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GENERALISED SPIN STRUCTURES ON 2-DIMENSIONAL ORBIFOLDS

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Abstract

Generalised spin structures, or r-spin structures, on a 2-dimensional orbifold Σ are r-fold fibrewise connected coverings (also called r^{th} roots) of its unit tangent bundle $ST\Sigma$. We investigate such structures on hyperbolic orbifolds. The conditions on r for such structures to exist are given. The action of the diffeomorphism group of Σ on the set of r-spin structures is described, and we determine the number of orbits under this action and their size. These results are then applied to describe the moduli space of taut contact circles on left-quotients of the 3-dimensional geometry $\widetilde{\mathrm{SL}}_2$.

1. Introduction

Spin structures on manifolds have been studied extensively, not least because of their relevance to physics. A spin structure on a Riemann surface Σ may be thought of as a square root of the tangent bundle $T\Sigma$, that is, a holomorphic line bundle $\mathcal{L}$ with $\mathcal{L}\otimes\mathcal{L} = T\Sigma$. On the level of the unit tangent bundle $ST\Sigma$, a spin structure can be interpreted as a fibrewise connected double covering $M \to ST\Sigma$ by another S^1-bundle M over Σ.

It is this last definition which most easily generalises to 2-dimensional orbifolds and coverings of higher order. This is not just generalisation for generalisation's sake. For instance, such objects appear in the work of Witten [20] on matrix models of 2-dimensional quantum gravity, see also [15]. Here the viewpoint is that of Algebraic Geometry, where an r^{th} root of the tangent bundle of a Riemann surface Σ is considered to be a holomorphic line bundle whose r^{th} tensor power equals $T\Sigma$. In that framework, questions of moduli have been studied by Jarvis [9] and others.

Our personal motivation for investigating such r^{th} roots comes from the moduli problem for taut contact circles on 3-manifolds. These structures were introduced in [6], where we also classified the 3-manifolds which admit such structures. The moduli question was largely settled in [8], but certain details as to the precise geometry of the moduli spaces had been left open. These details hinge on the classification of r^{th} roots of the unit tangent bundle of 2-dimensional hyperbolic orbifolds.

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Here is an outline of the paper. In Section 2 we present the basics of 2-dimensional hyperbolic orbifolds, mostly to set up notation. In Section 3 we recall the definition of the unit tangent bundle of an orbifold. Roots of such unit tangent bundles are defined in Section 4, where we determine the conditions on r (in terms of the genus and multiplicities of the cone points of the orbifold Σ) for r^{th} roots to exist. We also set up a one-to-one correspondence between r^{th} roots and certain homomorphisms on the fundamental group of the unit tangent bundle of Σ (Theorem 3). In Section 5 this is used to investigate the action by the diffeomorphism group of Σ on the set of r^{th} roots. The number of orbits under this action is determined (Proposition 5), as well as the length of the orbits (Proposition 8). In Section 6 we reformulate this action by the diffeomorphism group in algebraic terms as an action by the outer automorphism group of the orbifold fundamental group. Finally, in Section 7 we use this algebraic reformulation and the results of the previous sections to describe the Teichmüller space (Theorem 10) and moduli space (Theorem 11) of taut contact circles on left-quotients of the 3-dimensional Thurston geometry $\widetilde{\mathrm{SL}}_2$. In particular, we are interested in the enumeration of the connected components of the moduli space; this gives the number of distinct taut contact circles up to diffeomorphism and deformation.

Sections 2 to 5 are completely self-contained. The final two sections depend to some degree on our earlier work [8], but except for the algebraic reformulation of the moduli problem we do not need to quote any details from that paper.

2. Hyperbolic orbifolds

Throughout this paper, let Σ be a fixed (closed, orientable, 2-dimensional) orbifold of genus g and with n cone points of multiplicity $\alpha_1, \ldots, \alpha_n$. Moreover, it is assumed that Σ is of hyperbolic type, i.e. its orbifold Euler characteristic, defined as

$$\chi^{\text{orb}}(\Sigma) = 2 - 2g - n + \sum_{j=1}^{n} \frac{1}{\alpha_j},$$

is assumed to be negative. This condition on the orbifold Euler characteristic determines those orbifolds which admit a hyperbolic metric; however, as yet we do not fix such a hyperbolic structure.

The orbifold fundamental group π^{orb} of Σ is defined as the deck transformation group of the universal covering $\tilde{\Sigma} \to \Sigma$. We briefly recall the geometric realisation of this group and its standard presentation. To that end, choose a base point $x_0 \in \Sigma$ distinct from all the cone points, and a lift $\tilde{x}_0 \in \tilde{\Sigma}$ of x_0 in the universal covering space. Choose a system of $2g$ loops on Σ, based at x_0, and a curve from x_0 to each of the cone points, such that Σ looks as in Fig. 1 when cut open along these $2g + n$ curves. We may interpret that figure as a fundamental region in $\tilde{\Sigma}$; it is determined (amongst all possible fundamental regions whose boundary polygon maps to the chosen system of curves) by the indicated placement of $\tilde{x}_0$ on its boundary. Notice that the sides of

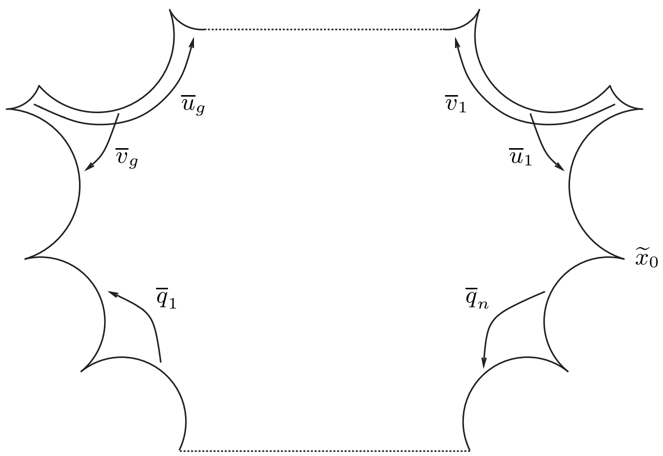

Fig. 1. A fundamental domain for Σ.

this polygon identified by the deck transformation $\bar{q}_j$ meet at a vertex mapping to the j^{th} cone point in Σ; all other vertices are lifts of x_0.

Let $\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g, \bar{q}_1, \ldots, \bar{q}_n$ be the deck transformations of $\tilde{\Sigma}$ which effect the gluing maps of the sides of the chosen fundamental polygon as indicated in Fig. 1. From the figure we see that the deck transformation $\prod_i [\bar{u}_i, \bar{v}_i] \prod_j \bar{q}_j$ (read from the right as a composition of maps) fixes the point $\tilde{x}_0$, which is not the lift of a cone point, so we conclude

$$\prod_i [\bar{u}_i, \bar{v}_i] \prod_j \bar{q}_j = 1.$$

Similarly, we have

$$\bar{q}_j^{\alpha_j} = 1, \quad j = 1, \ldots, n.$$

These relations give the standard presentation of π^{orb} as

$$\pi^{\text{orb}} = \left\{ \bar{u}_1, \bar{v}_1, \ldots, \bar{u}_g, \bar{v}_g, \bar{q}_1, \ldots, \bar{q}_n \colon \prod_i [\bar{u}_i, \bar{v}_i] \prod_j \bar{q}_j = 1, \; \bar{q}_j^{\alpha_j} = 1 \right\}.$$

Once Σ has been equipped with a hyperbolic structure and an orientation, then $\tilde{\Sigma} = \mathbb{H}^2$ and the $\bar{u}_i, \bar{v}_i, \bar{q}_j$ are orientation preserving isometries of $\mathbb{H}^2$, i.e. elements of $\text{PSL}_2\mathbb{R}$. The identification of $\tilde{\Sigma}$ with $\mathbb{H}^2$ is uniquely determined if we specify, for instance, the lift $\tilde{x}_0 \in \mathbb{H}^2$, the initial direction of one of the edges of the fundamental polygon emanating from that point, and require that the orientation lifted from Σ coincide with a chosen orientation of $\mathbb{H}^2$. In this way an oriented hyperbolic structure on Σ defines an element of the Weil space $\mathcal{R}(\pi^{\text{orb}}, \text{PSL}_2\mathbb{R})$ of faithful representations of π^{orb} in $\mathrm{PSL}_2\mathbb{R}$ with discrete and cocompact image. Conversely, any representation $\bar{\rho} \in \mathcal{R}(\pi^{\text{orb}}, \mathrm{PSL}_2\mathbb{R})$ determines a diffeomorphic copy $\bar{\rho}(\pi^{\text{orb}}) \setminus \mathbb{H}^2$ of Σ with a hyperbolic structure and an orientation.

It is possible to designate one of the orientations on any given Σ as *positive* and the other as *negative* in the following way. If there are cone points, it suffices to observe that $\bar{\rho}(\bar{q}_j)$ is a rotation by $\pm 2\pi/\alpha_j$, with the same sign for each $j = 1,\ldots,n$. (The sign is well defined even for $\alpha_j = 2$ when we regard the rotation as being through the interior of the fundamental domain.) Observe in Fig. 1 how the direction of rotation around the cone points relates to the orientation given by the pairs of arrows indicating the action of $\bar{u}_i$ and $\bar{v}_i$; thus, any such pair of arrows allows us to determine the orientation of Σ, also when no cone points are present. We write $\mathcal{R}^{\pm}(\pi^{\text{orb}}, \mathrm{PSL}_2\mathbb{R})$ for the corresponding components of $\mathcal{R}(\pi^{\text{orb}}, \mathrm{PSL}_2\mathbb{R})$. Any two representations $\bar{\rho}_1, \bar{\rho}_2 \in \mathcal{R}(\pi^{\text{orb}}, \mathrm{PSL}_2\mathbb{R})$ are related via conjugation with a diffeomorphism of $\mathbb{H}^2$. This diffeomorphism will be orientation preserving or reversing, depending on whether $\bar{\rho}_1, \bar{\rho}_2$ lie in the same component or not, see also [8, pp. 59/60]. This orientation issue will only become relevant in Section 7 of the present paper.

3. The unit tangent bundle of an orbifold

The unit tangent bundle of an oriented hyperbolic orbifold Σ is defined as follows, see [17, p. 466]. Write $\widetilde{\mathrm{SL}}_2$ for the universal cover of $\mathrm{PSL}_2\mathbb{R}$. There is a short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}}_2 \xrightarrow{p} \mathrm{PSL}_2\mathbb{R} \to 1.$$

Realise the given hyperbolic structure and orientation on Σ by a choice of representation $\bar{\rho} \in \mathcal{R}(\pi^{\text{orb}}, \mathrm{PSL}_2\mathbb{R})$. Then set

$$ST\Sigma = p^{-1}(\bar{\rho}(\pi^{\text{orb}})) \setminus \widetilde{\mathrm{SL}}_2;$$

this is the unit tangent bundle of Σ. It is in a natural way the total space of a Seifert bundle over Σ with normalised Seifert invariants

$$\{g;\, b = 2g-2;\, (\alpha_1, \alpha_1 - 1), \ldots, (\alpha_n, \alpha_n - 1)\}.$$

REMARK. There is a tricky orientation issue here. The group $\mathrm{PSL}_2\mathbb{R}$ of orientation preserving isometries of $\mathbb{H}^2$ acts, via the differential, transitively and with trivial point stabilisers on the unit tangent bundle $ST\mathbb{H}^2$ of $\mathbb{H}^2$ (see Scott's survey [17]), which allows us to identify $\mathrm{PSL}_2\mathbb{R}$ with $ST\mathbb{H}^2$. A given orientation on $\mathbb{H}^2$ thus induces an orientation on the S^1-fibres of $\mathrm{PSL}_2\mathbb{R} = ST\mathbb{H}^2 \to \mathbb{H}^2$, and hence on the $\mathbb{R}$-fibres of $\widetilde{\mathrm{SL}}_2 \to \mathbb{H}^2$. When we pass to a left-quotient of $\widetilde{\mathrm{SL}}_2$, these oriented $\mathbb{R}$-fibres descend to oriented Seifert fibres. With this orientation convention, the invariants of the multiple fibres are $(\alpha_j, 1)$, see [17, p. 467]. On the other hand, there is a natural *right* S^1-action on compact left-quotients of $\widetilde{\mathrm{SL}}_2$. When this right action is turned into a left action by the inverse elements (while keeping the orientation of $\widetilde{\mathrm{SL}}_2$ and its quotient), the Seifert invariants become $(\alpha_i, \alpha_i - 1)$. This is the convention of Raymond and Vasquez [16, pp. 169/70], which is the more suitable one for our more algebraic considerations in our earlier paper [8] and below.

A presentation of the fundamental group π of $ST\Sigma$ is given by

$$\pi = \Big\{ u_1, v_1, \ldots, u_g, v_g, q_1, \ldots, q_n, h\colon \prod_i [u_i, v_i] \prod_j q_j = h^{2g-2}, \\ q_j^{\alpha_j} h^{\alpha_j - 1} = 1,\ h \text{ central} \Big\}.$$

Under the projection $ST\Sigma \to \Sigma$, the generators of π and π^{orb} correspond to each other as suggested by our choice of notation. In other words, there is a representation $\rho \in \mathcal{R}(\pi, \widetilde{\mathrm{SL}}_2)$ with $\rho(\pi) = p^{-1}(\bar\rho(\pi^{\mathrm{orb}}))$ and $p(\rho(u_i)) = \bar\rho(\bar u_i))$ etc. For further details see [8, Section 4].

The Seifert fibration $ST\Sigma \to \Sigma$, up to equivalence, does not depend on the choice of hyperbolic structure on Σ. This allows us to speak of the unit tangent bundle $ST\Sigma$ (as a Seifert manifold) even when we have not fixed a metric on Σ.

4. Roots of the unit tangent bundle

Our aim is to classify r^{th} roots of $ST\Sigma$ for Σ an oriented orbifold of hyperbolic type, by which we mean the following.

DEFINITION. An r^{th} *root* of the unit tangent bundle $ST\Sigma$ is an r-fold fibrewise connected and orientation preserving covering $M \to ST\Sigma$ of $ST\Sigma$ by a Seifert manifold M. In other words, we require that each S^1-fibre of $ST\Sigma$ is covered r times positively by a single S^1-fibre of M.

REMARKS. (1) For $r = 2$, such coverings are precisely the spin structures on Σ. Spin structures on orbifolds of arbitrary dimension were defined and studied from the differential geometric point of view (index theory, twistor theory) in [5] and [3]. The latter paper contains a general existence and classification statement for spin structures on orbifolds, albeit only for orbifolds whose singular set is of codimension at least 4.

(2) In the case of a principal S^1-bundle without multiple fibres, one can pass to the associated complex line bundle. An r^{th} root then corresponds to a complex line bundle whose r^{th} tensor power is the given line bundle.

For the purpose of classifying such r^{th} roots $M \to ST\Sigma$ we need to specify a notion of equivalence. Both M and $ST\Sigma$ come equipped with an effective S^1-action that induces the Seifert fibre structure. The covering map $M \to ST\Sigma$ may be regarded as the quotient map under the $\mathbb{Z}_r$-action on M induced by this S^1-action on M and the natural inclusion $\mathbb{Z}_r \subset S^1$. In particular, the covering map $M \to ST\Sigma$ is regular, and M is a principal $\mathbb{Z}_r$-bundle over $ST\Sigma$.

Two r^{th} roots $q\colon M \to ST\Sigma$ and $q'\colon M' \to ST\Sigma$ will be regarded as equivalent if there is an S^1-equivariant diffeomorphism $\psi\colon M \to M'$ with $q' \circ \psi = q$. Since the S^1-actions on M and M' are lifted from the S^1-action on $ST\Sigma$, this amounts to the same as requiring the existence of a $\mathbb{Z}_r$-equivariant diffeomorphism $\psi\colon M \to M'$.

The equivalence classes of *arbitrary* principal $\mathbb{Z}_r$-bundles over $ST\Sigma$ are in natural one-to-one correspondence with the set $\mathrm{Hom}(\pi, \mathbb{Z}_r)$ of homomorphisms from the fundamental group π of $ST\Sigma$ into $\mathbb{Z}_r$, the correspondence being given by associating with a principal $\mathbb{Z}_r$-bundle its monodromy homomorphism [19, §13.9]. The r^{th} roots $M \to ST\Sigma$ are precisely those principal $\mathbb{Z}_r$-bundles over $ST\Sigma$ for which the $\mathbb{Z}_r$-action extends to an S^1-action covering the S^1-action on $ST\Sigma$. In other words, each S^1-fibre of $ST\Sigma$ lifts to a (positive) path of length $2\pi/r$ in the corresponding S^1-fibre of M. This is the same as saying that the monodromy homomorphism takes the value 1 on the fibre class h.

Thus, the r^{th} roots $M \to ST\Sigma$ are classified by the subset

$$\mathrm{Hom}_1(\pi, \mathbb{Z}_r) := \{\delta \in \mathrm{Hom}(\pi, \mathbb{Z}_r)\colon \delta(h) = 1\}.$$

If we drop the condition on orientations, we also have to allow homomorphisms δ with $\delta(h) = -1$. This will become relevant in Section 6.

On a given M there are other structures as principal $\mathbb{Z}_r$-bundles over $ST\Sigma$ with each $\mathbb{Z}_r$-orbit lying in an S^1-fibre of M. These correspond to homomorphisms $\delta \in \mathrm{Hom}(\pi, \mathbb{Z}_r)$ with $\delta(h)$ a generator of $\mathbb{Z}_r$. Such more general $\mathbb{Z}_r$-bundles play no role in our discussion.

REMARK. There is a well-known isomorphism between, on the one hand, the deck transformation group of the universal covering $\tilde{X} \to X$ of a topological space X and, on the other hand, the fundamental group $\pi_1(X, x_0)$. This isomorphism depends, up to an inner automorphism, on the choice of a lift $\tilde{x}_0 \in \tilde{X}$ of the base point x_0, cf. [8, Remark 4.10]. This dependence becomes irrelevant once we consider homomorphisms into the abelian group $\mathbb{Z}_r$. Thus, while we usually think of π as a deck transformation group, one may still interpret the monodromy homomorphism $\pi \to \mathbb{Z}_r$ as being defined in terms of loops as in [19, §13].

We now want to give a characterisation of the homomorphisms $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ in terms of the allowable values on the generators in the standard presentation of π. In order to do so, we need to recall a theorem of Raymond and Vasquez [16] about the Seifert invariants of left-quotients of Lie groups, cf. [7]. We have seen in the preceding section that, once we equip Σ with a hyperbolic structure, its unit tangent bundle $ST\Sigma$ can be written as a left-quotient of $\widetilde{\mathrm{SL}}_2$, and so the same is true for its

r-fold covering M. Indeed, the fundamental group of the manifold M corresponding to $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ is $\tilde{\pi} = \ker \delta$. A representation $\rho \in \mathcal{R}(\pi, \widetilde{\mathrm{SL}}_2)$ as described at the end of Section 3 induces a representation $\tilde{\rho} \in \mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2)$ of $\tilde{\pi}$ as the deck transformation group of M.

By construction, M is a Seifert manifold with n multiple fibres of multiplicities $\alpha_1, \ldots, \alpha_n$ (just like the Seifert manifold $ST\Sigma$), but whereas the fibre index (see [7, Definition 6] of $ST\Sigma$ equals 1, the fibre index of M is r. Then, according to [16] or [7], the normalised Seifert invariants

$$\{g, b, (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$$

of M (where b is an integer and each β_j an integer between 1 and $\alpha_j - 1$) are subject to the condition that there exist integers $k_1, \ldots, k_n$ such that

$$rb = 2g - 2 - \sum_{j=1}^{n} k_j, \tag{1}$$

$$r\beta_j = \alpha_j - 1 + k_j\alpha_j, \quad j = 1, \ldots, n. \tag{2}$$

(Observe that these conditions are satisfied for $ST\Sigma$ with $r = 1$, $b = 2g - 2$, and all k_j equal to zero.) For a given Σ, these conditions impose severe restrictions on the possible values of r. These restrictions are implicit in [16]; for the reader's convenience we deduce them directly from the equations (1) and (2).

Lemma 1. *If $r \in \mathbb{N}$ satisfies the Raymond–Vasquez relations* (1) *and* (2), *then r is prime to $\alpha_1 \cdots \alpha_n$ and divides the integer $\alpha_1 \cdots \alpha_n \cdot \chi^{\mathrm{orb}}$.*

Conversely, if $r \in \mathbb{N}$ satisfies these latter conditions (for given g, n and α_j), then there are integers b, k_j and β_j (with $1 \leq \beta_j \leq \alpha_j - 1$) such that equations (1) *and* (2) *are satisfied.*

Proof. From (2) we see that r must be prime to α_j. With (1) and (2) one computes

$$r \cdot \alpha_1 \cdots \alpha_n \cdot \left(b + \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} \right) = -\alpha_1 \cdots \alpha_n \cdot \chi^{\mathrm{orb}},$$

which proves the claimed divisibility.

For the converse, the condition $\gcd(r, \alpha_j) = 1$ allows us to choose integers $1 \leq \beta_j \leq \alpha_j - 1$ and k_j such that (2) holds. One then computes

$$r \cdot \alpha_1 \cdots \alpha_n \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j} = -\alpha_1 \cdots \alpha_n \cdot \chi^{\mathrm{orb}} - \alpha_1 \cdots \alpha_n \cdot \left(2g - 2 - \sum_{j=1}^{n} k_j \right),$$

which shows that r divides $2g - 2 - \sum_{j=1}^{n} k_j$, as was to be shown. $\square$

REMARK. Equation (2) and the fact that r and α_j are coprime imply that multiple fibres with the same α_j also have the same β_j (and hence the same k_j). This is a unique feature of left-quotients of $\widetilde{\mathrm{SL}}_2$.

The converse implication of the preceding lemma has the following consequence. Given an $r \in \mathbb{N}$ satisfying the divisibility assumptions, we find—by the lemma—a set of normalised Seifert invariants satisfying the Raymond–Vasquez relations. In particular, the Euler number

$$e = -\left(b + \sum_{j=1}^{n} \frac{\beta_j}{\alpha_j}\right)$$

of the Seifert fibration must be non-zero, since $re = \chi^{\mathrm{orb}} < 0$. This means that the Seifert manifold M defined by these invariants is diffeomorphic to a left-quotient of $\widetilde{\mathrm{SL}}_2$. The projection $\widetilde{\mathrm{SL}}_2 \to \mathrm{PSL}_2\mathbb{R}$ induces the Seifert fibration $M \to \Sigma$ over a hyperbolic orbifold Σ and gives M the structure of an r^{th} root of $ST\Sigma$.

Lemma 2. *The homomorphisms* $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$, *where* $r \in \mathbb{N}$ *is supposed to satisfy the Raymond–Vasquez relations* (1) *and* (2), *can take arbitrary values on the generators* $u_1, v_1, \ldots, u_n, v_n$, *but the value on the* q_j *is determined by* $\delta(q_j) = k_j \bmod r$.

Proof. In $\mathbb{Z}_r$ we compute

$$0 = \delta(1) = \delta(q_j^{\alpha_j} h^{\alpha_j - 1}) = \delta(q_j)\alpha_j + \alpha_j - 1.$$

From equation (2) we see that, first of all, α_j must be prime to r, and secondly, that $\delta(q_j) = k_j \bmod r$, as claimed. Equation (1) implies that this condition on δ is consistent with the other relation in the presentation of π. It is then easy to see that we may define a homomorphism $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ by prescribing arbitrary values on the u_i and v_i. $\square$

We summarise our discussion in the following theorem.

Theorem 3. *The unit tangent bundle* $ST\Sigma$ *of an orbifold* Σ *of hyperbolic type admits an* r^{th} *root if and only if* $r \in \mathbb{N}$ *is prime to the multiplicities* $\alpha_1, \ldots, \alpha_n$ *of the cone points and a divisor of the integer* $\alpha_1 \cdots \alpha_n \cdot \chi^{\mathrm{orb}}(\Sigma)$. *In that case, the distinct* r^{th} *roots are in natural one-to-one correspondence with the elements of* $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$.

REMARK. There is a simple geometric explanation why r needs to be prime to the multiplicities $\alpha_1, \ldots, \alpha_n$ for an r^{th} root of $ST\Sigma$ to exist. From the local model for a fibre of multiplicity α_j one sees that when we pass to an r-fold cover with connected covering of the multiple fibre, then for r not prime to α_j the covering of the regular fibres will fail to be connected.

Lemma 2 implies that any two homomorphisms in $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ differ by a homomorphism $\pi \to \mathbb{Z}_r$ that sends h and the q_j to zero. Such a homomorphism may be interpreted as an element of

$$\mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}_r) \cong \mathrm{Hom}(H_1(\Sigma), \mathbb{Z}_r) \cong H^1(\Sigma; \mathbb{Z}_r) \cong \mathbb{Z}_r^{2g}.$$

(In the first term we really do mean the fundamental group $\pi_1(\Sigma)$, not the orbifold fundamental group π^{orb}.) This one-to-one correspondence of r^{th} roots of $ST\Sigma$ with elements of $H^1(\Sigma; \mathbb{Z}_r)$, however, is not natural. All we have is a free and transitive action of $H^1(\Sigma; \mathbb{Z}_r)$ on the set of r^{th} roots.

One way to give an explicit one-to-one correspondence between $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ and $\mathbb{Z}_r^{2g}$ is to fix a presentation for π, and then to associate with $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ the tuple

$$(\delta(u_1), \delta(v_1), \ldots, \delta(u_g), \delta(v_g)) \in \mathbb{Z}_r^{2g}.$$

REMARKS. (1) As observed by Johnson [10], there is a natural geometric lifting of mod 2 homology classes from a surface to its unit tangent bundle. Thus, spin structures on surfaces are *naturally* classified both by $\mathrm{Hom}_1(\pi_1(\Sigma), \mathbb{Z}_2)$ and $H_1(\Sigma; \mathbb{Z}_2)$. There is no such natural lifting of mod r classes for r greater than 2. However, given a smooth simple closed curve on Σ, we can consider its tangential lift to $ST\Sigma$. This will be used in the next section to help us understand the action of the diffeomorphism group of Σ on $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$.

(2) For an honest S^1-bundle over an arbitrary manifold X, one can classify r^{th} roots by mimicking the spectral sequence argument of [13, Chapter II.1] with $\mathbb{Z}_2$-coefficients replaced by $\mathbb{Z}_r$-coefficients, cf. [11]. This allows one to show that an r^{th} root exists if and only if the mod r reduction of the Euler class of the S^1-bundle vanishes (which can also be seen by more simple means), and then r^{th} roots are in (non-natural) one-to-one correspondence with the elements of $H^1(X; \mathbb{Z}_r)$.

We close this section by giving an explicit presentation of the fundamental group $\tilde{\pi} = \ker \delta$ of the manifold M corresponding to $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$. Choose integers s_i, t_i with $s_i \equiv \delta(u_i)$ and $t_i \equiv \delta(v_i) \bmod r$, $i = 1, \ldots, g$. Then $\tilde{\pi}$ is generated by

$$\tilde{u}_i := u_i h^{-s_i},\ \tilde{v}_i := v_i h^{-t_i},\ i = 1, \ldots, g, \quad \tilde{q}_j := q_j h^{-k_j},\ j = 1, \ldots, n, \quad \text{and} \quad \tilde{h} := h^r.$$

With the help of the Raymond–Vasquez relations one sees that this yields the presentation

$$\tilde{\pi} = \left\{ \tilde{u}_1, \tilde{v}_1, \ldots, \tilde{u}_g, \tilde{v}_g, \tilde{q}_1, \ldots, \tilde{q}_n, \tilde{h} \colon \prod_i [\tilde{u}_i, \tilde{v}_i] \prod_j \tilde{q}_j = \tilde{h}^b, \right.$$
$$\left. \tilde{q}_j^{\alpha_j} \tilde{h}^{\beta_j} = 1,\ \tilde{h} \text{ central} \right\}.$$

5. The action of diffeomorphisms on roots

We are now going to define an action of the diffeomorphism group of Σ on the set of r^{th} roots of $ST\Sigma$. For $g \geq 2$, it will be shown that this action is transitive for r odd, and that it has exactly two orbits for r even; the case $g = 1$ will require an *ad hoc* treatment; on a given hyperbolic orbifold of genus $g = 0$ and for each r there is at most one r^{th} root, since in that case $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ is trivial. Throughout, we fix an orientation of Σ, and the diffeomorphisms we consider are always understood to be *orientation preserving*. A diffeomorphism of an orbifold may at best permute cone points of the same multiplicity. By Lemma 2 and the remark following the proof of Lemma 1, any such permutation can be achieved by a diffeomorphism that induces the trivial action on $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$. So in order to understand the action of the diffeomorphism group of Σ on $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$, it suffices to consider diffeomorphisms that fix a neighbourhood of each cone point.

Let $M \to ST\Sigma$ be an r^{th} root of $ST\Sigma$, corresponding to some homomorphism $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$, and let f be a diffeomorphism of Σ as described. By slight abuse of notation, we may regard the differential Tf as a diffeomorphism of $ST\Sigma$; the composition of the projection $M \to ST\Sigma$ with Tf is then a new r^{th} root of $ST\Sigma$. We denote the corresponding element in $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ by $f_*\delta$.

Geometrically this means the following. Given $u \in \pi$, represent it by a loop in $ST\Sigma$. Then $(f_*\delta)(u) \in \mathbb{Z}_r$ is given by the monodromy of the covering $M \to ST\Sigma$ along the *preimage* of that loop under Tf.

We make one further abuse of notation. If u is an oriented, smooth closed curve on Σ (avoiding the cone points), we also write u for its tangential lift to a closed curve in $ST\Sigma$. Up to conjugation, this represents a well-defined element in $\pi = \pi_1(ST\Sigma)$, so it makes sense to speak of $\delta(u) \in \mathbb{Z}_r$. This abuse of notation is justified by the fact that for f a diffeomorphism of Σ, the tangential lift of $f(u)$ equals the image of the tangential lift of u under the differential Tf.

Consider a topological model for Σ as in Fig. 2. Here Σ is given the standard orientation, so that the simple closed curves u_i, v_i representing the standard generators of $H_1(\Sigma)$ intersect positively in a single point. The notation u_i, v_i has been chosen in accordance with the presentation of π in Section 3. In the sequel we identify a homomorphism $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ (representing an r^{th} root of $ST\Sigma$) with the corresponding $2g$-tuple of integers mod r, that is, we write

$$\delta = (s_1, t_1, \dots, s_g, t_g) \in \mathbb{Z}_r^{2g},$$

where $s_i = \delta(u_i)$ and $t_i = \delta(v_i)$.

For our discussion below we note that the element $h \in \pi$ corresponds to a positively oriented regular fibre of $ST\Sigma$, so it can be represented by a small positively oriented circle on Σ.

Next we want to show that for any given $\delta \in \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ there is a diffeomorphism f of Σ such that $f_*\delta$ is in a very simple standard form. This is done by studying the

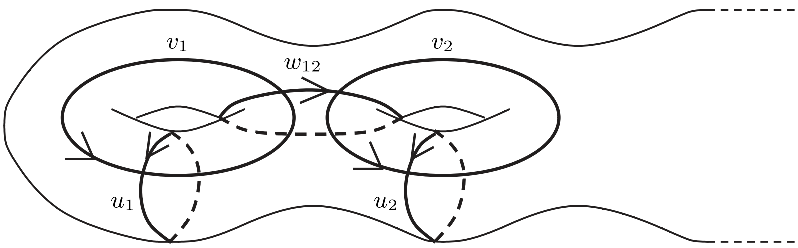

Fig. 2. Loops on Σ representing loops in $ST\Sigma$.

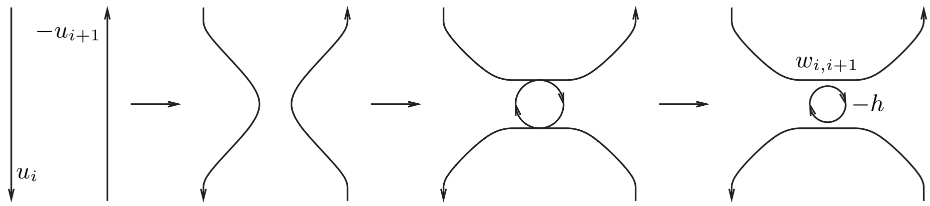

Fig. 3. The smooth deformation from $u_i - u_{i+1}$ to $w_{i,i+1} - h$.

transformation behaviour of δ under certain Dehn twists on Σ.

For u a *simple* closed curve on Σ, write f^u for the right-handed Dehn twist along u.

Lemma 4. *Under the basic Dehn twists f^{u_i}, f^{v_i} and $f^{w_{i,i+1}}$, the tuple $\delta = (s_1, t_1, \ldots, s_g, t_g)$ transforms as follows*:

$$f_*^{u_i}\delta = (\ldots, s_i, t_i - s_i, \ldots),$$

$$f_*^{v_i}\delta = (\ldots, s_i + t_i, t_i, \ldots),$$

$$f_*^{w_{i,i+1}}\delta = (\ldots, s_i, t_i - s_i + s_{i+1} - 1, s_{i+1}, t_{i+1} + s_i - s_{i+1} + 1, \ldots).$$

Proof. The Dehn twist f^{u_i} sends u_i to itself and v_i to $u_i + v_i$; the differential Tf^{u_i} has the same effect, when those curves are regarded as loops in $ST\Sigma$. So the inverse diffeomorphism sends u_i to itself and v_i to $v_i - u_i$. This gives the formula for $f_*^{u_i}$. The argument for $f_*^{v_i}$ is analogous.

In order to investigate $f_*^{w_{i,i+1}}$, we need to compute $\delta(w_{i,i+1})$.

CLAIM. $\delta(w_{i,i+1}) = \delta(u_i) - \delta(u_{i+1}) + 1$.

This can be seen as follows (cf. [10] for an analogous idea). The disjoint union of the smooth curves u_i and $-u_{i+1}$ (that is, u_{i+1} with reversed orientation) can be deformed smoothly into the union of $w_{i,i+1}$ and a small circle oriented negatively (see Fig. 3 for a schematic illustration). The tangential lift of the latter equals $-h$. This implies the claim.

Now, the inverse of $f^{w_{i,i+1}}$ sends v_i to $v_i - w_{i,i+1}$, and v_{i+1} to $v_{i+1} + w_{i,i+1}$; the other basic loops remain unchanged. In conjunction with the claim, this gives the formula for $f_*^{w_{i,i+1}}$. □

REMARK. The formulae of Lemma 4—for the case $r = 2$, where signs do not matter—were derived earlier by Dąbrowski and Percacci [4] by quite involved calculations in local coordinates. Related considerations can also be found in the work of Sipe [18]. She studied r^{th} roots of the unit tangent bundle of hyperbolic *surfaces* with the aim of describing certain finite quotients of their mapping class group.

The signs in the formulae of Lemma 4 change when we perform left-handed Dehn twists. Therefore, Dehn twists along u_i and v_i enable us to perform Euclid's algorithm on any pair of integers representing the pair (s_i, t_i) of mod r classes. This implies that we can reduce one component to zero and the other to the unique element $d_i \in \mathbb{Z}_r$ determined by the conditions that the principal ideal in $\mathbb{Z}_r$ generated by d_i equal the ideal generated by s_i and t_i, and that the integer representative of d_i lying between 1 and r be a divisor of r. By slight abuse of notation we write this last condition as $d_i | r$. The pair $(d_i, 0)$ can be changed to the pair $(0, d_i)$ by further such Dehn twists. In total, we can find a composition of Dehn twists of Σ that transforms δ to

$$(0, d_1, \dots, 0, d_g).$$

In order to simplify this further, we have to bring the curves $w_{i,i+1}$ into play. By the claim, the transformed δ takes the value 1 on $w_{i,i+1}$. Thus, when we perform d_1 right-handed Dehn twist along w_{12}, the tuple $(0, d_1, 0, d_2, \dots)$ changes to $(0, 0, 0, d_1 + d_2, \dots)$. Continuing with the appropriate Dehn twists along w_{23} up to $w_{g-1,g}$, we find a diffeomorphism transforming δ to

$$(0, \dots, 0, d_1 + \dots + d_g).$$

We shall presently describe further Dehn twists that bring δ into one of the forms listed in the next proposition.

Proposition 5. *By a sequence of Dehn twists, we can bring δ into one of the following standard forms*:

- $(0, \dots, 0, 0)$ *if* $g \geq 2$ *and* r *odd*,
- $(0, \dots, 0, 0)$ *or* $(0, \dots, 0, 1)$ *if* $g \geq 2$ *and* r *even*,
- $(0, d)$ *with* $d | r$ *if* $g = 1$ (*beware that this includes* $d = r \equiv 0$).

Of course, for $g = 1$ the surface Σ will be of hyperbolic type only if there is at least one cone point. For $g = 0$ (and at least three cone points), $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ is trivial.

Proof of Proposition 5. The case $g = 1$ has been settled by the discussion preceding the proposition. In the case $g \geq 2$, we may assume that δ has already been transformed into the form $(0, \dots, 0, d)$, as yet without any information on d.

We only write the last four components of the $2g$-tuple in $\mathbb{Z}_r^{2g}$. We claim that there are Dehn twists giving the following sequence of transformations:

$$(0, 0, 0, d) \to (0, \mp 1, 0, d \pm 1) \to (0, \pm 1, 0, d \pm 1) \to (0, 0, 0, d \pm 2).$$

Indeed, the first and third step are given by a Dehn twist (of the appropriate sign) along $w_{g-1,g}$, the second by a sequence of Dehn twists along u_{g-1} and v_{g-1}.

So we can always reduce the last component to 0 or 1. If r is odd, then Dehn twists along u_g and v_g allow us to transform from $(0, 1)$ (in the last two components) to $(0, 2)$—since either of 1 or 2 generates the same principal ideal in $\mathbb{Z}_r$, namely the full ring. □

The standard forms listed in the preceding proposition turn out to be pairwise inequivalent under the action of the diffeomorphism group. We first show this for the case $g = 1$.

Lemma 6. *For $g = 1$, two standard forms $(0, d)$ and $(0, d')$, where we think of d, d' as integers between 1 and r (which divide r), are equivalent if and only if $d = d'$.*

Proof. Assume without loss of generality that $d' \leq d$. The action of the diffeomorphism group of a hyperbolic orbifold Σ of genus 1 translates into the standard $\mathrm{SL}(2, \mathbb{Z})$-action on $\mathbb{Z}_r^2 = \mathrm{Hom}_1(\pi, \mathbb{Z}_r)$. The orbit of $(0, d)$ under this action consists of elements of the form (md, nd) (with m and n coprime). Since d is a divisor of r, the number nd (thought of as an integer) can be congruent to d' mod r only if d is a divisor of d', which forces $d = d'$. □

The $\mathbb{Z}_2$-invariant that distinguishes the standard forms in the case $g \geq 2$ (and r even) goes back to Atiyah [2]. A spin structure on an honest surface Σ has an associated complex line bundle L. Once a complex structure has been chosen on Σ, one can speak of holomorphic sections of L. The dimension mod 2 of the vector space of holomorphic sections turns out to be independent of the chosen complex structure; this is Atiyah's invariant of spin structures. As remarked earlier, Johnson [10] defined a natural lifting of mod 2 homology classes from a surface Σ to its unit tangent bundle. A spin structure on Σ then gives rise to a quadratic form on $H_1(\Sigma; \mathbb{Z}_2)$. Johnson goes on to show that the Arf invariant of that quadratic form (whose definition can be found on any Turkish 10 Lira note) equals Atiyah's invariant.

REMARK. The 2-dimensional spin cobordism group Ω_2^{spin} is isomorphic to $\mathbb{Z}_2$; the Atiyah invariant distinguishes the two cobordism classes.

Now we allow once again arbitrary orbifolds Σ of hyperbolic type. Motivated by Johnson's work, we define a $\mathbb{Z}_2$-valued invariant of an r^{th} root δ of $ST\Sigma$ (with r even), which we write as $\delta = (s_1, t_1, \dots, s_g, t_g) \in \mathbb{Z}_r^{2g}$, by

$$A(\delta) = \sum_{i=1}^{g} (s_i + 1)(t_i + 1) \mod 2.$$

Note that, for r even, this mod 2 reduction is well defined. The definition of this invariant can also be phrased as follows. Given a principal $\mathbb{Z}_r$-bundle $M \to ST\Sigma$ with r even, there is an intermediate double covering of $ST\Sigma$. Thus, an r^{th} root (with r even) induces in a natural way a spin structure. The A-invariant is simply the Atiyah invariant of that spin structure.

Lemma 7. *The number $A(\delta) \in \mathbb{Z}_2$ is a diffeomorphism invariant, i.e. for any (orientation preserving) diffeomorphism f of Σ one has $A(f_*\delta) = A(\delta)$.*

Proof. We need only consider diffeomorphisms that fix the cone points. The group of such diffeomorphisms is generated by the Dehn twists along u_i, v_i and $w_{i,i+1}$. The invariance of $A(\delta)$ under these Dehn twists can be checked easily with the formulae in Lemma 4. □

Obviously, the two standard forms $(0, \dots, 0, 0)$ and $(0, \dots, 0, 1)$ (for $g \geq 2$ and r even) are distinguished by the A-invariant.

DEFINITION. For $g \geq 2$ and r even, we say an r^{th} root δ is of *even* (resp. *odd*) *type* if $A(\delta)$ equals 0 (resp. 1).

So the standard form $(0, \dots, 0, 0)$ is of even type for g even, and of odd type for g odd; the standard form $(0, \dots, 0, 1)$ has the complementary type.

Proposition 8. *For $g \geq 2$ and r even, the number of r^{th} roots of even (resp. odd) type equals $r^{2g}(2^g \pm 1)/2^{g+1}$.*

Proof. Write $r = 2s$. An r^{th} root $\delta = (s_1, t_1, \dots, s_g, t_g)$ will be even if and only if an even number of summands $(s_i + 1)(t_i + 1)$ in $A(\delta)$ are odd. Such a summand is odd if and only if both s_i and t_i are even, which gives us s^2 possibilities for choosing s_i and t_i. On the other hand, there are $3s^2$ possibilities for choosing s_i and t_i such that $(s_i + 1)(t_i + 1)$ becomes even. It follows that the number of roots of even type is given by

$$\begin{aligned}\sum_{k \text{ even}} \binom{g}{k} (s^2)^k (3s^2)^{g-k} &= \frac{1}{2}((s^2 + 3s^2)^g + (-s^2 + 3s^2)^g) \\ &= \frac{1}{2}((4s^2)^g + (2s^2)^g) \\ &= \frac{r^{2g}(2^g + 1)}{2^{g+1}}.\end{aligned}$$

For roots of odd type, the calculation is analogous. □

Remark. In the case $r = 2$, i.e. for spin structures, Propositions 5 and 8 are well known—especially, it seems, among mathematical physicists. Our arguments for deriving them generalise those of Dąbrowski and Percacci [4]. An alternative approach can be found in the work of Alvarez-Gaumé, Moore and Vafa [1]. They appeal to the relation between spin structures and theta functions in order to describe the action of the diffeomorphism group.

6. An algebraic reformulation

The Baer–Nielsen theorem for the orbifold Σ says, in essence, that the group of all (not just orientation preserving) diffeomorphisms of Σ modulo those isotopic to the identity can be identified with the group $\mathrm{Out}(\pi^{\mathrm{orb}})$ of outer automorphisms of π^{orb}; see [21]. We now want to use this to reformulate the action of the diffeomorphism group on the space of r^{th} roots of $ST\Sigma$ in an algebraic way. This serves as a preparation for the next section, where we tie up our discussion of r^{th} roots with the moduli problem for so-called taut contact circles, which was addressed in our earlier paper [8]. As announced there, the results of the present note allow us to count the connected components of the moduli spaces in question.

There is an obvious action of $\mathrm{Aut}(\pi)$ on

$$\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r) := \{\delta \in \mathrm{Hom}(\pi, \mathbb{Z}_r) \colon \delta(h) = \pm 1\}.$$

(The fact that we now allow $\delta(h) = -1$ corresponds to having orientation reversing diffeomorphisms included in the discussion.) This descends to an action of $\mathrm{Out}(\pi)$, since $\mathbb{Z}_r$ is abelian. Thus, in order to define an action of $\mathrm{Out}(\pi^{\mathrm{orb}})$ on $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$, we should first define a suitable lift from $\mathrm{Aut}(\pi^{\mathrm{orb}})$ to $\mathrm{Aut}(\pi)$. Recall from [8, Lemma 4.13] that there is a short exact sequence

$$0 \to \mathbb{Z}^{2g} \to \mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi^{\mathrm{orb}}) \to 1.$$

(This holds true for the fundamental group π of any Seifert manifold which is a left quotient of $\widetilde{\mathrm{SL}}_2$ and has base orbifold Σ.) Thus, algebraically, it is not clear how to define a lifting. Instead, we find a suitable lift by a direct appeal to the Baer–Nielsen theorem. Put briefly, we represent a given element of $\mathrm{Out}(\pi^{\mathrm{orb}})$ by an orbifold diffeomorphism f of Σ, and then find the lift as the automorphism corresponding to the differential Tf. From that construction it is clear that our algebraic definition of the action by the diffeomorphism group on the set of r^{th} roots corresponds to the geometric definition in the preceding sections (except that we have replaced a left action by a right action, which is owed to the conventions in the algebraic setting of the next section).

Lemma 9. *There is a natural right action of* $\mathrm{Out}(\pi^{\mathrm{orb}})$ *on* $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$, *defined as follows. Given a class* $[\bar{\vartheta}] \in \mathrm{Out}(\pi^{\mathrm{orb}})$, *represented by an automorphism* $\bar{\vartheta} \in \mathrm{Aut}(\pi^{\mathrm{orb}})$, *there is a geometrically defined lifting of this representative to an automorphism* $\vartheta \in \mathrm{Aut}(\pi)$. *Then the action of* $[\bar{\vartheta}]$ *on* $\delta \in \mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$ *is defined by* $\delta \mapsto \delta \circ \vartheta$.

Proof. By the Nielsen theorem [21, Theorem 8.1] there is a diffeomorphism f of Σ (fixing a base point x_0) covered by a diffeomorphism $\tilde{f}$ of $\tilde{\Sigma}$ (fixing a chosen lift $\tilde{x}_0$ of x_0) such that

$$\tilde{f} \circ \bar{u} \circ \tilde{f}^{-1} = \bar{\vartheta}(\bar{u}) \quad \text{for all} \quad \bar{u} \in \pi^{\mathrm{orb}}.$$

Regard the differential Tf as a diffeomorphism of $ST\Sigma$, and let $\widetilde{Tf}$ be a lift to a diffeomorphism of $\widetilde{\mathrm{SL}}_2$. Define $\vartheta \in \mathrm{Aut}(\pi)$ by

$$\vartheta(u) = \widetilde{Tf} \circ u \circ \widetilde{Tf}^{-1} \quad \text{for all} \quad u \in \pi.$$

Since the fibre class h generates the centre of π, we have $\vartheta(h) = h^{\pm 1}$. So the homomorphism $\delta \circ \vartheta$ is still an element of $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$. Moreover, the definitions imply that $\bar{\vartheta}_1 \circ \bar{\vartheta}_2$ lifts to $\vartheta_1 \circ \vartheta_2$, so the prescription $\delta \mapsto \delta \circ \vartheta$ does indeed define a *right* action, provided we can establish independence of choices.

Two different lifts of Tf differ by a deck transformation of $ST\Sigma$, i.e. an element of π. So the corresponding lifts ϑ differ by an inner automorphism of π. Thus, the homomorphism $\delta \circ \vartheta$ into the abelian group $\mathbb{Z}_r$ is independent of this choice of lift.

Next, we show that $\delta \circ \vartheta$ depends only on the class $[\bar{\vartheta}]$, not on the choice of representative $\bar{\vartheta}$, or in other words, that for any *inner* automorphism $\bar{\vartheta}$ we have $\delta \circ \vartheta = \delta$. By the Baer theorem [21, Theorem 3.1], the Nielsen realisation f of any inner automorphism $\bar{\vartheta}$ of π^{orb} is isotopic to the identity (by an isotopy *not* fixing the base point, in general). Then Tf is likewise isotopic to the identity. This isotopy lifts to a fibre isotopy between $\widetilde{Tf}$ and a deck transformation of $\widetilde{\mathrm{SL}}_2 \to ST\Sigma$. This implies that the resulting ϑ will be an inner automorphism of π, and hence $\delta \circ \vartheta = \delta$, as we wanted to show.

Finally, it remains to verify that the construction does not depend on the choice of Nielsen realisation f. Two such realisations differ by a diffeomorphism whose lift to $\tilde{\Sigma}$ induces the identity on π^{orb}. Then the argument concludes as before by an appeal to Baer's theorem. $\square$

7. The moduli space of taut contact circles

Let M be a given closed, orientable 3-manifold diffeomorphic to a left quotient of $\widetilde{\mathrm{SL}}_2$ with fundamental group $\tilde{\pi}$. This is in a unique way a Seifert manifold over an orbifold Σ of hyperbolic type, with a well-defined fibre index r. Recall from the end of Section 4 the presentation of $\tilde{\pi}$ involving the normalised Seifert invariants of M.

As shown in our paper [8], the Teichmüller space $\mathcal{T}(M)$ of taut contact circles, i.e. the space of taut contact circles on M modulo diffeomorphisms isotopic to the identity, can be identified with $\mathrm{Inn}(\widetilde{\mathrm{SL}}_2) \setminus \mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2)$, where $\mathcal{R}$ stands for the Weil space of representations as in Section 2. The moduli space $\mathcal{M}(M)$ of taut contact circles, i.e. the space of taut contact circles on M modulo all diffeomorphisms of M, is in turn given by $\mathcal{T}(M)/\mathrm{Out}(\tilde{\pi})$. With this algebraic translation taken for granted, nothing further needs to be known about taut contact circles (not even their definition), i.e. the following can be read as a discussion of these algebraically defined spaces, where we want to understand the action of $\mathrm{Out}(\tilde{\pi})$ on $\mathcal{T}(M) = \mathrm{Inn}(\widetilde{\mathrm{SL}}_2) \setminus \mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2)$ with the help of the geometry of r^{th} roots of $ST\Sigma$. See also [12] for the relevance of such questions to the deformation theory of Seifert manifolds.

REMARK. To a large extent we follow the notational conventions of [8]. The one difference that needs to be pointed out is that in our previous paper, π denoted the fundamental group of M, as $ST\Sigma$ did not play much of a role in our discussion there. In the present paper, π denotes the fundamental group of $ST\Sigma$, and $\tilde{\pi}$ that of M.

Write $\mathcal{T}(\Sigma)$ for the Teichmüller space of hyperbolic metrics on the base orbifold Σ, together with a choice of orientation. This means that $\mathcal{T}(\Sigma)$ has two connected components $\mathcal{T}^+(\Sigma)$ and $\mathcal{T}^-(\Sigma)$. Algebraically, $\mathcal{T}(\Sigma)$ may be thought of as $\mathrm{Inn}(\mathrm{PSL}_2\mathbb{R}) \setminus \mathcal{R}(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R})$. In Section 4 of [8] it was shown that $\mathcal{T}(M)$ is a trivial principal $\mathbb{Z}^{2g}$-bundle over $\mathcal{T}(\Sigma)$. For $\mathrm{Aut}(\tilde{\pi})$ there is a short exact sequence as for $\mathrm{Aut}(\pi)$ in the previous section. The normal subgroup $\mathbb{Z}^{2g} \subset \mathrm{Aut}(\tilde{\pi})$ acts as $(r\mathbb{Z})^{2g}$ on the mentioned principal bundle. This implies that $\mathcal{T}(M)/\mathbb{Z}^{2g}$—where the quotient is taken under the action of $\mathbb{Z}^{2g} \subset \mathrm{Aut}(\tilde{\pi})$—is a trivial r^{2g}-fold covering of $\mathcal{T}(\Sigma)$, and the moduli space of taut contact circles on M can be described as

$$\mathcal{M}(M) = (\mathcal{T}(M)/\mathbb{Z}^{2g})/\mathrm{Out}(\pi^{\mathrm{orb}}).$$

So the following theorem essentially settles the moduli problem for taut contact circles on left quotients of $\widetilde{\mathrm{SL}}_2$. Here π denotes, as before, the fundamental group of $ST\Sigma$. For the proof below, notice that there are quotient maps $\tilde{\pi} \to \pi^{\mathrm{orb}}$ and $\pi \to \pi^{\mathrm{orb}}$, given by quotienting out the normal subgroup generated by the central element $\tilde{h}$ and h, respectively.

Theorem 10. *The quotient $\mathcal{T}(M)/\mathbb{Z}^{2g}$ of the Teichmüller space of taut contact circles on M under the action of $\mathbb{Z}^{2g} \subset \mathrm{Aut}(\tilde{\pi})$ has a natural description as follows*:

$$\mathcal{T}(M)/\mathbb{Z}^{2g} = \mathrm{Hom}_1(\pi, \mathbb{Z}_r) \times \mathcal{T}^+(\Sigma) \sqcup \mathrm{Hom}_{-1}(\pi, \mathbb{Z}_r) \times \mathcal{T}^-(\Sigma)$$

On the second factors $\mathcal{T}^\pm(M)$, the right action of $\mathrm{Out}(\pi^{\mathrm{orb}})$ *is the obvious one*; *on the first factors* $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$, *the group* $\mathrm{Out}(\pi^{\mathrm{orb}})$ *acts from the right as described in* Section 6.

REMARK. If the Nielsen realisation f of an automorphism $\bar{\vartheta}$ of π^{orb} is orientation reversing (so that $\bar{\vartheta}$ will exchange the components $\mathcal{T}^\pm(\Sigma)$), then the differential Tf, regarded as a diffeomorphism of $ST\Sigma$, will reverse the fibre direction, so $\bar{\vartheta}$ will also exchange $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$. In fact, no left-quotient of $\widetilde{\mathrm{SL}}_2$ admits any orientation reversing diffeomorphism [14].

Proof of Theorem 10. First we are going to define a map from the left-hand side $\mathcal{T}(M)/\mathbb{Z}^{2g}$ to the right factors $\mathcal{T}^+(\Sigma) \sqcup \mathcal{T}^-(\Sigma) = \mathcal{T}(\Sigma)$ on the right-hand side. Recall from [8, Section 4] that the projection $\widetilde{\mathrm{SL}}_2 \to \mathrm{PSL}_2\mathbb{R}$ induces a covering map $\mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2) \to \mathcal{R}(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R})$, which in turn induces a well-defined map $\mathcal{T}(M) \to \mathcal{T}(\Sigma)$, since any inner automorphism of $\widetilde{\mathrm{SL}}_2$ induces an inner automorphism of $\mathrm{PSL}_2\mathbb{R}$. The action of $\mathbb{Z}^{2g} \subset \mathrm{Aut}(\tilde{\pi})$ on $\tilde{\pi}$ is given by multiplying the generators $\tilde{u}_i, \tilde{v}_i$ with the corresponding power of the central element $\tilde{h}$. Since this central element generates the kernel of the quotient map $\tilde{\pi} \to \pi^{\mathrm{orb}}$, we get an induced map $\mathcal{T}(M)/\mathbb{Z}^{2g} \to \mathcal{T}(\Sigma)$.

Next we want to define a map $\mathcal{T}(M)/\mathbb{Z}^{2g} \to \mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$ to the left factors on the right-hand side. This means that, given $\tilde{\rho} \in \mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2)$ representing an element $[\tilde{\rho}] \in \mathcal{T}(M)/\mathbb{Z}^{2g}$, and given $u \in \pi$, we want to define $\delta(u) \in \mathbb{Z}_r$ in such a way that δ becomes a homomorphism $\pi \to \mathbb{Z}_r$ sending h to ± 1, and such that δ is independent of the chosen representative $\tilde{\rho}$.

Thus, start with $\tilde{\rho}$ and u as described. The element $u \in \pi$ projects to an element $\bar{u} \in \pi^{\mathrm{orb}}$, which in turn lifts to an element $\tilde{u} \in \tilde{\pi}$, unique up to powers of $\tilde{h}$. Likewise, the representation $\tilde{\rho} \in \mathcal{R}(\tilde{\pi}, \widetilde{\mathrm{SL}}_2)$ projects to a representation

$$\bar{\rho} \in \mathcal{R}(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R}) = \mathcal{R}^+(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R}) \sqcup \mathcal{R}^-(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R}),$$

as observed in the first part of the proof, and then can be lifted in a preferred way to a representation $\rho \in \mathcal{R}(\pi, \widetilde{\mathrm{SL}}_2)$; cf. Section 2 for the notation $\mathcal{R}^\pm$.

In [8, Section 4] we gave a definition of such a preferred lift that also allowed us to lift from a representation of π^{orb} to one of $\tilde{\pi}$. Here, where we only want to lift to a representation of π, we shall make a choice that leads to a natural description of the $\mathrm{Out}(\pi^{\mathrm{orb}})$-action. In order to allow unique lifting of maps to universal covers, we choose base points in Σ, $ST\Sigma$ and their universal covers in such a way that all relevant projections are base point preserving. Likewise, we choose a base point in

$\widetilde{\mathrm{SL}}_2 = \widetilde{ST\mathbb{H}}$ over a base point in $\mathbb{H}$; this determines a base point in any discrete quotient of these spaces.

Now to the definition of ρ. In the sequel it is understood that all diffeomorphisms are base point preserving. Choose a diffeomorphism $g\colon \Sigma \to \bar{\rho}(\pi^{\mathrm{orb}})\backslash\mathbb{H}^2$ whose (unique) lift $\tilde{g}$ to the universal cover satisfies

$$\tilde{g} \circ \bar{u} \circ \tilde{g}^{-1} = \bar{\rho}(\bar{u}) \quad \text{for all} \quad \bar{u} \in \pi^{\mathrm{orb}}.$$

This is possible by the Nielsen theorem again; observe the formal similarity with the argument in the proof of Lemma 9. Now, with L denoting left multiplication in $\widetilde{\mathrm{SL}}_2$, define the preferred lift ρ of $\bar{\rho}$ by

$$L_{\rho(u)} = \widetilde{Tg} \circ u \circ \widetilde{Tg}^{-1} \quad \text{for all} \quad u \in \pi.$$

REMARK. The preferred lift as defined in [8] depended on a choice of presentation of π. If we take the u_i and v_i as the tangential lifts of $\bar{u}_i$ and $\bar{v}_i$, then the preferred lift defined here is the same as that in [8].

When we identify $\mathbb{H}$ with the upper half-plane in $\mathbb{C}$, and $\widetilde{\mathrm{SL}}_2$ with $\mathbb{H}^2 \times \mathbb{R}$ with coordinates (z, θ), cf. [8, p. 58], we can describe the left action of $\rho(u)$ on $\widetilde{\mathrm{SL}}_2$ explicitly (at least for some elements $u \in \pi$). For h there is no choice in the lifting; one has

$$\rho(h)(z, \theta) = (z, \theta \pm 2\pi),$$

where the sign is determined by $\bar{\rho} \in \mathcal{R}^{\pm}(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R})$. Similarly, one has

$$\tilde{\rho}(\tilde{h})(z, \theta) = (z, \theta \pm 2\pi r).$$

The lift $\rho(q_j)$ is completely determined by the relation which q_j satisfies in the group π. For u_i resp. v_i, any lift other than the preferred one $\rho(u_i)$ resp. $\rho(v_i)$ would differ from it by an arbitrary translation in the θ-component by integer multiples of 2π. Moreover, the action of $\mathbf{w} \in \mathbb{Z}^{2g} \subset \mathrm{Aut}(\tilde{\pi})$ on $\tilde{\rho}$ is given by $\tilde{\rho} \mapsto \tilde{\rho}_{r\mathbf{w}}$, with

$$\tilde{\rho}_{r\mathbf{w}}(\tilde{u}_i)(z, \theta) = \tilde{\rho}(\tilde{u}_i)(z, \theta) + (0, 2\pi r w_{2i-1}),$$
$$\tilde{\rho}_{r\mathbf{w}}(\tilde{v}_i)(z, \theta) = \tilde{\rho}(\tilde{v}_i)(z, \theta) + (0, 2\pi r w_{2i}).$$

Now back to the construction of the homomorphism δ corresponding to the class $[\tilde{\rho}] \in \mathcal{T}(M)/\mathbb{Z}^{2g}$. Since both $\rho(u)$ and $\tilde{\rho}(\tilde{u})$ are lifts of $\bar{\rho}(\bar{u}) \in \mathrm{PSL}_2\mathbb{R}$ to $\widetilde{\mathrm{SL}}_2$, their actions on the θ-component differ by a shift by some integer multiple of 2π, so we can define $\delta(u) \in \mathbb{Z}$ by

$$\rho(u)(z, \theta) = \tilde{\rho}(\tilde{u})(z, \theta) + (0, 2\pi\delta(u)). \tag{3}$$

Since ρ is fixed to be the preferred lift of $\bar{\rho}$, the only ambiguity in this equation is the lift $\tilde{u}$ of $\bar{u}$, which may be changed by powers of $\tilde{h}$. From the described action of $\tilde{\rho}(\tilde{h})$ we conclude that $\delta(u)$ is well defined mod r, so we may regard it as a map into $\mathbb{Z}_r$. By construction it is clear that δ has the homomorphism property. For $u = h$ we may choose $\tilde{u} = 1$; this gives $\delta(h) = \pm 1$, where the sign again corresponds to $\bar{\rho} \in \mathcal{R}^{\pm}$, as it should. Hence $\delta \in \mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$.

Inner automorphisms of $\widetilde{\mathrm{SL}}_2$ act trivially on the θ-component, so δ only depends on the class of $\tilde{\rho}$ in $\mathcal{T}(M)$. Moreover, $\delta(u)$ does not change mod r when $\tilde{\rho}$ is replaced by some $\tilde{\rho}_{r\mathbf{w}}$ in the same orbit under the $\mathbb{Z}^{2g}$-action on $\mathcal{T}(M)$. This finishes the construction of the map

$$\mathcal{T}(M)/\mathbb{Z}^{2g} \to \mathrm{Hom}_1(\pi, \mathbb{Z}_r) \times \mathcal{T}^+(\Sigma) \sqcup \mathrm{Hom}_{-1}(\pi, \mathbb{Z}_r) \times \mathcal{T}^-(\Sigma).$$

We show this map to be a bijection by exhibiting an explicit inverse. The defining equation (3) for δ can be read backwards, as it were, in order to define the desired inverse map. Thus, given $\bar{\rho} \in \mathcal{R}^{\pm}(\pi^{\mathrm{orb}}, \mathrm{PSL}_2\mathbb{R})$ and $\delta \in \mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$ (with matching signs), we would like to use (3) to define $\tilde{\rho}$. This is indeed possible, if we take a little care. First of all, we know that there is no choice in defining $\tilde{\rho}(\tilde{h})$ and $\tilde{\rho}(\tilde{q}_j)$, so we only need to consider elements $\tilde{u} \in \tilde{\pi}$ which are not stabilised under the $\mathbb{Z}^{2g}$-action $\tilde{\rho}(\tilde{u}) \mapsto \tilde{\rho}_{r\mathbf{w}}(\tilde{u})$. Let $\bar{u} \in \pi^{\mathrm{orb}}$ be the projection of $\tilde{u}$, and $u \in \pi$ a lift of $\bar{u}$. In the equation

$$\tilde{\rho}(\tilde{u})(z, \theta) = \rho(u)(z, \theta) - (0, 2\pi\delta(u)),$$

with ρ taken as the preferred lift of $\bar{\rho}$, the right-hand side can be made sense of if the θ-component is read as lying in $\mathbb{R}/2\pi r\mathbb{Z}$, and it does not depend on the choice of lift u. So for $\tilde{u}$ of the described kind, we can use this equation (given $\bar{\rho}$, δ and $\tilde{u}$) to get a well-defined element $[\tilde{\rho}] \in \mathcal{T}(M)/\mathbb{Z}^{2g}$. This prescription obviously defines an inverse of the previously constructed map.

It remains to show that the right action of $\mathrm{Out}(\pi^{\mathrm{orb}})$ is as claimed in the theorem. Given $[\bar{\vartheta}] \in \mathrm{Out}(\pi^{\mathrm{orb}})$, let $\tilde{\vartheta} \in \mathrm{Aut}(\tilde{\pi})$ be any lift of $\bar{\vartheta}$, and $\vartheta \in \mathrm{Aut}(\pi)$ the lift constructed in the proof of Lemma 9. The action of $[\bar{\vartheta}]$ on $\mathcal{T}(M)/\mathbb{Z}^{2g}$ is given by $\tilde{\rho} \mapsto \tilde{\rho} \circ \tilde{\vartheta}$. This is indeed well defined: the choice of representative $\bar{\vartheta}$ of the class $[\bar{\vartheta}]$ is irrelevant, because in $\mathcal{T}(M)$ we have taken the quotient under $\mathrm{Inn}(\widetilde{\mathrm{SL}}_2)$; the specific lifting to $\tilde{\vartheta}$ is of no importance in the quotient $\mathcal{T}(M)/\mathbb{Z}^{2g}$. That the action of $\mathrm{Out}(\pi^{\mathrm{orb}})$ on the right-hand side of the identity in the theorem is also as claimed now follows from equation (3) and the observation that our construction of the preferred lift of $\bar{\rho}$ entails that $\rho \circ \vartheta$ is the preferred lift of $\bar{\rho} \circ \bar{\vartheta}$.

This concludes the proof of Theorem 10. □

REMARK. With ρ being the preferred lift of $\bar{\rho}$, equation (3) is precisely the algebraic reformulation of the geometric definition of δ as a monodromy homomorphism given in Section 4.

When we take the quotient under the action of $\mathrm{Out}(\pi^{\mathrm{orb}})$, the trivial covering $\mathcal{T}(M)/\mathbb{Z}^{2g} \to \mathcal{T}^+(\Sigma) \sqcup \mathcal{T}^-(\Sigma)$ given by Theorem 10 becomes a possibly branched covering $\mathcal{M}(M) \to \mathcal{M}(\Sigma)$, where $\mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{Out}(\pi^{\mathrm{orb}})$ denotes the moduli space of hyperbolic metrics on Σ.

We are now interested in the number of connected components of $\mathcal{M}(M)$, and the number of sheets in each connected component of the covering $\mathcal{M}(M) \to \mathcal{M}(\Sigma)$. The space $\mathcal{M}(\Sigma)$ is connected, so the number of connected components of $\mathcal{M}(M)$ equals the number of orbits of the $\mathrm{Out}(\pi^{\mathrm{orb}})$-action on $\mathrm{Hom}_{\pm 1}(\pi, \mathbb{Z}_r)$. Geometrically, this corresponds to the number of orbits of the action on $\mathrm{Hom}_1(\pi, \mathbb{Z}_r)$ given by the orientation preserving diffeomorphisms of Σ. Moreover, the number of sheets in each connected component of the covering $\mathcal{M}(M) \to \mathcal{M}(\Sigma)$ is given by the length of the corresponding orbit.

So the following theorem, the larger part of which was announced in [8], is a direct consequence of Propositions 5 and 8, and Lemma 6. (As before, we write r for the fibre index of the unique Seifert fibration $M \to \Sigma$; the genus of Σ is denoted by g.)

Theorem 11. *The moduli space $\mathcal{M}(M)$ of taut contact circles on M is a branched covering over the moduli space $\mathcal{M}(\Sigma)$ of hyperbolic metrics on Σ.*

For $g = 0$, the covering map $\mathcal{M}(M) \to \mathcal{M}(\Sigma)$ is a homeomorphism.

For $g = 1$, the number of connected components of $\mathcal{M}(M)$ equals the number of divisors of r. The number of sheets in the component of $\mathcal{M}(M)$ corresponding to $d \mid r$ equals the number of ordered pairs (s,t) of integers mod r that generate the same ideal in $\mathbb{Z}_r$ as d.

For $g \geq 2$ and r odd, $\mathcal{M}(M)$ is connected, and the branched covering $\mathcal{M}(M) \to \mathcal{M}(\Sigma)$ has r^{2g} sheets.

For $g \geq 2$ and r even, $\mathcal{M}(M)$ has two connected components, and the number of sheets in the two components equals $r^{2g}(2^g \pm 1)/2^{g+1}$.

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