GENERALIZATION OF A BINOMIAL IDENTITY OF SIMONS

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Abstract

We give a generalization of a binomial identity due to S. Simons using Cauchy's integral formula.

In [2] Simons proved a binomial identity which can be equivalently written as

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(1+x)^k . \tag{1}$$

In [1] and [3] this identity is proved in different short ways. In this note, using Cauchy's integral formula as in [3], we give a generalization of (1). Consider the term

$$f_n = [t^n]\frac{(1+yt)^\alpha}{(1-xt)^\beta} = \sum_{k=0}^{n} \binom{\alpha}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right) x^k y^{n-k}$$

where $\left(\!\binom{\beta}{k}\!\right) = \beta(\beta+1)\cdots(\beta+k-1)/k!$ and α, β, x, y are indeterminates. By Cauchy's integral formula we have

$$f_n = [t^n]\frac{(1+yt)^\alpha}{(1-xt)^\beta} = \frac{1}{2\pi\mathrm{i}}\oint \frac{(1+yz)^\alpha}{(1-xz)^\beta}\frac{\mathrm{d}z}{z^{n+1}} .$$

With the substitution $z = w/(1-sw)$, where s is an indeterminate, we have $\mathrm{d}z = \mathrm{d}w/(1-sw)^2$ and

$$\begin{aligned} f_n &= \frac{1}{2\pi\mathrm{i}}\oint \frac{(1+(y-s)w)^\alpha}{(1-(x+s)w)^\beta}\,(1-sw)^{\beta-\alpha+n-1}\frac{\mathrm{d}w}{w^{n+1}} \\ &= [t^n]\frac{(1+(y-s)t)^\alpha}{(1-(x+s)t)^\beta}(1-st)^{\beta-\alpha+n-1} . \end{aligned}$$

We now distinguish some cases. First let $s=y$. Then

$$f_n=[t^n]\frac{(1-yt)^{\beta-\alpha+n-1}}{(1-(x+y)t)^{\beta}}=\sum_{k=0}^{n}\binom{\beta-\alpha+n-1}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right)(-1)^{n-k}(x+y)^k y^{n-k}\,.$$

Hence we have the identity

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right)x^k y^{n-k}=\sum_{k=0}^{n}\binom{\beta-\alpha+n-1}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right)(-1)^{n-k}(x+y)^k y^{n-k}\,. \qquad (2)$$

Substituting β with $\beta+1$ identity (2) becomes

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k}=\sum_{k=0}^{n}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-1)^{n-k}(x+y)^k y^{n-k}\,. \qquad (3)$$

In particular for $\alpha=\beta$ identity (3) becomes

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\alpha+k}{k}x^k y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k}{k}(-1)^{n-k}(x+y)^k y^{n-k}\,. \qquad (4)$$

In particular, when $\alpha=n$ we have Simons's identity (1).

Suppose now that $s=-x$. Then

$$f_n=[t^n](1+(x+y)t)^{\alpha}(1+xt)^{\beta-\alpha+n-1}=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta-\alpha+n-1}{n-k}(x+y)^k x^{n-k}$$

and hence

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\left(\!\!\binom{\beta}{k}\!\!\right)x^k y^{n-k}=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta-\alpha+n-1}{n-k}(x+y)^k x^{n-k}\,. \qquad (5)$$

Substituting β with $\beta+1$, we have

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k}=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta-\alpha+n}{n-k}(x+y)^k x^{n-k}\,. \qquad (6)$$

Finally for $\alpha=\beta$ we get

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\alpha+k}{k}x^k y^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha}{k}(x+y)^k x^{n-k}\,. \qquad (7)$$

Let now $y=2s$ and $\beta=2\alpha-n+1$. Then

$$f_n=[t^n]\frac{(1-s^2t^2)^{\alpha}}{(1-(x+s)t)^{2\alpha-n+1}}=\sum_{k\geq 0}\binom{\alpha}{k}\binom{2\alpha-2k}{n-2k}(-1)^k s^{2k}(x+s)^{n-2k}\,.$$

Hence

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{2\alpha-k}{n-k}(2s)^k x^{n-k} = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{k}\binom{2\alpha-2k}{n-2k}(-1)^k s^{2k}(x+s)^{n-2k}\,. \tag{8}$$

Similarly, for $x=-2s$ and $\alpha=2\beta+n-1$, after the substitution of β with $\beta+1$, we have

$$f_n = [t^n]\frac{(1+(y-s)t^2)^{2\beta+n+1}}{(1-s^2t^2)^{\beta+1}} = \sum_{k\ge 0}\binom{2\beta+n+1}{n-2k}\binom{\beta+k}{k}s^{2k}(y-s)^{n-2k}$$

and thus

$$\sum_{k=0}^{n}\binom{2\beta+n+1}{n-k}\binom{\beta+k}{k}(-2s)^k y^{n-k} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{2\beta+n+1}{n-2k}\binom{\beta+k}{k}s^{2k}(y-s)^{n-2k}\,. \tag{9}$$

References

[1] R. Chapman, *A curious identity revised*, The Mathematical Gazette **87** (2003), 139–141.

[2] S. Simons, *A curious identity*, The Mathematical Gazette **85** (2001), 296–298.

[3] H. Prodinger, *A curious identity proved by Cauchy's integral formula*, The Mathematical Gazette, to appear.