Generalization of Different Type Integral Inequalities for $(\alpha, m)-$Convex Functions via Fractional Integrals

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Abstract

In this paper, a general integral identity for a twice differentiable functions is derived. By using of this identity, the author establishes some new Hermite-Hadamard type and Simpson type inequalities for differentiable $(\alpha, m)-$convex functions via Riemann Liouville fractional integral.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined on the interval I of real numbers. Then f is called convex if

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0, 1]$. There are many results associated with convex functions in the area of inequalities, but one of those is the classical Hermite Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

$a, b \in I$ with $a < b$. The another important and well known inequality is Simpson's inequality. This inequality is stated as:

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_\infty = \sup\limits_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \left\| f^{(4)} \right\|_\infty (b-a)^4 . \quad (3)$$

In [6], V.G. Mihesan presented the class of (α, m)-convex functions as below:

Definition 1.1 *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f (tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y) \quad (4)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Note that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions respectively: increasing, α-starshaped, starshaped, m-convex, convex, α-convex functions.

Denote by $K_m^\alpha(b)$ the set of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning (α, m)-convex functions see [3, 4, 7, 8, 14].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2 *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\kappa f$ and $J_{b-}^\kappa f$ of oder $\kappa > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\kappa f(x) = \frac{1}{\Gamma(\kappa)} \int_a^x (x - t)^{\kappa-1} f(t)dt, \ x > a$$

and

$$J_{b-}^\kappa f(x) = \frac{1}{\Gamma(\kappa)} \int_x^b (t - x)^{\kappa-1} f(t)dt, \ x < b$$

respectively, where $\Gamma(\kappa)$ is the Gamma function defined by $\Gamma(\kappa) = \int_0^\infty e^{-t}t^{\kappa-1}dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\kappa = 1$, the fractional integral reduces to the classical integral. Recently, many authors have studied a number of inequalities by used the Riemann Liouville fractional integrals, see [5, 10, 11, 12, 13] and the references cited therein.

In [9], Sarikaya et.al established some new inequalities of the Simpson and the Hermite–Hadamard type for functions whose absolute values of derivatives are convex:

Theorem 1.3 *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \to \mathbb{R}$ be a twice differentiable mapping on I° such that f'' is integrable and $0 \le \lambda \le 1$. If $|f''|^q$ is a convex mapping on $[a, b]$, $q \ge 1$, then the following inequalities hold:*

$$\left| (1 - \lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a) + f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \tag{5}$$

$$\le \begin{cases} \dfrac{(b-a)^2}{2} \left(\dfrac{\lambda^3}{3} + \dfrac{1-3\lambda}{24}\right)^{1-\frac{1}{q}} \\ \quad \times \left\{ \left(\left[\dfrac{\lambda^4}{6} + \dfrac{3-8\lambda}{3.2^6}\right] |f''(a)|^q + \left[\dfrac{(2-\lambda)\lambda^3}{6} + \dfrac{5-16\lambda}{3.2^6}\right] |f''(b)|^q\right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\left[\dfrac{\lambda^4}{6} + \dfrac{3-8\lambda}{3.2^6}\right] |f''(b)|^q + \left[\dfrac{(1+\lambda)(1-\lambda)^3}{6} + \dfrac{48\lambda-27}{3.2^6}\right] |f''(b)|^q\right)^{\frac{1}{q}}\right\}, \; \text{for } 0 \le \lambda \le \frac{1}{2} \\ \dfrac{(b-a)^2}{2} \left(\dfrac{3\lambda-1}{24}\right)^{1-\frac{1}{q}} \left\{ \left(\dfrac{8\lambda-3}{3.2^6} |f''(a)|^q + \dfrac{16\lambda-5}{3.2^6} |f''(b)|^q\right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\dfrac{8\lambda-3}{3.2^6} |f''(b)|^q + \dfrac{16\lambda-5}{3.2^6} |f''(a)|^q\right)^{\frac{1}{q}}\right\}, \; \text{for } \frac{1}{2} \le \lambda \le 1 \end{cases}$$

Let us recall the following special functions:
(1) The Beta function:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x, y > 0, \tag{6}$$

(2) The incomplete Beta function:

$$\beta(a, x, y) = \int_0^a t^{x-1}(1-t)^{y-1} dt, \quad 0 < a < 1, \; x, y > 0, \tag{7}$$

(3) The hypergeometric function:

$$_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \; c > b > 0, \; |z| < 1 \text{ (see [1])}. \tag{8}$$

The main aim of this article is to establish new generalization of Hermite Hadamard-type and Simpson-type inequalities for functions whose second derivatives in absolutely

value at certain powers are $(\alpha, m)-$convex. To begin with, the author derives a general integral identity for twice differentiable mappings. By using this integral equality, the author establishes some new inequalities of the Simpson-like and the Hermite-Hadamard-like type for these functions.

2   Main Results

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$
\begin{aligned}
& I_f\left(x, \lambda, \kappa; a, mb\right) \\
= \; & (1-\lambda)\left[\frac{(x-a)^\kappa + (mb-x)^\kappa}{mb-a}\right] f(x) + \lambda\left[\frac{(x-a)^\kappa f(a) + (mb-x)^\kappa f(mb)}{mb-a}\right] \\
& + \left(\frac{1}{\kappa+1} - \lambda\right)\left[\frac{(mb-x)^{\kappa+1} - (x-a)^{\kappa+1}}{mb-a}\right] f'(x) - \frac{\Gamma\left(\alpha+1\right)}{mb-a}\left[J_{x^-}^\kappa f(a) + J_{x^+}^\kappa f(mb)\right]
\end{aligned}
$$

where $a, b \in I$ and $m \in (0,1]$ with $a < mb$, $x \in [a, mb]$, $\lambda \in [0,1]$, $\kappa > 0$ and Γ is Euler Gamma function. In order to prove our main results we need the following identity.

Lemma 2.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a, mb]$, $m \in (0,1]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:*

$$
\begin{aligned}
& I_f\left(x, \lambda, \kappa; a, mb\right) \tag{9} \\
= \; & \frac{(x-a)^{\kappa+2}}{(\kappa+1)(mb-a)} \int_0^1 t\left((\kappa+1)\lambda - t^\kappa\right) f''\left(tx + (1-t)a\right) dt \\
& + \frac{(mb-x)^{\kappa+2}}{(\kappa+1)(mb-a)} \int_0^1 t\left((\kappa+1)\lambda - t^\kappa\right) f''\left(tx + m(1-t)b\right) dt.
\end{aligned}
$$

A simple proof of the equality can be done by performing an integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader.

Theorem 2.2 *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a/m, b \in I^\circ$ with $a < mb$. If $\left|f''\right|^q$ is $(\alpha, m)-$convex on $[a,b]$ for some fixed $\alpha \in [0,1]$, $m \in (0,1]$ and $q \geq 1$, then for $x \in [a, mb]$, $\lambda \in [0,1]$ and $\kappa > 0$ the following inequality for fractional integrals holds*

$$
\left|I_f\left(x, \lambda, \kappa; a, mb\right)\right| \tag{10}
$$

$$\leq \; \varphi_1^{1-\frac{1}{q}}(\kappa,\lambda) \left\{ \frac{(x-a)^{\kappa+2}}{(\kappa+1)(mb-a)} \left(|f''(x)|^q \, \varphi_2(\kappa,\lambda,\alpha) + m \left| f''\left(\frac{a}{m}\right) \right|^q \varphi_3(\kappa,\lambda,\alpha) \right)^{\frac{1}{q}} \right. $$

$$\left. + \frac{(mb-x)^{\kappa+2}}{(\kappa+1)(mb-a)} \left(|f''(x)|^q \, \varphi_2(\kappa,\lambda,\alpha) + m|f''(b)|^q \, \varphi_3(\kappa,\lambda,\alpha) \right)^{\frac{1}{q}} \right\}, $$

where

$$\varphi_1(\kappa,\lambda) = \begin{cases} \frac{\kappa[(\kappa+1)\lambda]^{\frac{\kappa+2}{\kappa}}}{\kappa+2} - \frac{(\kappa+1)\lambda}{2} + \frac{1}{\kappa+2}, & 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ \frac{(\kappa+1)\lambda}{2} - \frac{1}{\kappa+2}, & \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases},$$

$$\varphi_2(\kappa,\lambda,\alpha) = \begin{cases} \frac{2\kappa[(\kappa+1)\lambda]^{\frac{\kappa+\alpha+2}{\kappa}}}{(\alpha+2)(\kappa+\alpha+2)} - \frac{(\kappa+1)\lambda}{\alpha+2} + \frac{1}{\kappa+\alpha+2}, & 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ \frac{(\kappa+1)\lambda}{\alpha+2} - \frac{1}{\kappa+\alpha+2}, & \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases},$$

$$\varphi_3(\kappa,\lambda,\alpha) = \begin{cases} \frac{\kappa[(\kappa+1)\lambda]^{\frac{\kappa+2}{\kappa}}}{(\kappa+2)} - \frac{2\kappa[(\kappa+1)\lambda]^{\frac{\kappa+\alpha+2}{\kappa}}}{(\alpha+2)(\kappa+\alpha+2)} - \frac{\alpha(\kappa+1)\lambda}{2(\alpha+2)} + \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)}, & 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ \frac{\alpha(\kappa+1)\lambda}{2(\alpha+2)} - \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)}, & \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases}.$$

Proof. From Lemma 2.1, using the property of the modulus and the power-mean inequality we have

$$|I_f(x,\lambda,\kappa,a,mb)| \leq \frac{(x-a)^{\kappa+2}}{(\kappa+1)(mb-a)} \int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,|f''(tx+(1-t)a)|\,dt$$

$$+ \frac{(mb-x)^{\kappa+2}}{(\kappa+1)(mb-a)} \int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,|f''(tx+m(1-t)b)|\,dt$$

$$\leq \; \frac{(x-a)^{\kappa+2}}{(\kappa+1)(mb-a)} \left(\int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,|f''(tx+(1-t)a)|^q\,dt \right)^{\frac{1}{q}}$$

$$+ \frac{(mb-x)^{\kappa+2}}{(\kappa+1)(mb-a)} \left(\int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 t\,|(\kappa+1)\lambda - t^\kappa|\,|f''(tx+(1-t)mb)|^q\,dt \right)^{\frac{1}{q}}. \tag{11}$$

Since $|f''|^q$ is $(\alpha, m)-$convex on $[a, b]$ we get

$$\int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\,|f''(tx+(1-t)\,a)|^q\,dt$$

$$\leq \int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\left(t^\alpha\,|f''(x)|^q + m\,(1-t\alpha)\left|f''\left(\frac{a}{m}\right)\right|^q\right)dt$$

$$= |f''(x)|^q\,\varphi_2(\kappa, \lambda, \alpha) + m\left|f''\left(\frac{a}{m}\right)\right|^q\varphi_3(\kappa, \lambda, \alpha), \tag{12}$$

$$\int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\,|f''(tx+m(1-t)\,b)|^q\,dt$$

$$\leq \int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\left(t^\alpha\,|f''(x)|^q + m\,(1-t^\alpha)\,|f''(b)|^q\right)dt$$

$$= |f''(x)|^q\,\varphi_2(\kappa, \lambda, \alpha) + m\,|f''(b)|^q\,\varphi_3(\kappa, \lambda, \alpha), \tag{13}$$

where we use the fact that

$$\varphi_3(\kappa, \lambda, \alpha)$$

$$= \int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\,(1 - t^\alpha)\,dt$$

$$= \begin{cases} (\kappa+1)\,\lambda\displaystyle\int_0^{[(\kappa+1)\lambda]^{\frac{1}{\kappa}}} t\,(1-t^\alpha)\,dt - \displaystyle\int_0^{[(\kappa+1)\lambda]^{\frac{1}{\kappa}}} t^{\kappa+1}\,(1-t^\alpha)\,dt \\ \quad - (\kappa+1)\,\lambda\displaystyle\int_{[(\kappa+1)\lambda]^{\frac{1}{\kappa}}}^1 t\,(1-t^\alpha)\,dt + \displaystyle\int_{[(\kappa+1)\lambda]^{\frac{1}{\kappa}}}^1 t^{\kappa+1}\,(1-t^\alpha)\,dt \\ \hspace{6cm}, \quad 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ (\kappa+1)\,\lambda\displaystyle\int_0^1 t\,(1-t^\alpha)\,dt - \displaystyle\int_0^1 t^{\kappa+1}\,(1-t^\alpha)\,dt, \quad \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases}$$

$$= \begin{cases} \frac{\kappa[(\kappa+1)\lambda]^{\frac{\kappa+2}{\kappa}}}{(\kappa+2)} - \frac{2\kappa[(\kappa+1)\lambda]^{\frac{\kappa+\alpha+2}{\kappa}}}{(\alpha+2)(\kappa+\alpha+2)} - \frac{\alpha(\kappa+1)\lambda}{2(\alpha+2)} + \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)}, & 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ \frac{\alpha(\kappa+1)\lambda}{2(\alpha+2)} - \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)}, & \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases},$$

$$\varphi_2(\kappa, \lambda, \alpha)$$

$$= \int_0^1 t\,|(\kappa+1)\,\lambda - t^\kappa|\,t^\alpha dt$$

$$= \begin{cases} \frac{2\kappa[(\kappa+1)\lambda]^{\frac{\kappa+\alpha+2}{\kappa}}}{(\alpha+2)(\kappa+\alpha+2)} - \frac{(\kappa+1)\lambda}{\alpha+2} + \frac{1}{\kappa+\alpha+2}, & 0 \leq \lambda \leq \frac{1}{\kappa+1} \\ \frac{(\kappa+1)\lambda}{\alpha+2} - \frac{1}{\kappa+\alpha+2}, & \frac{1}{\kappa+1} < \lambda \leq 1 \end{cases}$$

and

$$
\varphi_1(\kappa, \lambda) = \int_0^1 t\left|(\kappa+1)\lambda - t^\kappa\right| dt
$$

$$
= \begin{cases} \dfrac{\kappa[(\kappa+1)\lambda]^{1+\frac{2}{\kappa}}}{\kappa+2} - \dfrac{(\kappa+1)\lambda}{2} + \dfrac{1}{\kappa+2}, & 0 \le \lambda \le \dfrac{1}{\kappa+1} \\[2mm] \dfrac{(\kappa+1)\lambda}{2} - \dfrac{1}{\kappa+2}, & \dfrac{1}{\kappa+1} < \lambda \le 1 \end{cases} \tag{14}
$$

Hence, If we use (12), (13) and (14) in (11), we obtain the desired result. This completes the proof.

Corollary 2.3 *In Theorem 2.2,*

(a) If we choose $q = 1$, then we get:

$$
\begin{aligned}
|I_f(x, \lambda, \kappa; a, mb)| \le \; & \left\{ \frac{(x-a)^{\kappa+1}}{mb-a}\left(|f''(x)|\,\varphi_2(\kappa, \lambda, \alpha) + m\left|f''\left(\frac{a}{m}\right)\right|\varphi_3(\kappa, \lambda, \alpha)\right) \right. \\
& \left. + \frac{(mb-x)^{\kappa+1}}{mb-a}\left(|f''(x)|\,\varphi_2(\kappa, \lambda, \alpha) + m\,|f''(b)|\,\varphi_3(\kappa, \lambda, \alpha)\right) \right\}.
\end{aligned}
$$

(b) If we choose $x = \frac{a+mb}{2}$, then we get:

$$
\begin{aligned}
& \left| \frac{2^{\kappa-1}}{(mb-a)^{\kappa-1}} I_f\left(\frac{a+mb}{2}, \lambda, \kappa; a, mb\right) \right| \\
= \; & \left| (1-\lambda)f\left(\frac{a+mb}{2}\right) + \lambda\left(\frac{f(a)+f(mb)}{2}\right) \right. \\
& \left. - \frac{\Gamma(\kappa+1)\,2^{\kappa-1}}{(mb-a)^\kappa}\left[J^\kappa_{\left(\frac{a+mb}{2}\right)^-}f(a) + J^\kappa_{\left(\frac{a+mb}{2}\right)^+}f(mb) \right] \right|
\end{aligned}
$$

$$
\begin{aligned}
\le \; & \frac{(mb-a)^2}{8(\kappa+1)}\varphi_1^{\,1-\frac{1}{q}}(\kappa, \lambda)\left\{ \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2(\kappa, \lambda, \alpha) + m\left|f''\left(\frac{a}{m}\right)\right|^q \varphi_3(\kappa, \lambda, \alpha) \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2(\kappa, \lambda, \alpha) + m\,|f''(b)|^q \varphi_3(\kappa, \lambda, \alpha) \right)^{\frac{1}{q}} \right\}. \tag{15}
\end{aligned}
$$

(c) If we choose $x = \frac{a+mb}{2}$ and $\lambda = \frac{1}{3}$, then we get:

$$\left| \frac{2^{\kappa-1}}{(mb-a)^{\kappa-1}} I_f\left(\frac{a+mb}{2}, \frac{1}{3}, \kappa; a, mb\right) \right|$$

$$= \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb)\right] - \frac{\Gamma(\kappa+1)2^{\kappa-1}}{(mb-a)^{\kappa}}\left[J^{\kappa}_{\left(\frac{a+b}{2}\right)^-}f(a) + J^{\kappa}_{\left(\frac{a+b}{2}\right)^+}f(mb)\right] \right|$$

$$\leq \frac{(mb-a)^2}{8(\kappa+1)} \varphi_1^{1-\frac{1}{q}}\left(\kappa, \frac{1}{3}\right) \left\{ \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2\left(\kappa, \frac{1}{3}, \alpha\right) + m\left|f''\left(\frac{a}{m}\right)\right|^q \varphi_3\left(\kappa, \frac{1}{3}, \alpha\right)\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2\left(\kappa, \frac{1}{3}, \alpha\right) + m|f''(b)|^q \varphi_3\left(\kappa, \frac{1}{3}, \alpha\right)\right)^{\frac{1}{q}} \right\}.$$

(d) If we choose $x = \frac{a+mb}{2}$, $\lambda = \frac{1}{3}$, and $\kappa = 1$, then we get:

$$\left| I_f\left(\frac{a+mb}{2}, \frac{1}{3}, 1, a, mb\right) \right|$$

$$= \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+mb}{2}\right) + f(mb)\right] - \frac{1}{(mb-a)}\int_a^b f(x)dx \right|$$

$$\leq \frac{(mb-a)^2}{162}\left(\frac{81}{8}\right)^{\frac{1}{q}} \left\{ \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2\left(1, \frac{1}{3}, \alpha\right) + m\left|f''\left(\frac{a}{m}\right)\right|^q \varphi_3\left(1, \frac{1}{3}, \alpha\right)\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\left|f''\left(\frac{a+mb}{2}\right)\right|^q \varphi_2\left(1, \frac{1}{3}, \alpha\right) + m|f''(b)|^q \varphi_3\left(1, \frac{1}{3}, \alpha\right)\right)^{\frac{1}{q}} \right\}.$$

where

$$\varphi_2\left(1, \frac{1}{3}, \alpha\right) = \frac{2^{\alpha+4} - 23^{\alpha+2} + 3^{\alpha+3}(\alpha+2)}{3^{\alpha+3}(\alpha+2)(\alpha+3)}, \tag{16}$$

$$\varphi_3\left(1, \frac{1}{3}, \alpha\right) = \frac{-2^{\alpha+4} - \alpha 3^{\alpha+2}(\alpha+3) + 3^{\alpha+3}(\alpha+2) + 83^{\alpha-1}(\alpha+2)(\alpha+3)}{3^{\alpha+3}(\alpha+2)(\alpha+3)}. \tag{17}$$

(e) If we choose $x = \frac{a+mb}{2}$ and $\lambda = 0$, then we get:

$$\left| \frac{2^{\kappa-1}}{(mb-a)^{\kappa-1}} I_f\left(\frac{a+mb}{2}, 0, \kappa; a, mb\right) \right|$$

$$= \left| f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\kappa+1)2^{\kappa-1}}{(mb-a)^{\kappa}}\left[J^{\kappa}_{\left(\frac{a+mb}{2}\right)^-}f(a) + J^{\kappa}_{\left(\frac{a+mb}{2}\right)^+}f(b)\right] \right|$$

$$\leq \frac{(mb-a)^2}{8(\kappa+1)(\alpha\kappa+2)}\left(\frac{\kappa+2}{\kappa+s+2}\right)^{\frac{1}{q}} \left\{ \left[\left|f''\left(\frac{a+mb}{2}\right)\right|^q + \frac{\kappa m\left|f''\left(\frac{a}{m}\right)\right|^q}{(\kappa+2)}\right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\left|f''\left(\frac{a+mb}{2}\right)\right|^q + \frac{\kappa m|f''(b)|^q}{(\kappa+2)}\right]^{\frac{1}{q}} \right\}.$$

(f) If we choose $x = \frac{a+mb}{2}$, $\lambda = 0$, and $\kappa = 1$, then we get:

$$
\left| I_f \left(\frac{a+mb}{2}, 0, 1; a, mb \right) \right|
$$

$$
= \left| f \left(\frac{a+mb}{2} \right) - \frac{1}{(mb-a)} \int_a^{mb} f(x)dx \right|
$$

$$
\leq \frac{(mb-a)^2}{48} \left(\frac{1}{s+3} \right)^{\frac{1}{q}} \left\{ \left[3 \left| f'' \left(\frac{a+mb}{2} \right) \right|^q + m \left| f'' \left(\frac{a}{m} \right) \right|^q \right]^{\frac{1}{q}} \right.
$$

$$
\left. + \left[3 \left| f'' \left(\frac{a+mb}{2} \right) \right|^q + m \left| f''(b) \right|^q \right]^{\frac{1}{q}} \right\}.
$$

(g) If we choose $x = \frac{a+mb}{2}$ and $\lambda = 1$, then we get:

$$
\left| \frac{2^{\kappa-1}}{(mb-a)^{\kappa-1}} I_f \left(\frac{a+mb}{2}, 1, \kappa; a, mb \right) \right|
$$

$$
= \left| \frac{f(a)+f(mb)}{2} - \frac{\Gamma(\kappa+1) 2^{\kappa-1}}{(mb-a)^\kappa} \left[J^\kappa_{\left(\frac{a+mb}{2} \right)^-} f(a) + J^\kappa_{\left(\frac{a+mb}{2} \right)^+} f(mb) \right] \right|
$$

$$
\leq \frac{(mb-a)^2}{8(\kappa+1)} \left(\frac{\kappa(\kappa+3)}{2(\kappa+2)} \right)^{1-\frac{1}{q}}
$$

$$
\times \left\{ \left[\frac{\kappa(\kappa+\alpha+3) \left| f'' \left(\frac{a+mb}{2} \right) \right|^q}{(\alpha+2)(\kappa+\alpha+2)} + m \left| f'' \left(\frac{a}{m} \right) \right|^q \left(\frac{\alpha(\kappa+1)}{2(\alpha+2)} - \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)} \right) \right]^{\frac{1}{q}} \right.
$$

$$
\left. + \left[\frac{\kappa(\kappa+\alpha+3) \left| f'' \left(\frac{a+mb}{2} \right) \right|^q}{(\alpha+2)(\kappa+\alpha+2)} + m \left| f''(b) \right|^q \left(\frac{\alpha(\kappa+1)}{2(\alpha+2)} - \frac{\kappa}{(\kappa+2)(\kappa+\alpha+2)} \right) \right]^{\frac{1}{q}} \right\}.
$$

(h) If we choose $x = \frac{a+mb}{2}, \lambda = 1$ and $\kappa = 1$, then we get:

$$\left| I_f \left(\frac{a+b}{2}, 1, 1; a, mb \right) \right|$$

$$= \left| \frac{f(a) + f(mb)}{2} - \frac{1}{(mb-a)} \int_a^{mb} f(x)dx \right|$$

$$\leq \frac{(mb-a)^2}{16} \left(\frac{2}{3} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left[\frac{(\alpha+4) \left| f'' \left(\frac{a+mb}{2} \right) \right|^q}{(\alpha+2)(\alpha+3)} + m \left| f'' \left(\frac{a}{m} \right) \right|^q \frac{3\alpha^2 + 8\alpha - 2}{3(\alpha+2)(\alpha+3)} \right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\frac{(\alpha+4) \left| f'' \left(\frac{a+mb}{2} \right) \right|^q}{(\alpha+2)(\alpha+3)} + m \left| f''(b) \right|^q \frac{3\alpha^2 + 8\alpha - 2}{3(\alpha+2)(\alpha+3)} \right]^{\frac{1}{q}} \right\}.$$

Remark 2.4 In (b) of Corollary 2.3, if we choose $\kappa = m = \alpha = 1$, then the inequality (15) reduces to the following inequality which is better than the inequality (5)

$$\left| (1-\lambda) f \left(\frac{a+b}{2} \right) + \lambda \left(\frac{f(a) + f(b)}{2} \right) - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \qquad (18)$$

$$\leq \frac{(b-a)^2}{16} \varphi_1^{1-\frac{1}{q}}(1,\lambda) \left\{ \left(\left| f'' \left(\frac{a+b}{2} \right) \right|^q \varphi_2(1,\lambda,1) + |f''(a)|^q \varphi_3(1,\lambda,1) \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\left| f'' \left(\frac{a+b}{2} \right) \right|^q \varphi_2(1,\lambda,1) + |f''(b)|^q \varphi_3(1,\lambda,1) \right)^{\frac{1}{q}} \right\},$$

where

$$\varphi_1(1,\lambda) = \begin{cases} 8 \left(\frac{\lambda^3}{3} + \frac{1-3\lambda}{24} \right), & 0 \leq \lambda \leq \frac{1}{2} \\ \frac{3\lambda-1}{3}, & \frac{1}{2} < \lambda \leq 1 \end{cases}, \qquad (19)$$

$$\varphi_2(1,\lambda,1) = \begin{cases} 16 \left(\frac{\lambda^4}{6} + \frac{3-8\lambda}{3.2^6} \right), & 0 \leq \lambda \leq \frac{1}{2} \\ \frac{8\lambda-3}{12}, & \frac{1}{2} < \lambda \leq 1 \end{cases}, \qquad (20)$$

$$\varphi_3(1,\lambda,1) = \begin{cases} \frac{-8\lambda^4 + 8\lambda^3 - \lambda}{3} + \frac{1}{12}, & 0 \leq \lambda \leq \frac{1}{2} \\ \frac{4\lambda-1}{12}, & \frac{1}{2} < \lambda \leq 1 \end{cases}. \qquad (21)$$

Theorem 2.5 Let $f : I \subset [0, \infty) \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a/m, b \in I^\circ$ with $a < mb$. If $|f''|^q$ is $(\alpha, m)-$convex on $[a,b]$ for some

fixed $\alpha \in [0, 1]$, $m \in (0, 1]$ *and* $q > 1$, *then for* $x \in [a, mb]$, $\lambda \in [0, 1]$ *and* $\kappa > 0$ *the following inequality for fractional integrals holds*

$$|I_f (x, \lambda, \kappa; a, mb)| \tag{22}$$

$$\leq \; \varphi_4^{\frac{1}{p}} (\kappa, \lambda, p) \left\{ \frac{(x - a)^{\kappa+2}}{(\kappa + 1) (mb - a)} \left(\frac{|f'' (x)|^q + \alpha m \left| f'' \left(\frac{a}{m} \right) \right|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \frac{(mb - x)^{\kappa+2}}{(\kappa + 1) (mb - a)} \left(\frac{|f'' (x)|^q + \alpha m |f'' (b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\},$$

where $p = \frac{q}{q-1}$ *and*

$$\varphi_4 (\kappa, \lambda, p)$$

$$= \begin{cases} \frac{1}{p(\kappa+1)+1}, & \lambda = 0 \\[2ex] \left[\begin{array}{l} \frac{[(\kappa+1)\lambda]^{\frac{1+(\kappa+1)p}{\kappa}}}{\kappa} \beta \left(\frac{1+p}{\kappa}, 1 + p \right) \\[1ex] + \frac{[1-(\kappa+1)\lambda]^{p+1}}{p+1} \cdot {}_2F_1 \left(1 - \frac{1+p}{\kappa}, 1; p + 2; 1 - (\kappa + 1) \lambda \right) \end{array} \right], & 0 < \lambda < \frac{1}{\kappa+1} \\[3ex] \frac{[(\kappa+1)\lambda]^{\frac{p(\kappa+1)+1}{\kappa}}}{\kappa} \beta \left(\frac{1}{(\kappa+1)\lambda}; \frac{1+p}{\kappa}, 1 + p \right), & \frac{1}{\kappa+1} \leq \lambda \leq 1 \end{cases} .$$

Proof. *Proof.* From Lemma 2.1, property of the modulus and using the Hölder inequality we have

$$|I_f (x, \lambda, \kappa; a, mb)| \leq \frac{(x - a)^{\kappa+2}}{(\kappa + 1) (mb - a)} \int_0^1 t \left| (\kappa + 1) \lambda - t^\kappa \right| |f'' (tx + (1 - t) a)| \, dt$$

$$+ \frac{(mb - x)^{\kappa+2}}{(\kappa + 1) (mb - a)} \int_0^1 t \left| (\kappa + 1) \lambda - t^\kappa \right| |f'' (tx + m (1 - t) b)| \, dt$$

$$\leq \; \frac{(x - a)^{\kappa+2}}{(\kappa + 1) (mb - a)} \left(\int_0^1 t^p \left| (\kappa + 1) \lambda - t^\kappa \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'' (tx + (1 - t) a)|^q dt \right)^{\frac{1}{q}}$$

$$+ \frac{(mb - x)^{\kappa+2}}{(\kappa + 1) (mb - a)} \left(\int_0^1 t^p \left| (\kappa + 1) \lambda - t^\kappa \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'' (tx + m (1 - t) b)|^q dt \right)^{\frac{1}{q}} \tag{23}$$

Since $|f''|^q$ is $(\alpha, m) -$convex on $[a, b]$ we get

$$\int_0^1 |f'' (tx + (1 - t) a)|^q dt \; \leq \; \int_0^1 \left(t^\alpha |f'' (x)|^q + m (1 - t^\alpha) \left| f'' \left(\frac{a}{m} \right) \right|^q \right) dt$$

$$= \; \frac{|f'' (x)|^q + m\alpha \left| f'' \left(\frac{a}{m} \right) \right|^q}{\alpha + 1}, \tag{24}$$

$$\int_0^1 |f''(tx + m(1-t)b)|^q \, dt \;\; \leq \;\; \int_0^1 \left(t^\alpha |f''(x)|^q + m(1-t^\alpha)|f''(b)|^q \right) dt$$

$$= \;\; \frac{|f''(x)|^q + m\alpha |f''(b)|^q}{\alpha + 1}, \qquad\qquad (25)$$

and

$$\int_0^1 t^p \left| (\alpha+1)\lambda - t^\kappa \right|^p dt \qquad\qquad (26)$$

$$= \begin{cases} \int_0^1 t^{(\kappa+1)p} dt & \lambda = 0 \\[2mm] \int_0^{[(\kappa+1)\lambda]^{\frac{1}{\alpha}}} t^p \left[(\kappa+1)\lambda - t^\kappa \right]^p dt + \int_{[(\kappa+1)\lambda]^{\frac{1}{\alpha}}}^1 t^p \left[t^\kappa - (\kappa+1)\lambda \right]^p dt, & 0 < \lambda < \frac{1}{\kappa+1} \\[2mm] \int_0^1 t^p \left[(\kappa+1)\lambda - t^\kappa \right]^p dt, & \frac{1}{\kappa+1} \leq \lambda \leq 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{p(\kappa+1)+1}, & \lambda = 0 \\[2mm] \left[\begin{array}{l} \dfrac{[(\kappa+1)\lambda]^{\frac{1+(\kappa+1)p}{\kappa}}}{\kappa} \beta\left(\frac{1+p}{\kappa}, 1+p\right) \\[2mm] + \dfrac{[1-(\kappa+1)\lambda]^{p+1}}{p+1} \cdot {}_2F_1\left(1 - \frac{1+p}{\kappa}, 1; p+2; 1-(\kappa+1)\lambda\right) \end{array} \right], & 0 < \lambda < \frac{1}{\kappa+1} \\[2mm] \dfrac{[(\kappa+1)\lambda]^{\frac{1+(\kappa+1)p}{\kappa}}}{\kappa} \beta\left(\frac{1}{(\kappa+1)\lambda}; \frac{1+p}{\kappa}, 1+p\right), & \frac{1}{\kappa+1} \leq \lambda \leq 1 \end{cases}$$

Hence, If we use (24), (25) and (26) in (23), we obtain the desired result. This completes the proof. ∎

Corollary 2.6 *In Theorem 2.5*
(a) If we choose $x = \frac{a+mb}{2}$, then we get:

$$\left| (1-\lambda)f\left(\frac{a+mb}{2}\right) + \lambda\left(\frac{f(a)+f(mb)}{2}\right) - \frac{\Gamma(\kappa+1)2^{\kappa-1}}{(mb-a)^\kappa} \left[J^\kappa_{\left(\frac{a+mb}{2}\right)^-} f(a) + J^\kappa_{\left(\frac{a+mb}{2}\right)^+} f(mb) \right] \right|$$

$$\leq \;\; \varphi_4^{\frac{1}{p}}(\kappa,\lambda,p) \frac{(mb-a)^2}{8(\kappa+1)} \left\{ \left(\frac{\left|f''\left(\frac{a+mb}{2}\right)\right|^q + \alpha m \left|f''\left(\frac{a}{m}\right)\right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right.$$

$$+ \left. \left(\frac{\left|f''\left(\frac{a+mb}{2}\right)\right|^q + \alpha m |f''(b)|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}$$

(b) If we choose $x = \frac{a+mb}{2}$, $\lambda = \frac{1}{3}$, then we get:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+mb}{2} \right) + f(mb) \right] - \frac{\Gamma(\kappa+1) 2^{\kappa-1}}{(mb-a)^\kappa} \left[J^\kappa_{\left(\frac{a+mb}{2} \right)^-} f(a) + J^\kappa_{\left(\frac{a+mb}{2} \right)^+} f(mb) \right] \right|$$

$$\leq \varphi_4^{\frac{1}{p}} \left(\kappa, \frac{1}{3}, p \right) \frac{(mb-a)^2}{8(\kappa+1)} \left\{ \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''\left(\frac{a}{m} \right) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''(b) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}$$

(c) If we choose $x = \frac{a+mb}{2}$, $\lambda = \frac{1}{3}$, and $\kappa = 1$, then we get:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+mb}{2} \right) + f(mb) \right] - \frac{1}{(mb-a)} \int_a^{mb} f(x) dx \right|$$

$$\leq \frac{(mb-a)^2}{16} \varphi_4^{\frac{1}{p}} \left(1, \frac{1}{3}, p \right) \left\{ \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''\left(\frac{a}{m} \right) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''(b) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}$$

where

$$\varphi_4 \left(1, \frac{1}{3}, p \right) = \left(\frac{2}{3} \right)^{1+2p} \beta(1+p, 1+p) + \left(\frac{1}{3} \right)^{1+p} {}_2F_1 \left(-p, 1; p+2; \frac{1}{3} \right).$$

(d) If we choose $x = \frac{a+mb}{2}$ and $\lambda = 0$, then we get:

$$\left| f\left(\frac{a+mb}{2} \right) - \frac{\Gamma(\kappa+1) 2^{\kappa-1}}{(mb-a)^\kappa} \left[J^\kappa_{\left(\frac{a+mb}{2} \right)^-} f(a) + J^\kappa_{\left(\frac{a+mb}{2} \right)^+} f(mb) \right] \right|$$

$$\leq \frac{(mb-a)^2}{16} \left(\frac{1}{p(\kappa+1)+1} \right)^{\frac{1}{p}} \left\{ \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''\left(\frac{a}{m} \right) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\left| f''\left(\frac{a+mb}{2} \right) \right|^q + \alpha m \left| f''(b) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} \right\}$$

(e) If we choose $x = \frac{a+mb}{2}$ and $\lambda = 1$, then we get:

$$\left| \frac{f(a) + f(mb)}{2} - \frac{\Gamma(\kappa+1) 2^{\kappa-1}}{(mb-a)^\kappa} \left[J^\kappa_{\left(\frac{a+mb}{2}\right)^-} f(a) + J^\kappa_{\left(\frac{a+mb}{2}\right)^+} f(mb) \right] \right|$$

$$\leq \frac{(mb-a)^2}{16} \varphi_4^{\frac{1}{p}}(\kappa, 1, p) \left\{ \left(\frac{\left| f''\left(\frac{a+mb}{2}\right) \right|^q + \alpha m \left| f''\left(\frac{a}{m}\right) \right|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\left| f''\left(\frac{a+mb}{2}\right) \right|^q + \alpha m \left| f''(b) \right|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}$$

where

$$\varphi_4(\kappa, 1, p) = \frac{(1+\kappa)^{\frac{p(\kappa+1)+1}{\kappa}}}{\kappa} \beta\left(\frac{1}{1+\kappa}; \frac{1+p}{\kappa}, 1+p \right).$$

(g) If we choose $x = \frac{a+mb}{2}, \lambda = 1$ and $\kappa = 1$, then we get:

$$\left| \frac{f(a) + f(mb)}{2} - \frac{1}{(mb-a)} \int_a^{mb} f(x) dx \right|$$

$$\leq \frac{(mb-a)^2}{4} \left(2\beta\left(\frac{1}{2}; 1+p, 1+p\right) \right)^{\frac{1}{p}} \left\{ \left(\frac{\left| f''\left(\frac{a+mb}{2}\right) \right|^q + \alpha m \left| f''\left(\frac{a}{m}\right) \right|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\frac{\left| f''\left(\frac{a+mb}{2}\right) \right|^q + \alpha m \left| f''(b) \right|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}$$

where

$$2\beta\left(\frac{1}{2}; 1+p, 1+p\right) = \beta(1+p, 1+p).$$

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