega_1}{3}, (\alpha + \beta)^\vee \cdot \mu\right) x_0\left((3\alpha + 2\beta) \cdot q, -\beta^\vee \cdot \mu\right) \\ & + x_0\left((\alpha + \beta) \cdot q + \frac{4\omega_1}{3}, \alpha^\vee \cdot \mu\right) x_0\left(\beta \cdot q, \beta^\vee \cdot \mu\right), \end{aligned} \tag{A.32}$$

where the periodicity of $x_0(u, w)$, (A.22), is also used. Adding (A.30) and (A.32) together and using the symmetry of solutions in (A.2) implies that the following functions are solutions to the G_2 functional equation (4.39):

$$\begin{aligned} x_L(u, w) &= \frac{\sigma(w - u)}{\sigma(w)\sigma(u)} \exp[b w u], \\ x_S(u, w) &= \left[\frac{\sigma(w - u - \frac{2\omega_1}{3})}{\sigma(w)\sigma(u + \frac{2\omega_1}{3})} \exp[(2/3)\eta_1 w] \right] \end{aligned}$$

$$+ \frac{\sigma(w - u - \frac{4\omega_1}{3})}{\sigma(w)\sigma(u + \frac{4\omega_1}{3})} \exp[(4/3)\eta_1 w] \Big] \exp[b w u]. \tag{A.33}$$

Again, since $x_L(u, w)$ is the same function as in (4.37) an arbitrary linear combination of the $x_S(u, w)$ from (4.37) and (A.33) also satisfies the G_2 functional equation. Requiring that the linear combinations should satisfy the A_2 functional equation and that it has a simple pole with unit residue at $u = 0$, we obtain

$$x_S^0(u, w) = \left[\frac{\sigma(w - u)}{\sigma(w)\sigma(u)} + \frac{\sigma(w - u - \frac{2\omega_1}{3})}{\sigma(w)\sigma(u + \frac{2\omega_1}{3})} \exp[(2/3)\eta_1 w] + \frac{\sigma(w - u - \frac{4\omega_1}{3})}{\sigma(w)\sigma(u + \frac{4\omega_1}{3})} \exp[(4/3)\eta_1 w] \right] \exp[b w u]. \tag{A.34}$$

This function has the following monodromies:

$$\begin{aligned} x_S^0\left(u + \frac{2\omega_1}{3}, w\right) &= x_S^0(u, w) \exp\left[\frac{2}{3}(-\eta_1 + b\omega_1)w\right], \\ x_S^0(u + 2\omega_3, w) &= x_S^0(u, w) \exp[2(-\eta_3 + b\omega_3)w], \\ x_S^0(u, w + 2\omega_1) &= x_S^0(u, w) \exp[2(-\eta_1 + b\omega_1)u], \\ x_S^0(u, w + 6\omega_3) &= x_S^0(u, w) \exp[6(-\eta_3 + b\omega_3)u]. \end{aligned} \tag{A.35}$$

The following properties of $x_S^0(u, w)$ can be shown. It has poles and zeros in the fundamental regions of u and w : simple poles at $u = 0$ and $w = 0$ with residues 1 and -3 , respectively, and a zero at $u = w/3$. Since it may be shown that the twisted solution $x_S(u, w)$ to the G_2 functional equation in (4.41) has the same monodromies, poles and zeros as $x_S^0(u, w)$, by the same argument as in the B_2 case given above, $x_S^0(u, w) = x_S(u, w)$ and the solutions derived above are the same as the twisted solutions in (4.41).

A.5. Dihedral solutions (4.44) and (4.45)

Next, it will be shown that the solutions (4.44) and (4.45) satisfy the dihedral $I_2(m)$ functional equations (4.42) for odd or even m , respectively. We assume that all roots are of the same length, even for an even integer m , since they may be made so by a redefinition of the coupling constants. First consider the case of an odd integer m and arbitrary $N = 1, \dots, [m/2]$. Substituting the functions

$$x(u, w) = \left(\frac{1}{u} - \frac{1}{w}\right) \tag{A.36}$$

in (4.43) for $f_{I_2(m)}^N(q, \mu)$ and redefining $\xi \rightarrow \xi|\alpha_j|^2/2$ gives

$$\begin{aligned} f_{I_2(m)}^N(q, \mu) &= g^2 \sum_{j=1}^m x(\alpha_{j+N} \cdot q, (\alpha_{j+N} \cdot \mu)\xi) x(\alpha_j \cdot q, -(\alpha_{j+2N} \cdot \mu)\xi) \\ &= g^2 \sum_{j=1}^m \left(\frac{1}{\alpha_{j+N} \cdot q} - \frac{1}{(\alpha_{j+N} \cdot \mu)\xi}\right) \left(\frac{1}{\alpha_j \cdot q} + \frac{1}{(\alpha_{j+2N} \cdot \mu)\xi}\right) \end{aligned}$$

$$= g^2 \sum_{j=1}^m \left[\frac{1}{(\alpha_{j+N} \cdot q)(\alpha_j \cdot q)} + \frac{1}{(\alpha_{j+N} \cdot q)(\alpha_{j+2N} \cdot \mu)\xi} - \frac{1}{(\alpha_j \cdot q)(\alpha_{j+N} \cdot \mu)\xi} - \frac{1}{(\alpha_{j+N} \cdot \mu)(\alpha_{j+2N} \cdot \mu)\xi^2} \right]. \tag{A.37}$$

The $O(1/\xi)$ terms cancel pairwise using the identity

$$\sum_{j=1}^m \frac{1}{(\alpha_{j+d} \cdot q)(\alpha_{j+N+d} \cdot \mu)} = \sum_{j=1}^m \frac{1}{(\alpha_{j+d+h} \cdot q)(\alpha_{j+d+N+h} \cdot \mu)} \tag{A.38}$$

with h, d , and N arbitrary integers. This identity follows from the property of the $I_2(m)$ roots $\alpha_{j+m} = -\alpha_j$. The vanishing of the $O(\xi^0)$ and $O(1/\xi^2)$ terms follows from a simple trigonometric identity ($q = |q|(\cos t, \sin t)$):

$$\sum_{j=1}^m \frac{1}{\cos(t - j\pi/m) \cos(t - (j + N)\pi/m)} = 0. \tag{A.39}$$

The left-hand side is a meromorphic function in t with a period π and it is exponentially decreasing as $t \rightarrow \pm i\infty$. It is elementary to show that all the residues of the possible simple poles $\frac{\pi}{2} + \frac{j\pi}{m}$, $j = 1, 2, \dots, m$ vanish. Thus (A.39) vanishes. This shows that the dihedral functional equation (4.42) for odd m is satisfied by the functions in (A.36). Including the symmetry of solutions of (A.2) implies that the more general solution in (4.44) also satisfies (4.42).

Next consider the case of even m and odd N in the functional equation (4.42). Substituting the functions

$$x_O(u, w) = \left(\frac{1}{u} - \frac{1}{w} \right), \quad x_E(u, w) = \left(\frac{1}{u} - \frac{c}{w} \right) \tag{A.40}$$

in the equation for $f_{I_2(m)}^N(q, \mu)$, (4.43), and redefining $\xi \rightarrow \xi|\alpha_j|^2/2$, as before, gives

$$\begin{aligned} f_{I_2(m)}^N(q, \mu) &= g_O g_E \sum_{j=1}^{m/2} \left[x_E(\alpha_{2j} \cdot q, -(\alpha_{2j+2N} \cdot \mu)\xi) x_O(\alpha_{2j+N} \cdot q, (\alpha_{2j+N} \cdot \mu)\xi) \right. \\ &\quad \left. + x_E(\alpha_{2j+N-1} \cdot q, (\alpha_{2j+N-1} \cdot \mu)\xi) \right. \\ &\quad \left. \times x_O(\alpha_{2j-1} \cdot q, -(\alpha_{2j+2N-1} \cdot \mu)\xi) \right] \\ &= g_O g_E \left\{ \sum_{j=1}^m \left[\frac{1}{(\alpha_j \cdot q)(\alpha_{j+N} \cdot q)} - \frac{c}{(\alpha_{j+2N} \cdot \mu)(\alpha_{j+N} \cdot \mu)\xi^2} \right] \right. \\ &\quad \left. + \frac{1}{\xi} \sum_{j=1}^{m/2} \left[\frac{c}{(\alpha_{2j+N} \cdot q)(\alpha_{2j+2N} \cdot \mu)} - \frac{c}{(\alpha_{2j-1} \cdot q)(\alpha_{2j+N-1} \cdot \mu)} \right] \right\} \end{aligned}$$

$$\left. -\frac{1}{(\alpha_{2j} \cdot q)(\alpha_{2j+N} \cdot \mu)} + \frac{1}{(\alpha_{2j+N-1} \cdot q)(\alpha_{2j+2N-1} \cdot \mu)} \right\}. \quad (\text{A.41})$$

As before, the $O(1/\xi)$ terms in $f_{I_2(m)}^N(q, \mu)$ cancel pairwise using the identity

$$\sum_{j=1}^{m/2} \frac{1}{(\alpha_{2j+d} \cdot q)(\alpha_{2j+N+d} \cdot \mu)} = \sum_{j=1}^{m/2} \frac{1}{(\alpha_{2j+d+h} \cdot q)(\alpha_{2j+d+N+h} \cdot \mu)} \quad (\text{A.42})$$

with h an even integer and d and N arbitrary integers. The $O(\xi^0)$ and $O(1/\xi^2)$ terms are proportional to the corresponding terms for odd m given above and therefore vanish. Owing to the symmetry of (A.2), the solution of (4.45) satisfies (4.42) for even m and odd N .

For the case of the solutions for the $I_2(m)$ functional equation for even m and even N , note that $f_{I_2(m)}^N(q, \mu)$ may be written as

$$\begin{aligned} f_{I_2(m)}^N(q, \mu) &= \sum_{j=1}^{m/2} \left[g_O^2 x_O(\alpha_{2j+N} \cdot q, (\alpha_{2j+N} \cdot \mu)\xi) x_O(\alpha_{2j} \cdot q, -(\alpha_{2j+2N} \cdot \mu)\xi) \right. \\ &\quad \left. + g_E^2 x_E(\alpha_{2j+N-1} \cdot q, (\alpha_{2j+N-1} \cdot \mu)\xi) \right. \\ &\quad \left. \times x_E(\alpha_{2j-1} \cdot q, -(\alpha_{2j+2N-1} \cdot \mu)\xi) \right] \\ &= f_{I_2(m/2)}^{N/2}(q, \mu) + g_{I_2(m/2)}^{N/2}(q, \mu) \\ &= 0. \end{aligned} \quad (\text{A.43})$$

Here $g_{I_2(m/2)}^{N/2}(q, \mu)$ is proportional to $f_{I_2(m/2)}^{N/2}(q, \mu)$ with the $I_2(m/2)$ roots rotated by π/m and so the vanishing of $f_{I_2(m)}^N(q, \mu)$ follows using the previous equation and induction on N . Therefore the solutions in (4.45) solve the $I_2(m)$ functional equation (4.42) for all m and N .

References

- 1) F. Calogero, *J. Math. Phys.* **12** (1971), 419.
B. Sutherland, *Phys. Rev.* **A5** (1972), 1372.
J. Moser, *Adv. Math.* **16** (1975), 197; in *Dynamical Systems, Theory and Applications*, ed. J. Moser, *Lecture Notes in Physics* **38** (Springer-Verlag, 1975).
F. Calogero, C. Marchioro and O. Ragnisco, *Lett. Nuovo Cim.* **13** (1975), 383.
F. Calogero, *Lett. Nuovo Cim.* **13** (1975), 411.
- 2) M. A. Olshanetsky and A. M. Perelomov, *Phys. Rep.* **71** (1981), 314.
- 3) A. J. Bordner, E. Corrigan and R. Sasaki, *Prog. Theor. Phys.* **100** (1998), 1107, [hep-th/9805106](#).
A. J. Bordner, R. Sasaki and K. Takasaki, *Prog. Theor. Phys.* **101** (1999), 487, [hep-th/9809068](#).
A. J. Bordner and R. Sasaki, *Prog. Theor. Phys.* **101** (1999), 799, [hep-th/9812232](#).
- 4) C. F. Dunkl, *Trans. Amer. Math. Soc.* **311** (1989), 167.
- 5) V. M. Buchstaber, G. Felder and A. P. Veselov, *Duke Math. J.* **76** (1994), 885.
- 6) J. E. Humphreys, *Reflection groups and Coxeter groups* (Cambridge Univ. Press, Cambridge, 1990).

- 7) J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and representations* (Cambridge Univ. Press, Cambridge, 1997).
- 8) M. A. Olshanetsky and A. M. Perelomov, *Phys. Rep.* **94** (1983), 313.
- 9) I. M. Krichever, *Funct. Anal. Appl.* **14** (1980), 282.
- 10) E. D'Hoker and D. H. Phong, *Nucl. Phys.* **B530** (1998), 537, [hep-th/9804124](#); **B534** (1998), 697, [hep-th/9804126](#).
- 11) I. G. MacDonald, *Bull. London Math. Soc.* **4** (1972), 148.
- 12) L. Hawkins, *Sém. Lothar. Combin.* **34** (1995), Art. B34b (electronic).
- 13) V. I. Inozemtsev, *Lett. Math. Phys.* **17** (1989), 11.
- 14) D. F. Lawden, *Elliptic functions and applications* (Springer-Verlag, New York, 1989).
- 15) M. Bruschi and F. Calogero, *Siam J. Math. Anal.* **21** (1990), 1019.