Research Article
Generalised Interval-Valued Fuzzy Soft Set

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We introduce the concept of generalised interval-valued fuzzy soft set and its operations and study some of their properties. We give applications of this theory in solving a decision making problem. We also introduce a similarity measure of two generalised interval-valued fuzzy soft sets and discuss its application in a medical diagnosis problem: fuzzy set; soft set; fuzzy soft set; generalised fuzzy soft set; generalised interval-valued fuzzy soft set; interval-valued fuzzy set; interval-valued fuzzy soft set.

1. Introduction

Molodtsov [1] initiated the theory of soft set as a new mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, and so forth. Presently, work on the soft set theory is progressing rapidly. Maji et al. [2, 3], Roy and Maji [4] have further studied the theory of soft set and used this theory to solve some decision making problems. Maji et al. [5] have also introduced the concept of fuzzy soft set, a more general concept, which is a combination of fuzzy set and soft set and studied its properties. Zou and Xiao [6] introduced soft set and fuzzy soft set into the incomplete environment, respectively. Alkhazaleh et al. [7] introduced the concept of soft multiset as a generalisation of soft set. They also defined the concepts of fuzzy parameterized interval-valued fuzzy soft set [8] and possibility fuzzy soft set [9] and gave their applications in decision making and medical diagnosis. Alkhazaleh and Salleh [10] introduced the concept of a soft expert set, where the user can know the opinion of all experts in one model without any operations. Even after any operation, the user can know the opinion of all experts. In 2011, Salleh [11] gave a brief survey from soft set to intuitionistic fuzzy soft set. Majumdar and Samanta [12] introduced and studied generalised fuzzy soft set where the degree is

attached with the parameterization of fuzzy sets while defining a fuzzy soft set. Yang et al. [13] presented the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set [14, 15] and soft set models. In this paper, we generalise the concept of fuzzy soft set as introduced by Maji et al. [5] to generalised interval-valued fuzzy soft set. In our generalisation of fuzzy soft set, a degree is attached with the parameterization of fuzzy sets while defining an interval-valued fuzzy soft set. Also, we give some applications of generalised interval-valued fuzzy soft set in decision making problem and medical diagnosis.

2. Preliminary

In this section, we recall some definitions and properties regarding fuzzy soft set and generalised fuzzy soft set required in this paper.

Definition 2.1 (see [15]). An interval-valued fuzzy set $\widetilde{X}$ on a universe U is a mapping such that

$$\widetilde{X} : U \longrightarrow \text{Int}([0,1]), \tag{2.1}$$

where $\text{Int}([0,1])$ stands for the set of all closed subintervals of $[0,1]$, the set of all interval-valued fuzzy sets on U is denoted by $\widetilde{P}(U)$.

Suppose that $\widetilde{X} \in \widetilde{P}(U)$, for all $x \in U$, $\mu_x(x) = [\mu_x^-(x), \mu_x^+(x)]$ is called the degree of membership of an element x to X. $\mu_x^-(x)$ and $\mu_x^+(x)$ are referred to as the lower and upper degrees of membership of x to X where $0 \leqslant \mu_x^-(x) \leqslant \mu_x^+(x) \leqslant 1$.

Definition 2.2 (see [14]). The subset, complement, intersection, and union of the interval-valued fuzzy sets are defined as follows. Let $\widetilde{X}, \widetilde{Y} \in \widetilde{P}(U)$, then

(a) the complement of $\widetilde{X}$ is denoted by $\widetilde{X}^c$ where

$$\mu_{\widetilde{X}^c}(x) = 1 - \mu_{\widetilde{X}}(x) = \left[1 - \mu_{\widetilde{X}}^+(x), 1 - \mu_{\widetilde{X}}^-(x)\right], \tag{2.2}$$

(b) the intersection of $\widetilde{X}$ and $\widetilde{Y}$ is denoted by $\widetilde{X} \cap \widetilde{Y}$ where

$$\begin{aligned}
\mu_{\widetilde{X} \cap \widetilde{Y}}(x) &= \inf\left[\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\right] \\
&= \left[\inf\left(\mu_{\widetilde{X}}^-(x), \mu_{\widetilde{Y}}^-(x)\right), \inf\left(\mu_{\widetilde{X}}^+(x), \mu_{\widetilde{Y}}^+(x)\right)\right],
\end{aligned} \tag{2.3}$$

(c) the union of $\widetilde{X}$ and $\widetilde{Y}$ is denoted by $\widetilde{X} \cup \widetilde{Y}$ where

$$\begin{aligned}
\mu_{\widetilde{X} \cup \widetilde{Y}}(x) &= \sup\left[\mu_{\widetilde{X}}(x), \mu_{\widetilde{Y}}(x)\right] \\
&= \left[\sup\left(\mu_{\widetilde{X}}^-(x), \mu_{\widetilde{Y}}^-(x)\right), \sup\left(\mu_{\widetilde{X}}^+(x), \mu_{\widetilde{Y}}^+(x)\right)\right];
\end{aligned} \tag{2.4}$$

(d) X is a subset of Y denoted by $X \subseteq Y$ if $\mu_X^-(x) \leq \mu_Y^-(x)$ and $\mu_X^+(x) \leq \mu_Y^+(x)$.

Definition 2.3 (see [14]). The compatibility measure $\overline{\varphi}(A, B)$ of an interval-valued fuzzy set A with an interval-valued fuzzy set B (A is a reference set) is given by

$$\overline{\varphi}(A, B) = [\varphi^-(A, B), \varphi^+(A, B)], \tag{2.5}$$

such that

$$\varphi^-(A, B) = \min[\varphi_1(A, B), \varphi_2(A, B)],$$
$$\varphi^+(A, B) = \max[\varphi_1(A, B), \varphi_2(A, B)], \tag{2.6}$$

where

$$\varphi_1(A, B) = \frac{\max_{x \in X}\{\min[\mu_A^-(x), \mu_B^-(x)]\}}{\max_{x \in X}[\mu_A^-(x)]},$$

$$\varphi_2(A, B) = \frac{\max_{x \in X}\{\min[\mu_A^+(x), \mu_B^+(x)]\}}{\max_{x \in X}[\mu_A^+(x)]}. \tag{2.7}$$

Theorem 2.4 (see [14]). *Consider arbitrary, nonempty interval-valued fuzzy sets A, B, and C from the family of $ivf(X)$ and a compatibility measure in the sense of Definition 2.3. Then,*

(a) $A \ \overline{\varphi}(A, A) = [1, 1] = \{1\}$,

(b) $\overline{\varphi}(A, B) = [0, 0] = \{0\} \Leftrightarrow A \cap B = \emptyset$,

(c) *in general* $\overline{\varphi}(A, B) \neq \overline{\varphi}(B, A)$.

Let U be a universal set and E a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$. Molodtsov [1] defined soft set as follows.

Definition 2.5. A pair (F, E) is called a soft set over U, where F is a mapping given by $F : E \rightarrow P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.6 (see [5]). Let U be a universal set, and let E be a set of parameters. Let I^U denote the power set of all fuzzy subsets of U. Let $A \subseteq E$. A pair (F, E) is called a fuzzy soft set over U where F is a mapping given by

$$F : A \longrightarrow I^U. \tag{2.8}$$

Definition 2.7 (see [12]). Let $U = \{x_1, x_2, \ldots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \ldots, e_m\}$ be the universal set of parameters. The pair (U, E) will be called a soft universe.

Let $F : E \to I^U$, where I^U is the collection of all fuzzy subsets of U, and let μ be a fuzzy subset of E. Let $F_\mu : E \to I^U \times I$ be a function defined as follows:

$$F_\mu(e) = \left(F(e),\ \mu(e) \right). \tag{2.9}$$

Then, F_μ is called a generalised fuzzy soft set (GFSS in short) over the soft set (U, E). Here, for each parameter e_i, $F_\mu(e_i) = (F(e_i), \mu(e_i))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$ but also the degree of possibility of such belongingness which is represented by $\mu(e_i)$. So we can write $F_\mu(e_i)$ as follows:

$$F_\mu(e_i) = \left(\left\{ \frac{x_1}{F(e_i)(x_1)}, \frac{x_2}{F(e_i)(x_2)}, \dots, \frac{x_n}{F(e_i)(x_n)} \right\}, \mu(e_i) \right), \tag{2.10}$$

where $F(e_i)(x_1)$, $F(e_i)(x_2), \dots, F(e_i)(x_n)$ are the degree of belongingness and $\mu(e_i)$ is the degree of possibility of such belongingness.

Definition 2.8 (see [13]). Let U be an initial universe and E a set of parameters. $\widetilde{P}(U)$ denotes the set of all interval-valued fuzzy sets of U. Let $A \subseteq E$. A pair $(\widetilde{F}, A)$ is an interval-valued fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : A \to \widetilde{P}(U)$.

3. Generalised Interval-Valued Fuzzy Soft Set

In this section, we generalise the concept of interval-valued fuzzy soft sets as introduced in [13]. In our generalisation of interval-valued fuzzy soft set, a degree is attached with the parameterization of fuzzy sets while defining an interval-valued fuzzy soft set.

Definition 3.1. Let $U = \{x_1, x_2, \dots, x_n\}$ be the universal set of elements and $E = \{e_1, e_2, \dots, e_m\}$ the universal set of parameters. The pair (U, E) will be called a soft universe. Let $\widetilde{F} : E \to \widetilde{P}(U)$ and μ be a fuzzy set of E, that is, $\mu : E \to I = [0, 1]$, where $\widetilde{P}(U)$ is the set of all interval-valued fuzzy subsets on U. Let $\widetilde{F}_\mu : E \to \widetilde{P}(U) \times I$ be a function defined as follows:

$$\widetilde{F}_\mu(e) = \left(\widetilde{F}(e),\ \mu(e) \right). \tag{3.1}$$

Then, $\widetilde{F}_\mu$ is called a generalised interval-valued fuzzy soft set (GIVFSS in short) over the soft universe (U, E). For each parameter e_i, $\widetilde{F}_\mu(e_i) = (\widetilde{F}(e_i)(x), \mu(e_i))$ indicates not only the degree of belongingness of the elements of U in $\widetilde{F}(e_i)$ but also the degree of possibility of such belongingness which is represented by $\mu(e_i)$. So we can write $\widetilde{F}_\mu(e_i)$ as follows:

$$\widetilde{F}_\mu(e_i) = \left(\left\{ \frac{x_1}{\widetilde{F}(e_i)(x_1)}, \frac{x_2}{\widetilde{F}(e_i)(x_2)}, \dots, \frac{x_n}{\widetilde{F}(e_i)(x_n)} \right\}, \mu(e_i) \right). \tag{3.2}$$

Example 3.2. Let $U = \{x_1, x_2, x_3\}$ be a set of universe, $E = \{e_1, e_2, e_3\}$ a set of parameters, and let $\mu : E \to I$. Define a function $\widetilde{F}_\mu : E \to \widetilde{P}(U) \times I$ as follows:

$$\widetilde{F}_\mu(e_1) = \left(\left\{\frac{x_1}{[0.3, 0.6]}, \frac{x_2}{[0.7, 0.8]}, \frac{x_3}{[0.5, 0.8]}\right\}, 0.6\right),$$

$$\widetilde{F}_\mu(e_2) = \left(\left\{\frac{x_1}{[0.1, 0.4]}, \frac{x_2}{[0, 0.3]}, \frac{x_3}{[0.1, 0.5]}\right\}, 0.5\right), \tag{3.3}$$

$$\widetilde{F}_\mu(e_3) = \left(\left\{\frac{x_1}{[0.7, 0.8]}, \frac{x_2}{[0.1, 0.2]}, \frac{x_3}{[0, 0.4]}\right\}, 0.3\right).$$

Then, $\widetilde{F}_\mu$ is a GIVFSS over (U, E).

In matrix notation, we write

$$\widetilde{F}_\mu = \begin{pmatrix} [0.3, 0.6] & [0.7, 0.8] & [0.5, 0.8] & 0.6 \\ [0.1, 0.4] & [0, 0.3] & [0.1, 0.5] & 0.5 \\ [0.7, 0.8] & [0.1, 0.2] & [0, 0.4] & 0.3 \end{pmatrix}. \tag{3.4}$$

Definition 3.3. Let $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ be two GIVFSSs over (U, E). $\widetilde{F}_\mu$ is called a generalised interval-valued fuzzy soft subset of $\widetilde{G}_\delta$, and we write $\widetilde{F}_\mu \subseteq \widetilde{G}_\delta$ if

(a) $\mu(e)$ is a fuzzy subset of $\delta(e)$ for all $e \in E$,

(b) $\widetilde{F}(e)$ is an interval-valued fuzzy subset of $\widetilde{G}(e)$ for all $e \in E$.

Example 3.4. Let $U = \{x_1, x_2, x_3\}$ be a set of three cars, and let $E = \{e_1, e_2, e_3\}$ be a set of parameters where $e_1 = $ cheap, $e_2 = $ expensive, $e_3 = $ red. Let $\widetilde{F}_\mu$ be a GIVFSS over (U, E) defined as follows:

$$\widetilde{F}_\mu(e_1) = \left(\left\{\frac{x_1}{[0.1, 0.3]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.4\right),$$

$$\widetilde{F}_\mu(e_2) = \left(\left\{\frac{x_1}{[0, 0.3]}, \frac{x_2}{[0, 0.2]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.4\right), \tag{3.5}$$

$$\widetilde{F}_\mu(e_3) = \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.1, 0.1]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.1\right).$$

Let $\widetilde{G}_\delta$ be another GIVFSS over (U, E) defined as follows:

$$\widetilde{G}_\delta(e_1) = \left(\left\{\frac{x_1}{[0.3, 0.6]}, \frac{x_2}{[0.7, 0.8]}, \frac{x_3}{[0.5, 0.8]}\right\}, 0.6\right),$$

$$\widetilde{G}_\delta(e_2) = \left(\left\{\frac{x_1}{[0.2, 0.4]}, \frac{x_2}{[0.2, 0.3]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.5\right), \tag{3.6}$$

$$\widetilde{G}_\delta(e_3) = \left(\left\{\frac{x_1}{[0.7, 0.8]}, \frac{x_2}{[0.2, 0.4]}, \frac{x_3}{[0.2, 0.5]}\right\}, 0.3\right).$$

It is clear that $\widetilde{F}_\mu$ is a GIVFS subset of $\widetilde{G}_\delta$.

Definition 3.5. Two GIVFSSs $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ over (U, E) are said to be equal, and we write $\widetilde{F}_\mu = \widetilde{G}_\delta$ if $\widetilde{F}_\mu$ is a GIVFS subset of $\widetilde{G}_\delta$ and $\widetilde{G}_\delta$ is a GIVFS subset of $\widetilde{F}_\mu$. In other words, $\widetilde{F}_\mu = \widetilde{G}_\delta$ if the following conditions are satisfied:

   (a) $\mu(e)$ is equal to $\delta(e)$ for all $e \in E$,

   (b) $\widetilde{F}(e)$ is equal to $\widetilde{G}(e)$ for all $e \in E$.

Definition 3.6. A GIVFSS is called a generalised null interval-valued fuzzy soft set, denoted by $\widetilde{\emptyset}_\mu$ if $\widetilde{\emptyset}_\mu : E \to \widetilde{P}(U) \times I$ such that

$$\widetilde{\emptyset}_\mu(e) = \left(\widetilde{F}(e)(x), \mu(e) \right), \tag{3.7}$$

where $\widetilde{F}(e) = [0, 0] = [0]$ and $\mu(e) = 0$ for all $e \in E$.

Definition 3.7. A GIVFSS is called a generalised absolute interval-valued fuzzy soft set, denoted by $\widetilde{A}_\mu$ if $\widetilde{A}_\mu : E \to \widetilde{P}(U) \times I$ such that

$$\widetilde{A}_\mu(e) = \left(\widetilde{F}(e)(x), \mu(e) \right), \tag{3.8}$$

where $\widetilde{F}(e) = [1, 1] = [1]$, and $\mu(e) = 1$ for all $e \in E$.

4. Basic Operations on GIVFSS

In this section, we introduce some basic operations on GIVFSS, namely, complement, union and intersection and we give some properties related to these operations.

Definition 4.1. Let $\widetilde{F}_\mu$ be a GIVFSS over (U, E). Then, the complement of $\widetilde{F}_\mu$, denoted by $\widetilde{F}_\mu^c$ and is defined by $\widetilde{F}_\mu^c = \widetilde{G}_\delta$, such that $\delta(e) = c(\mu(e))$ and $\widetilde{G}(e) = \widetilde{c}(\widetilde{F}(e))$ for all $e \in E$, where c is a fuzzy complement and $\widetilde{c}$ is an interval-valued fuzzy complement.

Example 4.2. Consider a GIVFSS $\widetilde{F}_\mu$ over (U, E) as in Example 3.2:

$$\widetilde{F}_\mu = \begin{pmatrix} [0.3, 0.6] & [0.7, 0.8] & [0.5, 0.8] & 0.6 \\ [0.1, 0.4] & [0, 0.3] & [0.1, 0.5] & 0.5 \\ [0.7, 0.8] & [0.1, 0.2] & [0, 0.4] & 0.3 \end{pmatrix}. \tag{4.1}$$

By using the basic fuzzy complement for $\mu(e)$ and interval-valued fuzzy complement for $\widetilde{F}(e)$, we have $\widetilde{F}_\mu^c = G_\delta$ where

$$G_\delta = \begin{pmatrix} [0.4, 0.7] & [0.2, 0.3] & [0.2, 0.5] & 0.4 \\ [0.6, 0.9] & [0.7, 1] & [0.5, 0.9] & 0.5 \\ [0.2, 0.3] & [0.8, 0.9] & [0.6, 1] & 0.7 \end{pmatrix}. \tag{4.2}$$

Proposition 4.3. *Let $\tilde{F}_\mu$ be a GIVFSS over (U, E). Then, the following holds:*

$$\left(\tilde{F}_\mu^c\right)^c = \tilde{F}_\mu. \tag{4.3}$$

Proof. Since $\tilde{F}_\mu^c = \tilde{G}_\delta$, then

$$\left(\tilde{F}_\mu^c\right)^c = \tilde{G}_\delta^c \tag{4.4}$$

but, from Definition 4.1, $\tilde{G}_\delta = (\tilde{c}(\tilde{F}(e)), c(\mu(e)))$, then

$$\tilde{G}_\delta^c = \left(\tilde{c}\left(\tilde{c}\left(\tilde{F}(e)\right)\right), c(c(\mu(e)))\right)$$

$$= \left(\tilde{F}(e), \mu(e)\right) \tag{4.5}$$

$$= \tilde{F}_\mu. \qquad\qquad \square$$

Definition 4.4. Union of two GIVFSSs $(\tilde{F}_\mu, A)$ and $(\tilde{G}_\delta, B)$, denoted by $\tilde{F}_\mu \cup \tilde{G}_\delta$, is a GIVFSS $(\widetilde{H}_\nu, C)$ where $C = A \cup B$ and $\widetilde{H}_\nu : E \to \tilde{P}(U) \times I$ is defined by

$$\widetilde{H}_\nu(e) = \left(\widetilde{H}(e), \nu(e)\right) \tag{4.6}$$

such that $\widetilde{H}(e) = \tilde{F}(e) \cup \tilde{G}(e)$ and $\nu(e) = s(\mu(e), \delta(e))$, where s is an s-norm and $\widetilde{H}(e) = [\sup(\mu_{\tilde{F}(e)}^-, \mu_{\tilde{G}(e)}^-), \sup(\mu_{\tilde{F}(e)}^+, \mu_{\tilde{G}(e)}^+)]$.

Example 4.5. Consider GIVFSS $\tilde{F}_\mu$ and $\tilde{G}_\delta$ as in Example 3.4. By using the interval-valued fuzzy union and basic fuzzy union, we have $\tilde{F}_\mu \cup \tilde{G}_\delta = \widetilde{H}_\nu$, where

$$\widetilde{H}_\nu(e_1) = \left(\left\{\frac{x_1}{[\sup(0.1, 0.3), \sup(0.3, 0.6)]}, \frac{x_2}{[\sup(0.5, 0.7), \sup(0.7, 0.8)]},\right.\right.$$

$$\left.\left.\frac{x_3}{[\sup(0.3, 0.5), \sup(0.5, 0.8)]}\right\}, \max(0.4, 0.6)\right) \tag{4.7}$$

$$= \left(\left\{\frac{x_1}{[0.3, 0.6]}, \frac{x_2}{[0.7, 0.8]}, \frac{x_3}{[0.5, 0.8]}\right\}, 0.6\right).$$

Similarly, we get

$$\widetilde{H}_\nu(e_2) = \left(\left\{\frac{x_1}{[0.2, 0.4]}, \frac{x_2}{[0.2, 0.3]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.5\right),$$

$$\widetilde{H}_\nu(e_3) = \left(\left\{\frac{x_1}{[0.7, 0.8]}, \frac{x_2}{[0.2, 0.4]}, \frac{x_3}{[0.2, 0.5]}\right\}, 0.3\right). \tag{4.8}$$

In matrix notation, we write

$$
\widetilde{H}_v(e) = \begin{pmatrix} [0.3,0.6] & [0.7,0.8] & [0.5,0.8] & 0.6 \\ [0.2,0.4] & [0.2,0.3] & [0.3,0.5] & 0.5 \\ [0.7,0.8] & [0.2,0.4] & [0.2,0.5] & 0.3 \end{pmatrix}. \tag{4.9}
$$

Proposition 4.6. *Let $\widetilde{F}_\mu$, $\widetilde{G}_\delta$, and $\widetilde{H}_v$ be any three GIVFSSs. Then, the following results hold.*

(a) $\widetilde{F}_\mu \cup \widetilde{G}_\delta = \widetilde{G}_\delta \cup \widetilde{F}_\mu$.

(b) $\widetilde{F}_\mu \cup (\widetilde{G}_\delta \cup \widetilde{H}_v) = (\widetilde{F}_\mu \cup \widetilde{G}_\delta) \cup \widetilde{H}_v$.

(c) $\widetilde{F}_\mu \cup \widetilde{F}_\mu \subseteq \widetilde{F}_\mu$.

(d) $\widetilde{F}_\mu \cup \widetilde{A}_\mu = \widetilde{A}_\mu$.

(e) $\widetilde{F}_\mu \cup \widetilde{\emptyset}_\mu = \widetilde{F}_\mu$.

Proof. (a) $\widetilde{F}_\mu \cup \widetilde{G}_\delta = \widetilde{H}_v$.

From Definition 4.4, we have $\widetilde{H}_v(e) = (\widetilde{H}(e)(x), v(e))$ such that $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$ and $v(e) = s(\mu(e), \delta(e))$.

But $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e) = \widetilde{G}(e) \cup \widetilde{F}(e)$ (since union of interval-valued fuzzy sets is commutative) and $v(e) = s(\mu(e), \delta(e)) = s(\delta(e), \mu(e))$ (since s-norm is commutative), then $\widetilde{G}_\delta \cup \widetilde{F}_\mu = \widetilde{H}_v$.

(b) The proof is straightforward from Definition 4.4.
(c) The proof is straightforward from Definition 4.4.
(d) The proof is straightforward from Definition 4.4.
(e) The proof is straightforward from Definition 4.4.   □

Definition 4.7. Intersection of two GIVFSSs $(\widetilde{F}_\mu, A)$ and $(\widetilde{G}_\delta, B)$, denoted by $\widetilde{F}_\mu \cap \widetilde{G}_\delta$, is a GIVFSS $(\widetilde{H}_v, C)$ where $C = A \cup B$ and $\widetilde{H}_v : E \to \widetilde{P}(U) \times I$ is defined by

$$
\widetilde{H}_v(e) = \left(\widetilde{H}(e), v(e) \right) \tag{4.10}
$$

such that $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$ and $v(e) = t(\mu(e), \delta(e))$, where t is a t-norm and $\widetilde{H}(e) = [\inf(\mu^-_{\widetilde{F}(e)}, \mu^-_{\widetilde{G}(e)}), \inf(\mu^+_{\widetilde{F}(e)}, \mu^+_{\widetilde{G}(e)})]$.

Example 4.8. Consider GIVFSS $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ as in Example 4.5. By using the interval-valued fuzzy intersection and basic fuzzy intersection, we have $\widetilde{F}_\mu \cap \widetilde{G}_\delta = \widetilde{H}_v$, where

$$
\begin{aligned}
\widetilde{H}_v(e_1) \quad &= \left(\left\{ \frac{x_1}{[\inf(0.1,0.3), \inf(0.3,0.6)]}, \frac{x_2}{[\inf(0.5,0.7), \inf(0.7,0.8)]}, \right. \right. \\
&\qquad \left. \left. \frac{x_3}{[\inf(0.3,0.5), \inf(0.5,0.8)]} \right\} \min(0.4,0.6) \right) \\
&= \left(\left\{ \frac{x_1}{[0.1,0.3]}, \frac{x_2}{[0.5,0.7]}, \frac{x_3}{[0.3,0.5]} \right\}, 0.4 \right).
\end{aligned} \tag{4.11}
$$

Similarly, we get

$$\widetilde{H}_v(e_2) = \left(\left\{\frac{x_1}{[0,0.3]}, \frac{x_2}{[0,0.2]}, \frac{x_3}{[0.1,0.3]}\right\}, 0.4\right),$$

$$\widetilde{H}_v(e_3) = \left(\left\{\frac{x_1}{[0.5,0.6]}, \frac{x_2}{[0.1,0.1]}, \frac{x_3}{[0.1,0.3]}\right\}, 0.1\right). \tag{4.12}$$

In matrix notation, we write

$$\widetilde{H}_v(e) = \begin{pmatrix} [0.1,0.3] & [0.5,0.7] & [0.3,0.5] & 0.4 \\ [0,0.3] & [0,0.2] & [0.1,0.3] & 0.4 \\ [0.5,0.6] & [0.1,0.1] & [0.1,0.3] & 0.1 \end{pmatrix}. \tag{4.13}$$

Proposition 4.9. *Let $\widetilde{F}_\mu$, $\widetilde{G}_\delta$, and $\widetilde{H}_v$ be any three GIVFSSs. Then, the following results hold.*

(a) $\widetilde{F}_\mu \cap \widetilde{G}_\delta = \widetilde{G}_\delta \cap \widetilde{F}_\mu$.

(b) $\widetilde{F}_\mu \cap (\widetilde{G}_\delta \cap \widetilde{H}_v) = (\widetilde{F}_\mu \cap \widetilde{G}_\delta) \cap \widetilde{H}_v$.

(c) $\widetilde{F}_\mu \cap \widetilde{F}_\mu \subseteq \widetilde{F}_\mu$.

(d) $\widetilde{F}_\mu \cap \widetilde{A}_\mu = \widetilde{F}_\mu$.

(e) $\widetilde{F}_\mu \cap \widetilde{\emptyset}_\mu = \widetilde{\emptyset}_\mu$.

Proof. (a) $\widetilde{F}_\mu \cap \widetilde{G}_\delta = \widetilde{H}_v$.

From Definition 4.7, we have $\widetilde{H}_v(e) = (\widetilde{H}(e)(x), v(e))$ such that $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$ and $v(e) = t(\mu(e), \delta(e))$.

But $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e) = \widetilde{G}(e) \cap \widetilde{F}(e)$ (since intersection of interval-valued fuzzy sets is commutative) and $v(e) = t(\mu(e), \delta(e)) = t(\delta(e), \mu(e))$ (since t-norm is commutative), then $\widetilde{G}_\delta \cap \widetilde{F}_\mu = \widetilde{H}_v$.

(b) The proof is straightforward from Definition 4.7.

(c) The proof is straightforward from Definition 4.7.

(d) The proof is straightforward from Definition 4.7.

(e) The proof is straightforward from Definition 4.7. □

Proposition 4.10. *Let $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ be any two GIVFSSs. Then the DeMorgan's Laws hold:*

(a) $(\widetilde{F}_\mu \cup \widetilde{G}_\delta)^c = \widetilde{G}_\delta^c \cap \widetilde{F}_\mu^c$.

(b) $(\widetilde{F}_\mu \cap \widetilde{G}_\delta)^c = \widetilde{G}_\delta^c \cup \widetilde{F}_\mu^c$.

Proof. (a) Consider

$$\widetilde{F}_\mu^c \cap \widetilde{G}_\delta^c = ((\widetilde{c}(F(e)), c(\mu(e))) \cap (\widetilde{c}(G(e)), c(\delta(e))))$$

$$= ((\widetilde{c}(F(e)) \cap \widetilde{c}(G(e))), (c(\mu(e)) \cap c(\delta(e)))) \tag{4.14}$$

$$= \left(((F(e)) \cup (G(e)))^{\widetilde{c}}, (\mu(e) \cup \delta(e))^c\right) = \left(\widetilde{F}_\mu \cup \widetilde{G}_\delta\right)^c.$$

(b) The proof is similar to the above. □

Proposition 4.11. *Let $\widetilde{F}_\mu$, $\widetilde{G}_\delta$, and $\widetilde{H}_v$ be any three GIVFSSs. Then, the following results hold.*

    (a) $\widetilde{F}_\mu \cup (\widetilde{G}_\delta \cap \widetilde{H}_v) = (\widetilde{F}_\mu \cup \widetilde{G}_\delta) \cap (\widetilde{F}_\mu \cup \widetilde{H}_v)$.

    (b) $\widetilde{F}_\mu \cap (\widetilde{G}_\delta \cup \widetilde{H}_v) = (\widetilde{F}_\mu \cap \widetilde{G}_\delta) \cup (\widetilde{F}_\mu \cap \widetilde{H}_v)$.

Proof. (a) For all $x \in E$,

$$
\begin{aligned}
\lambda_{\widetilde{F}(x) \cup (\widetilde{G}(x) \cap \widetilde{H}(x))}(x) &= \left[\sup\left(\lambda^-_{\widetilde{F}(x)}(x), \lambda^-_{\widetilde{G}(x) \cap \widetilde{H}(x)}(x) \right), \sup\left(\lambda^+_{\widetilde{F}(x)}(x), \lambda^+_{\widetilde{G}(x) \cap \widetilde{H}(x)}(x) \right) \right] \\
&= \left[\sup\left(\lambda^-_{\widetilde{F}(x)}(x), \inf\left(\lambda^-_{\widetilde{G}(x)}(x), \lambda^-_{\widetilde{H}(x)}(x) \right) \right), \right. \\
&\qquad \left. \sup\left(\lambda^+_{\widetilde{F}(x)}(x), \inf\left(\lambda^+_{\widetilde{G}(x)}(x), \lambda^+_{\widetilde{H}(x)}(x) \right) \right) \right] \\
&= \left[\inf\left(\sup\left(\lambda^-_{\widetilde{F}(x)}(x), \lambda^-_{\widetilde{G}(x)}(x) \right), \sup\left(\lambda^-_{\widetilde{F}(x)}(x), \lambda^-_{\widetilde{H}(x)}(x) \right) \right), \right. \\
&\qquad \left. \inf\left(\sup\left(\lambda^+_{\widetilde{F}(x)}(x), \lambda^+_{\widetilde{G}(x)}(x) \right), \sup\left(\lambda^+_{\widetilde{F}(x)}(x), \lambda^+_{\widetilde{H}(x)}(x) \right) \right) \right] \\
&= \lambda_{(\widetilde{F}(x) \cup \widetilde{G}(x)) \cap (\widetilde{F}(x) \cup \widetilde{H}(x))}(x), \\
\gamma_{\mu(x) \cup (\delta(x) \cap v(x))}(x) &= \max\left\{ \gamma_{\mu(x)}(x), \gamma_{\delta(x) \cap v(x)}(x) \right\} \\
&= \max\left\{ \gamma_{\mu(x)}(x), \min\left(\gamma_{\delta(x)}(x), \gamma_{v(x)}(x) \right) \right\} \\
&= \min\left\{ \max\left(\gamma_{\mu(x)}(x), \gamma_{\delta(x)}(x) \right), \max\left(\gamma_{\mu(x)}(x), \gamma_{v(x)}(x) \right) \right\} \\
&= \min\left\{ \gamma_{\mu(x) \cup \delta(x)}(x), \gamma_{\mu(x) \cup v(x)}(x) \right\} \\
&= \gamma_{(\mu(x) \cup \delta(x)) \, \widetilde{\cap} \, (\mu(x) \cup v(x))}(x).
\end{aligned}
\tag{4.15}
$$

    (b) Similar to the proof of (a).   □

5. AND and OR Operations on GIVFSS with Application

In this section, we give the definitions of AND and OR operations on GIVFSS. An application of this operations in decision making problem has been shown.

Definition 5.1. If $(\widetilde{F}_\mu, A)$ and $(\widetilde{G}_\delta, B)$ are two GIVFSSs, then "$(\widetilde{F}_\mu, A)$ AND $(\widetilde{G}_\delta, B)$" denoted by $(\widetilde{F}_\mu, A) \wedge (\widetilde{G}_\delta, B)$ is defined by

$$
\left(\widetilde{F}_\mu, A \right) \wedge \left(\widetilde{G}_\delta, B \right) = \left(\widetilde{H}_\lambda, A \times B \right),
\tag{5.1}
$$

where $\widetilde{H}_\lambda(\alpha, \beta) = (H(\alpha, \beta), \lambda(\alpha, \beta))$ for all $(\alpha, \beta) \in A \times B$, such that $\widetilde{H}(\alpha, \beta) = \widetilde{F}(\alpha) \cap \widetilde{G}(\beta)$ and $\lambda(\alpha, \beta) = t(\mu(\alpha), \delta(\beta))$, for all $(\alpha, \beta) \in A \times B$, where t is a t-norm.

Example 5.2. Suppose the universe consists of three machines x_1, x_2, x_3, that is, $U = \{x_1, x_2, x_3\}$, and consider the set of parameters $E = \{e_1, e_2, e_3\}$ which describe their perfor-mances according to certain specific task. Suppose a firm wants to buy one such machine

depending on any two of the parameters only. Let there be two observations $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ by two experts A and B, respectively, defined as follows:

$$
\begin{aligned}
\widetilde{F}_\mu(e_1) &= \left(\left\{\frac{x_1}{[0.1, 0.3]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.4\right), \\
\widetilde{F}_\mu(e_2) &= \left(\left\{\frac{x_1}{[0, 0.3]}, \frac{x_2}{[0, 0.2]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.4\right), \\
\widetilde{F}_\mu(e_3) &= \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.1, 0.1]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.1\right), \\
\widetilde{G}_\mu(e_1) &= \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.2, 0.6]}, \frac{x_3}{[0.4, 0.5]}\right\}, 0.3\right), \\
\widetilde{G}_\mu(e_2) &= \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.4, 0.6]}, \frac{x_3}{[0, 0.3]}\right\}, 0.1\right), \\
\widetilde{G}_\mu(e_3) &= \left(\left\{\frac{x_1}{[0.1, 0.6]}, \frac{x_2}{[0.4, 0.7]}, \frac{x_3}{[0.2, 0.3]}\right\}, 0.2\right).
\end{aligned}
\tag{5.2}
$$

To find the AND between the two GIVFSSs, we have $(\widetilde{F}_\mu, A)$ AND $(\widetilde{G}_\delta, B) = (\widetilde{H}_\lambda, A \times B)$, where

$$
\begin{aligned}
\widetilde{H}_\lambda(e_1, e_1) &= \left(\left\{\frac{x_1}{[0.1, 0.3]}, \frac{x_2}{[0.2, 0.6]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.3\right), \\
\widetilde{H}_\lambda(e_1, e_2) &= \left(\left\{\frac{x_1}{[0.1, 0.3]}, \frac{x_2}{[0.4, 0.6]}, \frac{x_3}{[0, 0.3]}\right\}, 0.1\right), \\
\widetilde{H}_\lambda(e_1, e_3) &= \left(\left\{\frac{x_1}{[0.1, 0.3]}, \frac{x_2}{[0.4, 0.7]}, \frac{x_3}{[0.2, 0.3]}\right\}, 0.2\right), \\
\widetilde{H}_\lambda(e_2, e_1) &= \left(\left\{\frac{x_1}{[0, 0.3]}, \frac{x_2}{[0, 0.2]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.3\right), \\
\widetilde{H}_\lambda(e_2, e_2) &= \left(\left\{\frac{x_1}{[0, 0.3]}, \frac{x_2}{[0, 0.2]}, \frac{x_3}{[0, 0.3]}\right\}, 0.1\right), \\
\widetilde{H}_\lambda(e_2, e_3) &= \left(\left\{\frac{x_1}{[0, 0.3]}, \frac{x_2}{[0, 0.2]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.2\right), \\
\widetilde{H}_\lambda(e_3, e_1) &= \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.1, 0.1]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.1\right), \\
\widetilde{H}_\lambda(e_3, e_2) &= \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.1, 0.1]}, \frac{x_3}{[0, 0.3]}\right\}, 0.1\right), \\
\widetilde{H}_\lambda(e_3, e_3) &= \left(\left\{\frac{x_1}{[0.1, 0.6]}, \frac{x_2}{[0.1, 0.1]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.1\right).
\end{aligned}
\tag{5.3}
$$

Table 1: $(\widetilde{H}_\lambda, A \times B)$.

	x_1	x_2	x_3	μ
(e_1, e_1)	[0.1, 0.3]	[0.2, 0.6]	[0.3, 0.5]	0.3
(e_1, e_2)	[0.1, 0.3]	[0.4, 0.6]	[0, 0.3]	0.1
(e_1, e_3)	[0.1, 0.3]	[0.4, 0.7]	[0.2, 0.3]	0.2
(e_2, e_1)	[0, 0.3]	[0, 0.2]	[0.1, 0.3]	0.3
(e_2, e_2)	[0, 0.3]	[0, 0.2]	[0, 0.3]	0.1
(e_2, e_3)	[0, 0.3]	[0, 0.2]	[0.1, 0.3]	0.2
(e_3, e_1)	[0.3, 0.5]	[0.1, 0.1]	[0.1, 0.3]	0.1
(e_3, e_2)	[0.3, 0.5]	[0.1, 0.1]	[0, 0.3]	0.1
(e_3, e_3)	[0.1, 0.6]	[0.1, 0.1]	[0.1, 0.3]	0.1

Table 2: Numerical grade $r_{p \in P}(x_i)$.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
x_1	−0.8	−0.5	−0.8	0	0	0	(1)	(1.1)	(0.8)
x_2	(0.4)	(1.3)	(1.3)	−0.3	−0.2	−0.3	−0.8	−0.7	−0.7
x_3	(0.4)	−0.8	−0.5	(0.3)	(0.1)	(0.3)	−0.2	−0.4	−0.1
μ	0.3	0.1	0.2	0.3	0.1	0.2	0.1	0.1	0.1

Now, to determine the best machine, we first compute the numerical grade $r_{p \in P}(x_i)$ for each $p \in P$ such that

$$r_{p \in P}(x_i) = \sum_{x \in U} \left(\left(c_i^- - \mu_{H(p_i)}^-(x) \right) + \left(c_i^+ - \mu_{H(p_i)}^+(x) \right) \right). \tag{5.4}$$

The result is shown in Tables 1 and 2.

Let $P = \{p_1 = (e_1, e_1), p_2 = (e_1, e_2), \ldots, p_9 = (e_3, e_3)\}$.

Now, we mark the highest numerical grade (indicated in parenthesis) in each row excluding the last row which is the grade of such belongingness of a machine against each pair of parameters (see Table 3). Now, the score of each such machine is calculated by taking the sum of the products of these numerical grades with the corresponding value of μ. The machine with the highest score is the desired machine. We do not consider the numerical grades of the machine against the pairs (e_i, e_i), $i = 1, 2, 3$, as both the parameters are the same:

Score $(x_1) = (1 * 0.1) + (1.1 * 0.1) = 0.21$,

Score $(x_2) = (1.3 * 0.1) + (1.3 * 0.2) = 0.39$,

Score $(x_3) = (0.3 * 0.3) + (0.3 * 0.2) = 0.15$.

The firm will select the machine with the highest score. Hence, they will buy machine x_2.

Table 3: Grade table.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
x_i	x_2, x_3	x_2	x_2	x_3	x_3	x_3	x_1	x_1	x_1
Highest grade	—	1.3	1.3	0.3	—	0.3	1	1.1	—
μ	0.3	0.1	0.2	0.3	0.1	0.2	0.1	0.1	0.1

Definition 5.3. If $(\tilde{F}_\mu, A)$ and $(\tilde{G}_\delta, B)$ are two GIVFSSs, then "$(\tilde{F}_\mu, A)$ OR $(\tilde{G}_\delta, B)$" denoted by $(\tilde{F}_\mu, A) \vee (\tilde{G}_\delta, B)$ is defined by

$$\left(\tilde{F}_\mu, A\right) \vee \left(\tilde{G}_\delta, B\right) = \left(\widetilde{H}_\lambda, A \times B\right), \tag{5.5}$$

where $\widetilde{H}_\lambda(\alpha, \beta) = (H(\alpha, \beta), \lambda(\alpha, \beta))$ for all $(\alpha, \beta) \in A \times B$, such that $\widetilde{H}(\alpha, \beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta)$ and $\lambda(\alpha, \beta) = s(\mu(\alpha), \delta(\beta))$, for all $(\alpha, \beta) \in A \times B$, where s is an s-norm.

Example 5.4. Consider Example 5.2. To find the OR between the two GIVFSSs, we have $(\tilde{F}_\mu, A)$ OR $(\tilde{G}_\delta, B) = (\widetilde{H}_\lambda, A \times B)$, where

$$\widetilde{H}_\lambda(e_1, e_1) = \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.4, 0.5]}\right\}, 0.4\right),$$

$$\widetilde{H}_\lambda(e_1, e_2) = \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.4, 0.5]}\right\}, 0.4\right),$$

$$\widetilde{H}_\lambda(e_1, e_3) = \left(\left\{\frac{x_1}{[0.1, 0.6]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.3, 0.5]}\right\}, 0.4\right),$$

$$\widetilde{H}_\lambda(e_2, e_1) = \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0., 0.6]}, \frac{x_3}{[0.4, 0.5]}\right\}, 0.4\right),$$

$$\widetilde{H}_\lambda(e_2, e_2) = \left(\left\{\frac{x_1}{[0.3, 0.5]}, \frac{x_2}{[0.4, 0.6]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.4\right), \tag{5.6}$$

$$\widetilde{H}_\lambda(e_2, e_3) = \left(\left\{\frac{x_1}{[0.1, 0.6]}, \frac{x_2}{[0.4, 0.7]}, \frac{x_3}{[0.2, 0.3]}\right\}, 0.4\right),$$

$$\widetilde{H}_\lambda(e_3, e_1) = \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.2, 0.6]}, \frac{x_3}{[0.4, 0.5]}\right\}, 0.3\right),$$

$$\widetilde{H}_\lambda(e_3, e_2) = \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.4, 0.6]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.1\right),$$

$$\widetilde{H}_\lambda(e_3, e_3) = \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.4, 0.7]}, \frac{x_3}{[0.2, 0.3]}\right\}, 0.2\right).$$

Remark 5.5. We use the same method in Example 5.2 for the OR operation if the firm wants to buy one such machine depending on any one of the parameters only.

Proposition 5.6. *Let $(\widetilde{F}_\mu, A)$ and $(\widetilde{G}_\delta, B)$ be any two GIVFSSs. Then, the following results hold:*

(a) $((\widetilde{F}_\mu, A) \wedge (\widetilde{G}_\delta, B))^c = (\widetilde{F}_\mu, A)^c \vee (\widetilde{G}_\delta, B)^c$,

(b) $((\widetilde{F}_\mu, A) \vee (\widetilde{G}_\delta, B))^c = (\widetilde{F}_\mu, A)^c \wedge (\widetilde{G}_\delta, B)^c$.

Proof. Straightforward from Definitions 4.1, 5.1, and 5.3. ☐

Proposition 5.7. *Let $(\widetilde{F}_\mu, A)$, $(\widetilde{G}_\delta, B)$, and $(\widetilde{H}_\lambda, C)$ be any three GIVFSSs. Then, the following results hold:*

(a) $(\widetilde{F}_\mu, A) \wedge ((\widetilde{G}_\delta, B) \wedge (\widetilde{H}_\lambda, C)) = ((\widetilde{F}_\mu, A) \wedge (\widetilde{G}_\delta, B)) \wedge (\widetilde{H}_\lambda, C)$,

(b) $(\widetilde{F}_\mu, A) \vee ((\widetilde{G}_\delta, B) \vee (\widetilde{H}_\lambda, C)) = ((\widetilde{F}_\mu, A) \vee (\widetilde{G}_\delta, B)) \vee (\widetilde{H}_\lambda, C)$,

(c) $(\widetilde{F}_\mu, A) \vee ((\widetilde{G}_\delta, B) \wedge (\widetilde{H}_\lambda, C)) = ((\widetilde{F}_\mu, A) \vee (\widetilde{G}_\delta, B)) \wedge ((\widetilde{F}_\mu, A) \vee (\widetilde{H}_\lambda, C))$,

(d) $(\widetilde{F}_\mu, A) \wedge ((\widetilde{G}_\delta, B) \vee (\widetilde{H}_\lambda, C)) = ((\widetilde{F}_\mu, A) \wedge (\widetilde{G}_\delta, B)) \vee ((\widetilde{F}_\mu, A) \wedge (\widetilde{H}_\lambda, C))$.

Proof. Straightforward from Definitions 5.1 and 5.3. ☐

Remark 5.8. The commutativity property does not hold for AND and OR operations since $A \times B \neq B \times A$.

6. Similarity between Two GIVFSS

In this section, we give a measure of similarity between two GIVFSSs. We are taking the set theoretic approach because it is easier to calculate on and is a very popular method too.

Definition 6.1. Similarity between two GIVFSSs $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$, denoted by $S(\widetilde{F}_\mu, \widetilde{G}_\delta)$, is defined by

$$
\begin{aligned}
S\left(\widetilde{F}_\mu, \ \widetilde{G}_\delta\right) &= \left[\varphi^-\left(\widetilde{F}, \widetilde{G}\right) \cdot m(\mu, \ \delta), \varphi^+\left(\widetilde{F}, \widetilde{G}\right) \cdot m(\mu, \ \delta)\right], \text{ such that} \\
\varphi^-\left(\widetilde{F}, \ \widetilde{G}\right) &= \min\left(\varphi_1\left(\widetilde{F}, \widetilde{G}\right), \varphi_2\left(\widetilde{F}, \widetilde{G}\right)\right), \\
\varphi^+\left(\widetilde{F}, \ \widetilde{G}\right) &= \max\left(\varphi_1\left(\widetilde{F}, \widetilde{G}\right), \varphi_2\left(\widetilde{F}, \widetilde{G}\right)\right),
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
\varphi_1\left(\widetilde{F}, \ \widetilde{G}\right) &= \begin{cases} 0, & \text{if } \mu_{\widetilde{F}_i}^-(x) = 0, \ \forall i, \\[2mm] \dfrac{\sum_i^n \max_{x \in X}\left\{\min\left(\mu_{\widetilde{F}_i}^-(x), \mu_{\widetilde{G}_i}^-(x)\right)\right\}}{\sum_i^n \max_{x \in X}\left[\mu_{\widetilde{F}_i}^-(x)\right]}, & \text{otherwise,} \end{cases} \\[4mm]
\varphi_2\left(\widetilde{F}, \ \widetilde{G}\right) &= \dfrac{\sum_i^n \max_{x \in X}\left\{\min\left(\mu_{\widetilde{F}_i}^+(x), \mu_{\widetilde{G}_i}^+(x)\right)\right\}}{\sum_i^n \max_{x \in X}\left(\mu_{\widetilde{F}_i}^+(x)\right)}, \\[4mm]
m(\mu(e), \ \delta(e)) &= 1 - \dfrac{\sum\left|\mu(e) - \delta(e)\right|}{\sum\left|\mu(e) + \delta(e)\right|}.
\end{aligned}
\tag{6.2}
$$

Definition 6.2. Let $\widetilde{F}_\mu$ and $\widetilde{G}_\delta$ be two GIVFSSs over the same universe (U, E). We say that two GIVFSS are significantly similar if $\varphi^-(\widetilde{F}, \widetilde{G}) \cdot m(\mu, \ \delta) \geqslant 1/2$.

Theorem 6.3. *Let $\widetilde{F}_\mu$, $\widetilde{G}_\delta$, and $\widetilde{H}_\lambda$ be any three GIVFSSs over (U, E). Then, the following hold:*

(a) *in general $S(\widetilde{F}_\mu, \widetilde{G}_\delta) \neq S(\widetilde{G}_\delta, \widetilde{F}_\mu)$,*

(b) *$\varphi^-(\widetilde{F}, \ \widetilde{G}) \geq 0$ and $\varphi^+(\widetilde{F}, \ \widetilde{G}) \leq 1$,*

(c) *$\widetilde{F}_\mu = \widetilde{G}_\delta \Rightarrow S(\widetilde{F}_\mu, \widetilde{G}_\delta) = 1$,*

(d) *$\widetilde{F}_\mu \subseteq \widetilde{G}_\delta \subseteq \widetilde{H}_\lambda \Rightarrow S(\widetilde{F}_\mu, \widetilde{H}_\lambda) \leqslant S(\widetilde{G}_\delta, \widetilde{H}_\lambda)$,*

(e) *$\widetilde{F}_\mu \widetilde{\cap} \widetilde{G}_\delta = \emptyset \Leftrightarrow S(\widetilde{F}_\mu, \widetilde{G}_\delta) = 0$.*

Proof. (a) The proof is straightforward and follows from Definition 6.1.
  (b) From Definition 6.1, we have

$$\varphi_1\left(\widetilde{F}, \ \widetilde{G}\right) = \begin{cases} 0, & \text{if } \mu^-_{\widetilde{F}_i}(x) = 0, \ \forall i, \\ \dfrac{\sum_i^n \max_{x \in X}\left\{\min\left(\mu^-_{\widetilde{F}_i}(x), \mu^-_{\widetilde{G}_i}(x)\right)\right\}}{\sum_i^n \max_{x \in X}\left(\mu^-_{\widetilde{F}_i}(x)\right)}, & \text{otherwise.} \end{cases} \tag{6.3}$$

If $\mu^-_{\widetilde{F}_i}(x) = 0$, for all i, then $\varphi^-(\widetilde{F}, \ \widetilde{G}) = 0$, and, if $\mu^-_{\widetilde{F}_i}(x) \neq 0$, for some i, then it is clear that $\varphi^-(\widetilde{F}, \ \widetilde{G}) \geq 0$.
  Also since $\varphi^+(\widetilde{F}, \widetilde{G}) = \max(\varphi_1(\widetilde{F}, \widetilde{G}), \varphi_2(\widetilde{F}, \widetilde{G}))$, suppose that $\varphi_1(\widetilde{F}, \widetilde{G}) = 1$ and $\varphi_2(\widetilde{F}, \widetilde{G}) = 1$, then $\varphi^+(\widetilde{F}, \ \widetilde{G}) = 1$, that means, if $\varphi_1(\widetilde{F}, \widetilde{G}) < 1$ and $\varphi_2(\widetilde{F}, \widetilde{G}) < 1$, then $\varphi^+(\widetilde{F}, \widetilde{G}) \leq 1$.
  (c) The proof is straightforward and follows from Definition 6.1.
  (d) The proof is straightforward and follows from Definition 6.1.
  (e) The proof is straightforward and follows from Definition 6.1. $\qquad\square$

Example 6.4. Let $\widetilde{F}_\mu$ be GIVFSS over (U, E) defined as follows:

$$\widetilde{F}_\mu(e_1) = \left(\left\{\frac{x_1}{[0.3, 0.7]}, \frac{x_2}{[0.4, 0.8]}, \frac{x_3}{[0.1, 0.3]}\right\}, 0.4\right),$$

$$\widetilde{F}_\mu(e_2) = \left(\left\{\frac{x_1}{[0.5, 0.6]}, \frac{x_2}{[0.1, 0.3]}, \frac{x_3}{[0, 0.4]}\right\}, 0.6\right), \tag{6.4}$$

$$\widetilde{F}_\mu(e_3) = \left(\left\{\frac{x_1}{[0.7, 0.9]}, \frac{x_2}{[0.1, 0.5]}, \frac{x_3}{[0.8, 1]}\right\}, 0.8\right).$$

Let $\widetilde{G}_\delta$ be another GIVFSS over (U, E) defined as follows:

$$\widetilde{G}_\delta(e_1) = \left(\left\{\frac{x_1}{[0.1, 0.4]}, \frac{x_2}{[0.5, 0.7]}, \frac{x_3}{[0.2, 0.3]}\right\}, 0.3\right),$$

$$\widetilde{G}_\delta(e_2) = \left(\left\{\frac{x_1}{[0.6, 0.8]}, \frac{x_2}{[0.5, 0.6]}, \frac{x_3}{[0.4, 0.8]}\right\}, 0.7\right), \tag{6.5}$$

$$\widetilde{G}_\delta(e_3) = \left(\left\{\frac{x_1}{[0.4, 0.7]}, \frac{x_2}{[0.3, 0.5]}, \frac{x_3}{[0.5, 0.7]}\right\}, 0.6\right).$$

Here,

$$m(\mu(e),\ \delta(e)) = 1 - \frac{\sum|\mu(e) - \delta(e)|}{\sum|\mu(e) + \delta(e)|}$$

$$= 1 - \frac{|(0.4 - 0.3)| + |(0.6 - 0.7)| + |(0.8 - 0.6)|}{|(0.4 + 0.3)| + |(0.6 + 0.7)| + |(0.80 + 0.6)|}$$

$$\cong 0.8824,$$

$$\varphi_1\left(\widetilde{F},\ \widetilde{G}\right) = \frac{(\max\{\min(0.3, 0.1), \min(0.4, 0.5), \min(0.1, 0.2)\}}{\max(0.3, 0.4, 0.1) + \max(0.5, 0.1, 0) + \max(0.7, 0.1, 0.8)}$$

$$+ \frac{\max\{\min(0.5, 0.6), \min(0.1, 0.5), \min(0, 0.4)\}}{\max(0.3, 0.4, 0.1) + \max(0.5, 0.1, 0) + \max(0.7, 0.1, 0.8)}$$

$$+ \frac{\max\{\min(0.7, 0.4), \min(0.1, 0.3), \min(0.8, 0.5)\})}{\max(0.3, 0.4, 0.1) + \max(0.5, 0.1, 0) + \max(0.7, 0.1, 0.8)}$$

$$= \frac{\max\{0.1, 0.4, 0.1\} + \max\{0.5, 0.1, 0\} + \max\{0.4, 0.1, 0.5\}}{\max(0.3, 0.4, 0.1) + \max(0.5, 0.1, 0) + \max(0.7, 0.1, 0.8)} \tag{6.6}$$

$$= \frac{0.4 + 0.5 + 0.5}{0.4 + 0.5 + 0.8} = 0.824,$$

$$\varphi_2\left(\widetilde{F},\ \widetilde{G}\right) = \frac{(\max\{\min(0.7, 0.4), \min(0.8, 0.7), \min(0.3, 0.3)\}}{\max(0.7, 0.8, 0.3) + \max(0.6, 0.3, 0.4) + \max(0.9, 0.5, 1)}$$

$$+ \frac{\max\{\min(0.6, 0.8), \min(0.3, 0.6), \min(0.4, 0.8)\}}{\max(0.7, 0.8, 0.3) + \max(0.6, 0.3, 0.4) + \max(0.9, 0.5, 1)}$$

$$+ \frac{\max\{\min(0.9, 0.7), \min(0.5, 0.5), \min(1, 0.7)\})}{\max(0.7, 0.8, 0.3) + \max(0.6, 0.3, 0.4) + \max(0.9, 0.5, 1)}$$

$$= \frac{\max\{0.4, 0.7, 0.3\} + \max\{0.6, 0.3, 0.4\} + \max\{0.7, 0.5, 0.7\}}{\max(0.7, 0.8, 0.3) + \max(0.6, 0.3, 0.4) + \max(0.9, 0.5, 1)}$$

$$= \frac{0.7 + 0.6 + 0.7}{0.8 + 0.6 + 1} = 0.833.$$

Then,

$$\varphi^-\left(\tilde{F},\ \tilde{G}\right) = \min\left(\varphi_1\left(\tilde{F},\ \tilde{G}\right), \varphi_2\left(\tilde{F},\ \tilde{G}\right)\right) = \min(0.824, 0.833) = 0.824,$$

$$\varphi^+\left(\tilde{F},\ \tilde{G}\right) = \max\left(\varphi_1\left(\tilde{F},\ \tilde{G}\right), \varphi_2\left(\tilde{F},\ \tilde{G}\right)\right) = \max(0.824, 0.833) = 0.833. \tag{6.7}$$

Hence, the similarity between the two GIVFSSs $\tilde{F}_\mu$ and $\tilde{G}_\delta$ will be

$$\begin{aligned} S\left(\tilde{F}_\mu,\ \tilde{G}_\delta\right) &= \left[\varphi^-\left(\tilde{F}, \tilde{G}\right) \cdot m(\mu,\ \delta), \varphi^+\left(\tilde{F}, \tilde{G}\right) \cdot m(\mu,\ \delta)\right] \\ &= \left[(0.824) \cdot (0.8824), (0.833) \cdot (0.8824)\right] \\ &= [0.727, 0.735]. \end{aligned} \tag{6.8}$$

Therefore, $\tilde{F}_\mu$ and $\tilde{G}_\delta$ are significantly similar.

7. Application of Similarity Measure in Medical Diagnosis

In this section, we will try to estimate the possibility that a sick person having certain visible symptoms is suffering from dengue fever. For this, we first construct a GIVFSS model for dengue fever and the GIVFSS of symptoms for the sick person. Next, we find the similarity measure of these two sets. If they are significantly similar, then we conclude that the person is possibly suffering from dengue fever. Let our universal set contain only two elements "yes" and "no," that is, $U = \{y, n\}$. Here, the set of parameters E is the set of certain visible symptoms. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, where e_1 = body temperature, e_2 = cough with chest congestion, e_3 = loose motion, e_4 = chills, e_5 = headache, e_6 = low heart rate (bradycardia), and e_7 = pain upon moving the eyes. Our model GIVFSS for dengue fever M_μ is given in Table 4, and this can be prepared with the help of a physician.

Now, after talking to the sick person, we can construct his GIVFSS G_δ as in Table 5.

Now, we find the similarity measure of these two sets by using the same method as in Example 6.4, where, after the calculation, we get $\varphi^-(\widetilde{M}, \tilde{G})m(\mu,\ \delta) \cong 0.22 < 1/2$. Hence the two GIVFSSs are not significantly similar. Therefore, we conclude that the person is not suffering from dengue fever.

8. Conclusion

In this paper, we have introduced the concept of generalised interval-valued fuzzy soft set and studied some of its properties. The complement, union, intersection, "AND," and "OR" operations have been defined on the interval-valued fuzzy soft sets. An application of this theory is given in solving a decision making problem. Similarity measure of two generalised interval-valued fuzzy soft sets is discussed, and its application to medical diagnosis has been shown.

Table 4: Model GIVFSS for dengue fever.

M_μ	e_1	e_2	e_3	e_4	e_5	e_6	e_7
y	1	0	0	1	1	1	1
n	0	1	1	0	0	0	0
μ	1	1	1	1	1	1	1

Table 5: GIVFSS for the sick person.

F_α	e_1	e_2	e_3	e_4	e_5	e_6	e_7
y	$[0.3, 0.4]$	$[0.2, 0.5]$	$[0, 0.2]$	1	$[0.4, 0.6]$	0	$[0.3, 0.4]$
n	$[0.6, 0.9]$	$[0.5, 0.7]$	$[0.6, 0.8]$	$[0.3, 0.5]$	1	$[0.4, 0.6]$	$[0.3, 0.5]$
δ	0.3	0.5	0.4	0.6	0.1	0.5	0.2

Acknowledgment

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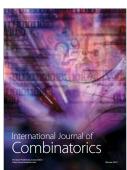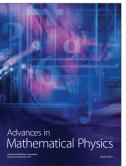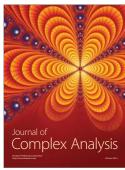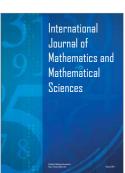[15] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-I," vol. 8, pp. 199–249, 1975.