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Generalization of Equation of Collective Submanifold

—— *A Theory of Large Amplitude Collective Motion and Its Coupling with Intrinsic Degrees of Freedom* ——

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A theory is proposed with the aim of describing large amplitude collective motion and its coupling with intrinsic degrees of freedom. A canonical transformation is investigated in the full time-dependent Hartree-Fock theory, i.e., in the classical image of boson expansion theory. With the aid of the transformation, the whole system is separated into collective and intrinsic degrees of freedom. Under the condition of the unique separation, two types of equations of collective submanifold, which are canonically invariant, are obtained. One is of the same form as that of the conventional equation of collective submanifold. A principle of the specification of coordinate system is discussed.

§ 1. Introduction

One of the most fundamental problems in nuclear structure theory is to construct theory, with the aid of which not only collective motion but also its coupling with intrinsic degrees of freedom is described. If we restrict ourselves only to collective motion, there exists a theory, which may be called theory of collective submanifold. A main business in this theory is to solve the equation of collective submanifold. One of the present authors (A.K.), together with Marumori, Maskawa and Sakata, proposed a concrete method for solving the equation.[1] In spite of this fact, it has been widely believed that the equation cannot be solved uniquely, for example, as was stressed by Mukherjee and Pal.[2] Recently, the present authors, together with Iida, showed that the discussion concerning the non-uniqueness is based on a certain misunderstanding of the equation of collective submanifold and the solution exists uniquely.[3] In this sense, the theory of collective submanifold is self-contained in the present form and a powerful method for the description of collective motion.

However, if we include intrinsic degrees of freedom in the description, the situation becomes different. In this case, the simplest approach may be the random phase approximation (RPA). The RPA theory gives us the equation of collective submanifold in small amplitude limit. Further, the degrees of freedom orthogonal to the collective one, which we will call generally intrinsic degrees of freedom, are introduced in a natural way. In RPA, fluctuations around the equilibrium point given by the static Hartree-Fock (-Bogoliubov) method are taken into account in terms of linear effects. Therefore, RPA is not applicable to large amplitude collective motion, in which non-linear effects of the fluctuations are expected to be large. With the aim of making the basic idea of RPA applicable to the collective motion with non-linear effects, boson expansion (BE) theory has been developed. The typical example is the Holstein-Primakoff type representation. The BE theory is a kind of quantum canonical theory constructed by boson operators or their equivalent coordinate and momentum operators. As one of theoretical interests, we can see that the c-number replacement of these operators leads us to full time-dependent

Hartree-Bogoliubov or Hartree-Fock (TDHF) theory. This fact was first pointed out by Marshalek and Holzwarth[4] and, on the basis of this fact, one of the present authors (M.Y.) and others made various analysis for the BE theory.[5]

On the other hand, by combining a Slater determinant with a certain kind of relations which the present authors have called canonicity condition, the full TDHF theory can be formulated in terms of canonical variables. An interesting point of this formulation is as follows: If the Poisson brackets of the canonical variables are replaced with the commutators, this classical theory becomes the Holstein-Primakoff type BE theory. This investigation was initiated by the present authors[6] and the process is just the inverse of Marshalek and Holzwarth's idea.[4] The canonicity condition was originally introduced in Ref.1) and it enables us to formulate the full TDHF theory in the canonical form. However, the TDHF theory leading us to the Holstein-Primakoff type BE theory is not suitable for the adiabatic treatment of collective motion, i.e., the ATDHF approach. Recently, the present authors also used a Slater determinant and a canonicity condition, the forms of which are different from the case of the Holstein-Primakoff BE theory.[7] As a result, another form of the full TDHF theory was obtained in the canonical form. This can be expected to be suitable for the ATDHF approach, because the time-reversal property of the original fermion system can be treated properly. However, we have to point out that any form of the full TDHF theory we have given contains a defect: There does not exist any scheme in the theory, with the aid of which we can select collective degrees of freedom. Therefore, as a natural consequence, this theory does not contain also a scheme which determines intrinsic degrees of freedom. For this reason, the BE theory has been regarded as not so powerful method as expected at the beginning.

In this paper, we will plan a revival of the BE theory in the framework of classical mechanics. The main aim is to demonstrate that, under a certain device, the BE theory is powerful for the study of collective motion and its coupling with intrinsic degrees of freedom. With the use of boson-coherent state, a classical image of the BE theory is obtained, i.e., the full TDHF theory in the canonical form. For simplicity, in this paper, we deal with the case in which only one degree of freedom is attributed to collective motion and the others to intrinsic ones. We imagine the following case: The collective motion is of the large amplitude and the fluctuations of the intrinsic motions are small. Under this imagination, we construct a canonical transformation expressed in terms of power series expansion for all degrees of freedom except a certain one. Clearly, the certain one corresponds to the collective degree of freedom. This transformation can separate the whole system into the collective and the intrinsic degrees of freedom. It may be self-evident that the separation should be uniquely performed. This requirement gives us a set of differential equations, the forms of which are completely identical with the conventional equations of collective submanifold.[8] In addition, we obtain another set of differential equations, with the use of which coupling between the collective and the intrinsic degrees of freedom can be described in the first order. This is just a generalization of the conventional equation of collective submanifold. We call the former and the latter as the first and the second equations of collective submanifold. One of the most important properties of these equations is that they are canonically invariant, i.e., the forms are invariant under any canonical transformation for the collective variables. This fact was first recongnized by the present authors in the case of the first equation.[9] Since the collective submanifold does not depend on the choice of its coordinate system, the invariance is quite natural. However, in order to express the Hamiltonian in a concrete form, it is necessary to fix a certain coordinate system. This fact gives us various possibilities for expressing the Hamiltonian in terms of the collective and the intrinsic variables. Then, if we succeed in solving the equation of collective submanifold under a certain coordinate system, the collective motion and its coupling with the intrinsic degrees of freedom can be described in the full TDHF theory, i.e., in the classical image of the BE theory.

In §§ 2 and 3, a possible form of canonical transformation is investigated in terms of power series expansion. In §4, we will obtain the equations of collective submanifold under the condition of unique separation of the degrees of freedom into the collective and the intrinsic ones. Section 5 will be devoted to proving the canonical invariance. Finally, in §6, we will discuss specification of coordinate system with some concluding remarks.

§ 2. Classical image of many-fermion system

Let us consider a many-fermion system which is mapped into a boson space spanned by the boson operators $\hat{B}_r$ and $\hat{B}_r^*(r=1, 2, \cdots, N)$. A typical example can be found in the Holstein-Primakoff type BE theory. Another one has been presented by the present authors with the aim of applying to the ATDHF approach.[7] In these cases, $\hat{B}_r(\hat{B}_r^*)$ is constructed by replacing fermion-pair or particle-hole annihilation operator with boson. It is well known that a coherent state $|c\rangle$ defined in Eq. (2·1) gives us a classical image of many-fermion system in terms of canonical form. The image is the so-called full TDHF theory. The coherent state $|c\rangle$ is defined by

$$|c\rangle=\exp[iU]\cdot\exp[i\hat{U}]|0\rangle, \qquad (\hat{B}_r|0\rangle=0) \tag{2·1}$$

$$\left.\begin{aligned} &U=W^0-\frac{1}{2}\sum_{r=1}^{N}(Q_r^0P_r^0-q_r^0p_r^0),\\ &W^0=W^0(q_1^0p_1^0\cdots q_N^0p_N^0),\\ &\hat{U}=\sum_{r=1}^{N}(\hat{Q}_rP_r^0-Q_r^0\hat{P}_r). \end{aligned}\right\} \tag{2·1a}$$

Here, $\hat{Q}_r$ and $\hat{P}_r$ are given by

$$\hat{Q}_r=(\hat{B}_r^*+\hat{B}_r)/\sqrt{2}, \qquad \hat{P}_r=i(\hat{B}_r^*-\hat{B}_r)/\sqrt{2}. \tag{2·1b}$$

The factor $\exp[iU]$ is a phase, which we will give the meaning, later. Clearly, $\hat{Q}_r$ and $\hat{P}_r$ satisfy

$$\langle c|\hat{Q}_r|c\rangle=Q_r^0, \qquad \langle c|\hat{P}_r|c\rangle=P_r^0. \tag{2·2}$$

The relations (2·2) tell us that Q_r^0 and P_r^0 are the classical images of the operators $\hat{Q}_r$ and $\hat{P}_r$, respectively. Therefore, if Q_r^0 and P_r^0 can be regarded as dynamical variables in classical mechanics, they are canonical. With the use of Q_r^0 and P_r^0, we can construct a canonical form of full TDHF theory. As was mentioned in §1, this fact was first pointed out by Marshalek and Holzwarth.[4] However, the canonical form constructed by the variables Q_r^0 and P_r^0 is generally not suitable for our present aim, because it is not expressed explicitly in terms of collective and intrinsic variables. Then, we have to investigate a canonical transformation of Q_r^0 and P_r^0 to the variables suitable for our present aim. The phase factor $\exp[iU]$ enables us to find the transformation. The symbols q_r^0 and $p_r^0(r=1, 2, \cdots, N)$ denote canonical variables obtained from Q_r^0 and P_r^0 by a canonical transformation.

Let us find the transformation from (Q_r^0, P_r^0) to (q_r^0, p_r^0). For this aim, we introduce the following set of the relations, which we have called canonicity condition:

$$\left.\begin{aligned} &\langle c|i\partial/\partial q_r^0|c\rangle = p_r^0/2, \\ &\langle c|-i\partial/\partial p_r^0|c\rangle = q_r^0/2. \end{aligned}\right\} \tag{2·3}$$

The straightforward calculation of Eqs. (2·3) gives us

$$\left.\begin{aligned} &\sum_{s=1}^{N} P_s^0 \frac{\partial Q_s^0}{\partial q_r^0} - \frac{\partial W^0}{\partial q_r^0} = p_r^0, \\ &\sum_{s=1}^{N} P_s^0 \frac{\partial Q_s^0}{\partial p_r^0} - \frac{\partial W^0}{\partial p_r^0} = 0. \end{aligned}\right\} \tag{2·3a}$$

As is well known in analytical dynamics, the relation (2·3a) supports that q_r^0 and p_r^0 $(r=1, 2, \cdots, N)$ are certainly canonical if Q_r^0 and P_r^0 $(r=1, 2, \cdots, N)$ are canonical. In the case of many-boson system, we can start directly from the relation (2·3a). In order to stress the parallel relation between the original fermion and the boson system, we started from the condition (2·3). In the case of the Slater determinant, the canonicity condition (2·3) is essential for obtaining a classical form of the full TDHF theory as a classical image of the BE theory.[6] If W^0 is specified in one form for the variables $q_1^0 p_1^0 \cdots q_N^0 p_N^0$, one type of the canonical transformation is fixed.

Our final aim is to solve Eqs. (2·3a), i.e., to obtain Q_r^0 and P_r^0 as functions of $q_1^0 p_1^0 \cdots q_N^0 p_N^0$. In order to get a possible method for the solution, first, we will rewrite Eqs. (2·3a) to slightly different forms. For rewriting, we consider the following canonical transformation from (q_r^0, p_r^0) to (q_r, p_r):

$$\begin{cases} q_l^0 = q_l, \\ p_l^0 = p_l; \quad (l=1, 2, \cdots, N-1) \end{cases} \qquad \begin{cases} q_N^0 = \alpha + q_N, \\ p_N^0 = \pi + p_N. \end{cases} \tag{2·4}$$

Here, α and π are arbitrary real numbers and, for a moment, they play a role of parameters. Then, one-step canonical transformation from (Q_r^0, P_r^0) to (q_r, p_r) can be given by the following relations:

$$\left.\begin{aligned} &\sum_{s=1}^{N} P_s^0 \frac{\partial Q_s^0}{\partial q_r} - \frac{\partial W^{(\alpha\pi)}}{\partial q_r} = p_r^0, \\ &\sum_{s=1}^{N} P_s^0 \frac{\partial Q_s^0}{\partial p_r} - \frac{\partial W^{(\alpha\pi)}}{\partial p_r} = 0. \end{aligned}\right\} \tag{2·5}$$

Here, $W^{(\alpha\pi)}$ is a function of $q_1 p_1 \cdots q_N p_N$ defined by

$$\begin{aligned} W^{(\alpha\pi)} &= W^{(\alpha\pi)}(q_1 p_1 \cdots q_N p_N) \\ &= W^0(q_1 p_1 \cdots q_{N-1} p_{N-1}\, \alpha + q_N\, \pi + p_N) + \pi q_N. \end{aligned} \tag{2·6}$$

Let us suppose that we have the solutions of Eqs. (2·3a) in the following forms:

$$\left.\begin{aligned} Q_r^0 &= \mathcal{Q}_r(q_1^0 p_1^0 \cdots q_N^0 p_N^0), \\ P_r^0 &= \mathcal{P}_r(q_1^0 p_1^0 \cdots q_N^0 p_N^0). \end{aligned}\right\} \tag{2·7}$$

On the other hand, let us denote the solutions of Eqs. (2·5) as

$$\left.\begin{aligned} Q_r^0 &= \mathcal{Q}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_N p_N), \\ P_r^0 &= \mathcal{P}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_N p_N). \end{aligned}\right\} \tag{2·8}$$

The arguments of the functions $(\mathcal{Q}_r, \mathcal{P}_r)$ and $(\mathcal{Q}_r^{(\alpha\pi)}, \mathcal{P}_r^{(\alpha\pi)})$ are connected with each other through the relations (2·4). Therefore, we have

$$\left.\begin{aligned} \mathcal{Q}_r(q_1 p_1 \cdots q_{N-1} p_{N-1}\, \alpha + q_N\, \pi + p_N) &= \mathcal{Q}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} q_N p_N), \\ \mathcal{P}_r(q_1 p_1 \cdots q_{N-1} p_{N-1}\, \alpha + q_N\, \pi + p_N) &= \mathcal{P}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} q_N p_N). \end{aligned}\right\} \tag{2·9}$$

By putting q_N and p_N equal to zero, we get the following relations:

$$\left.\begin{aligned} \mathcal{Q}_r(q_1 p_1 \cdots q_{N-1} p_{N-1} \alpha\pi) &= \mathcal{Q}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} 00), \\ \mathcal{P}_r(q_1 p_1 \cdots q_{N-1} p_{N-1} \alpha\pi) &= \mathcal{P}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} 00). \end{aligned}\right\} \tag{2·10}$$

The relations (2·10) tell us two important facts. First, if we regard α and π as dynamical variables, $\mathcal{Q}_r(q_1 p_1 \cdots \alpha\pi)$ and $\mathcal{P}_r(q_1 p_1 \cdots \alpha\pi)$ are the solutions of Eqs. (2·3a) under the correspondence

$$\begin{cases} q_l^0 = q_l, \\ p_l^0 = p_l; \quad (l = 1, 2, \cdots, N-1) \end{cases} \qquad \begin{cases} q_N^0 = \alpha, \\ p_N^0 = \pi. \end{cases} \tag{2·11}$$

Clearly, α and π are canonical. Second, by putting q_N and p_N equal to zero in the solutions of Eqs. (2·5), we can obtain the solutions of Eqs.(2·3a) under the correspondence (2·11). Thus, Q_r^0 and P_r^0 are expressed as

$$\left.\begin{aligned} Q_r^0 &= Q_r^0(q_1 p_1 \cdots q_{N-1} p_{N-1} \alpha\pi) \\ &= \mathcal{Q}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} 00), \\ P_r^0 &= P_r^0(q_1 p_1 \cdots q_{N-1} p_{N-1} \alpha\pi) \\ &= \mathcal{P}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_{N-1} p_{N-1} 00). \end{aligned}\right\} \tag{2·12}$$

We can see that the problem is reduced to solving Eqs. (2·5).

§ 3. Canonical transformation to collective and intrinsic variables

Let us start from solving Eqs. (2·5). We will give explicit forms of $\mathcal{Q}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_N p_N)$ and $\mathcal{P}_r^{(\alpha\pi)}(q_1 p_1 \cdots q_N p_N)$ shown in Eqs. (2·8) in terms of power series expansion for q_r and $p_r (r = 1, 2, \cdots, N)$:

$$\mathcal{Q}_r^{(\alpha\pi)} = Q_r(\alpha\pi) + Q_r^{(1)}(\alpha\pi) + Q_r^{(2)}(\alpha\pi) + \cdots, \tag{3·1a}$$

$$\mathcal{P}_r^{(\alpha\pi)} = P_r(\alpha\pi) + P_r^{(1)}(\alpha\pi) + P_r^{(2)}(\alpha\pi) + \cdots. \tag{3·1b}$$

Further, we expand $W^{(\alpha\pi)}$ in the form

$$W^{(\alpha\pi)} = W(\alpha\pi) + W^{(1)}(\alpha\pi) + W^{(2)}(\alpha\pi) + \cdots. \tag{3·1c}$$

Here, $Q_r(\alpha\pi)$, $P_r(\alpha\pi)$ and $W(\alpha\pi)$ denote the zero-th order terms and the indices (1), (2), etc. mean linear, quadratic, etc. for q_r and p_r. The coefficients of the expansions may be functions of the parameters α and π. In this paper, we will show the results calculated explicitly up to the quadratic order. For a moment, we will omit the symbol $(\alpha\pi)$

appearing in the expansions (3·1).

Substituting the expansions (3·1) into Eqs. (2·5), we pick up the zero-th order terms from both the sides:

$$\left.\begin{aligned} &\sum_{s=1}^{N} P_s \frac{\partial Q_s^{(1)}}{\partial q_r} - \frac{\partial W^{(1)}}{\partial q_r} = 0\,, \\ &\sum_{s=1}^{N} P_s \frac{\partial Q_s^{(1)}}{\partial p_r} - \frac{\partial W^{(1)}}{\partial p_r} = 0\,. \end{aligned}\right\} \tag{3·2}$$

Since P_s does not depend on q_r and p_r and $W^{(1)}$ is linear for them, $W^{(1)}$ is uniquely given as

$$W^{(1)}(\alpha\pi) = \sum_r P_r Q_r^{(1)}\,. \tag{3·3}$$

The linear parts of Eqs. (2·5) become

$$\left.\begin{aligned} &\sum_{s=1}^{N} \left(P_s \frac{\partial Q_s^{(2)}}{\partial q_r} + P_s^{(1)} \frac{\partial Q_s^{(1)}}{\partial q_r} \right) - \frac{\partial W^{(2)}}{\partial q_r} = p_r\,, \\ &\sum_{s=1}^{N} \left(P_s \frac{\partial Q_s^{(2)}}{\partial p_r} + P_s^{(1)} \frac{\partial Q_s^{(1)}}{\partial p_r} \right) - \frac{\partial W^{(2)}}{\partial p_r} = 0\,. \end{aligned}\right\} \tag{3·4}$$

The above equations are rewritten as

$$\left.\begin{aligned} &\sum_{s=1}^{N} P_s^{(1)} \frac{\partial Q_s^{(1)}}{\partial q_r} - \frac{\partial \bar{W}^{(2)}}{\partial q_r} = p_r\,, \\ &\sum_{s=1}^{N} P_s^{(1)} \frac{\partial Q_s^{(1)}}{\partial p_r} - \frac{\partial \bar{W}^{(2)}}{\partial p_r} = 0\,, \end{aligned}\right\} \tag{3·5}$$

where $\bar{W}^{(2)}$ is defined by

$$W^{(2)}(\alpha\pi) = \sum_r P_r Q_r^{(2)} + \bar{W}^{(2)}\,. \tag{3·6}$$

Eliminating $\bar{W}^{(2)}$ from Eqs. (3·5), we have

$$\left.\begin{aligned} &\sum_r \left(\frac{\partial Q_r^{(1)}}{\partial q_s} \frac{\partial P_r^{(1)}}{\partial p_t} - \frac{\partial Q_r^{(1)}}{\partial p_t} \frac{\partial P_r^{(1)}}{\partial q_s} \right) = \delta_{st}\,, \\ &\sum_r \left(\frac{\partial Q_r^{(1)}}{\partial q_s} \frac{\partial P_r^{(1)}}{\partial q_t} - \frac{\partial Q_r^{(1)}}{\partial q_t} \frac{\partial P_r^{(1)}}{\partial q_s} \right) = 0\,, \\ &\sum_r \left(\frac{\partial Q_r^{(1)}}{\partial p_s} \frac{\partial P_r^{(1)}}{\partial p_t} - \frac{\partial Q_r^{(1)}}{\partial p_t} \frac{\partial P_r^{(1)}}{\partial p_s} \right) = 0\,. \end{aligned}\right\} \tag{3·7}$$

Solution of Eqs. (3·7) is given as follows:

$$\left.\begin{aligned} Q_r^{(1)}(\alpha\pi) &= \sum_{s=1}^{N} (A_{rs} q_s + C_{rs} p_s)\,, \\ P_r^{(1)}(\alpha\pi) &= \sum_{s=1}^{N} (B_{rs} q_s + D_{rs} p_s)\,. \end{aligned}\right\} \tag{3·8}$$

In the $2N$-dimensional vector form, it is expressed as

$$\begin{bmatrix} Q^{(1)} \\ P^{(1)} \end{bmatrix} = S \begin{bmatrix} q \\ p \end{bmatrix}, \qquad S = \begin{bmatrix} A & C \\ B & D \end{bmatrix}. \tag{3·8a}$$

The $2N \times 2N$ matrix S is symplectic:

$$S^TJS=J\,,\qquad J=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{3·9a}$$

Here, the symbol T denotes the operation of transpose; 0 and 1 in J are the $N\times N$ null and the $N\times N$ unit matrix, respectively. In terms of the $N\times N$ matrices A, B, C and D, the relation (3·9a) can be expressed as

$$\left.\begin{aligned} A^TD-B^TC&=1\,, & D^TA-C^TB&=1\,,\\ A^TB-B^TA&=0\,, & D^TC-C^TD&=0\,. \end{aligned}\right\} \tag{3·9b}$$

The inverse relations are as follows:

$$SJS^T=J\,, \tag{3·10a}$$

$$\left.\begin{aligned} AD^T-CB^T&=1\,, & DA^T-BC^T&=1\,,\\ AC^T-CA^T&=0\,, & DB^T-BD^T&=0\,. \end{aligned}\right\} \tag{3·10b}$$

By the use of Eqs. (3·5) with the solution (3·8), $\bar{W}^{(2)}$ is obtained in the form

$$\bar{W}^{(2)}(\alpha\pi)=\frac{1}{2}\sum_r(Q_r^{(1)}P_r^{(1)}-q_rp_r). \tag{3·11}$$

Our next task is to obtain $Q_r^{(2)}$ and $P_r^{(2)}$. The quadratic parts of Eqs. (2·5) are

$$\left.\begin{aligned} &\sum_{s=1}^{N}\left(P_s\frac{\partial Q_s^{(3)}}{\partial q_r}+P_s^{(1)}\frac{\partial Q_s^{(2)}}{\partial q_r}+P_s^{(2)}\frac{\partial Q_s^{(1)}}{\partial q_r}\right)-\frac{\partial W^{(3)}}{\partial q_r}=0\,,\\ &\sum_{s=1}^{N}\left(P_s\frac{\partial Q_s^{(3)}}{\partial p_r}+P_s^{(1)}\frac{\partial Q_s^{(2)}}{\partial p_r}+P_s^{(3)}\frac{\partial Q_s^{(1)}}{\partial p_r}\right)-\frac{\partial W^{(3)}}{\partial p_r}=0\,. \end{aligned}\right\} \tag{3·12}$$

The above equation can be rewritten as

$$\left.\begin{aligned} &\sum_{s=1}^{N}\left(P_s^{(1)}\frac{\partial Q_s^{(2)}}{\partial q_r}+P_s^{(2)}\frac{\partial Q_s^{(1)}}{\partial q_r}\right)-\frac{\partial \bar{W}^{(3)}}{\partial q_r}=0\,,\\ &\sum_{s=1}^{N}\left(P_s^{(1)}\frac{\partial Q_s^{(2)}}{\partial p_r}+P_s^{(2)}\frac{\partial Q_s^{(1)}}{\partial p_r}\right)-\frac{\partial \bar{W}^{(3)}}{\partial p_r}=0\,, \end{aligned}\right\} \tag{3·13}$$

$$W^{(3)}(\alpha\pi)=\sum_r P_rO_r^{(3)}+\bar{W}^{(3)}\,. \tag{3·14}$$

Eliminating $\bar{W}^{(3)}$ from Eqs. (3·13), we have

$$\left.\begin{aligned} &\sum_r\left(\frac{\partial Q_r^{(2)}}{\partial q_s}\frac{\partial P_r^{(1)}}{\partial p_t}-\frac{\partial Q_r^{(2)}}{\partial p_t}\frac{\partial P_r^{(1)}}{\partial q_s}+\frac{\partial Q_r^{(1)}}{\partial q_s}\frac{\partial P_r^{(2)}}{\partial p_t}-\frac{\partial Q_r^{(1)}}{\partial p_t}\frac{\partial P_r^{(2)}}{\partial q_s}\right)=0\,,\\ &\sum_r\left(\frac{\partial Q_r^{(2)}}{\partial q_s}\frac{\partial P_r^{(1)}}{\partial q_t}-\frac{\partial Q_r^{(2)}}{\partial q_t}\frac{\partial P_r^{(1)}}{\partial q_s}+\frac{\partial Q_r^{(1)}}{\partial q_s}\frac{\partial P_r^{(2)}}{\partial q_t}-\frac{\partial Q_r^{(1)}}{\partial q_t}\frac{\partial P_r^{(2)}}{\partial q_s}\right)=0\,,\\ &\sum_r\left(\frac{\partial Q_r^{(2)}}{\partial p_s}\frac{\partial P_r^{(1)}}{\partial p_t}-\frac{\partial Q_r^{(2)}}{\partial p_t}\frac{\partial P_r^{(1)}}{\partial p_s}+\frac{\partial Q_r^{(1)}}{\partial p_s}\frac{\partial P_r^{(2)}}{\partial p_t}-\frac{\partial Q_r^{(1)}}{\partial p_t}\frac{\partial P_r^{(2)}}{\partial p_s}\right)=0\,. \end{aligned}\right\} \tag{3·15}$$

Equations (3·15) lead us to the relations

$$\frac{\partial q_s^{(2)}}{\partial q_t}+\frac{\partial p_t^{(2)}}{\partial p_s}=0,\quad \frac{\partial p_s^{(2)}}{\partial q_t}-\frac{\partial p_t^{(2)}}{\partial q_s}=0,\quad \frac{\partial q_s^{(2)}}{\partial q_t}-\frac{\partial q_t^{(2)}}{\partial p_s}=0\,. \tag{3·16}$$

Here, $q_r^{(2)}$ and $p_r^{(2)}$ are defined in the $2N$-dimensional vector form:

$$\begin{bmatrix} Q^{(2)} \\ P^{(2)} \end{bmatrix} = S \begin{bmatrix} q^{(2)} \\ p^{(2)} \end{bmatrix}. \tag{3·17}$$

The matrix S is the same as that given in Eqs. (3·8). The solution of Eqs. (3·16) is generally obtained by the use of function Φ:

$$q_r^{(2)} = \frac{\partial \Phi}{\partial p_r}, \qquad p_r^{(2)} = -\frac{\partial \Phi}{\partial q_r}. \tag{3·18}$$

Since $q_r^{(2)}$ and $p_r^{(2)}$ are quadratic, Φ should be cubic:

$$\Phi = \frac{1}{6}\sum_{rst=1}^{N}(A_{rst}q_rq_sq_t + 3B_{rs,t}q_rq_sp_t + 3C_{r,st}q_rp_sp_t + D_{rst}p_rp_sp_t). \tag{3·19}$$

Here, A_{rst}, $B_{rs,t}$, $C_{r,st}$ and D_{rst} have the following symmetry properties:

$$\left.\begin{aligned} &A_{rst} = A_{rts} = A_{str} = A_{srt} = A_{trs} = A_{tsr}, \\ &B_{rs,t} = B_{sr,t}, \qquad C_{r,st} = C_{r,ts}, \\ &D_{rst} = D_{rts} = D_{str} = D_{srt} = D_{trs} = D_{tsr}. \end{aligned}\right\} \tag{3·20}$$

Then, substituting Φ given in Eq. (3·19) into Eqs. (3·18), we get

$$\left.\begin{aligned} q_r^{(2)} &= \frac{1}{2}\sum_{st=1}^{N}(B_{ts,r}q_tq_s + 2C_{t,sr}q_tp_s + D_{tsr}p_tp_s), \\ p_r^{(2)} &= -\frac{1}{2}\sum_{st=1}^{N}(A_{rst}q_sq_t + 2B_{rs,t}q_sp_t + C_{r,st}p_sp_t). \end{aligned}\right\} \tag{3·21}$$

Therefore, $Q_r^{(2)}$ and $P_r^{(2)}$ are given in the following forms:

$$\left.\begin{aligned} Q_r^{(2)}(\alpha\pi) = \frac{1}{2}\sum_{st=1}^{N}\Big[&\sum_{r'}(A_{rr'}B_{st,r'} - C_{rr'}A_{r'st})q_sq_t \\ &+ 2\sum_{r'}(A_{rr'}C_{s,tr'} - C_{rr'}B_{r's,t})q_sp_t \\ &+ \sum_{r'}(A_{rr'}D_{str'} - C_{rr'}C_{r',st})p_sp_t\Big], \\ P_r^{(2)}(\alpha\pi) = \frac{1}{2}\sum_{st=1}^{N}\Big[&\sum_{r'}(B_{rr'}B_{st,r'} - D_{rr'}A_{r'st})q_sq_t \\ &+ 2\sum_{r'}(B_{rr'}C_{s,tr'} - D_{rr'}B_{r's,t})q_sp_t \\ &+ \sum_{r'}(B_{rr'}D_{str'} - D_{rr'}C_{r',st})p_sp_t\Big]. \end{aligned}\right\} \tag{3·22}$$

The above is a formal solution of Eqs. (2·5) calculated up to the quadratic order.

The expressions (3·1) mean the Tayler expansion of $\mathcal{Q}_r^{(\alpha\pi)}$ and $\mathcal{P}_r^{(\alpha\pi)}$ for $q_1p_1\cdots q_Np_N$ around $q=p=0$. As can be seen in Eqs. (2·9), $\mathcal{Q}_r^{(\alpha\pi)}$ and $\mathcal{P}_r^{(\alpha\pi)}$ are functions of $q_1p_1\cdots q_{N-1}p_{N-1}$ and $\alpha+q_N$, $\pi+p_N$. Therefore, the derivatives of $\mathcal{Q}_r^{(\alpha\pi)}$ and $\mathcal{P}_r^{(\alpha\pi)}$ related with q_N and p_N can be replaced with those for α and π, respectively. This means that certain parts of the coefficients in the expansions (3·1) are replaced with the derivatives for α and π. The replacements are explicitly given as follows:

$$A_{rN} = \frac{\partial Q_r}{\partial \alpha}, \quad B_{rN} = \frac{\partial P_r}{\partial \alpha}, \quad C_{rN} = \frac{\partial Q_r}{\partial \pi}, \quad D_{rN} = \frac{\partial P_r}{\partial \pi}, \tag{3·23}$$

$$\left.\begin{aligned}
&\sum_t (A_{rt}B_{Ns,t}-C_{rt}A_{stN})=\frac{\partial A_{rs}}{\partial\alpha}\,,\\
&\sum_t (B_{rt}B_{Ns,t}-D_{rt}A_{stN})=\frac{\partial B_{rs}}{\partial\alpha}\,,\\
&\sum_t (A_{rt}C_{N,ts}-C_{rt}B_{Nt,s})=\frac{\partial C_{rs}}{\partial\alpha}\,,\\
&\sum_t (B_{rt}C_{N,ts}-D_{rt}B_{Nt,s})=\frac{\partial D_{rs}}{\partial\alpha}\,,\\
&\sum_t (A_{rt}C_{s,tN}-C_{rt}B_{st,N})=\frac{\partial A_{rs}}{\partial\pi}\,,\\
&\sum_t (B_{rt}C_{s,tN}-D_{rt}B_{st,N})=\frac{\partial B_{rs}}{\partial\pi}\,,\\
&\sum_t (A_{rt}D_{Nts}-C_{rt}C_{t,sN})=\frac{\partial C_{rs}}{\partial\pi}\,,\\
&\sum_t (B_{rt}D_{Nts}-D_{rt}C_{t,sN})=\frac{\partial D_{rs}}{\partial\pi}\,.
\end{aligned}\right\} \tag{3·24}$$

The inverse relations of Eqs. (3·24) are

$$\left.\begin{aligned}
A_{Nst}&=\left(B^T\frac{\partial A}{\partial\alpha}-A^T\frac{\partial B}{\partial\alpha}\right)_{st}\,,\\
B_{Ns,t}&=\left(B^T\frac{\partial C}{\partial\alpha}-A^T\frac{\partial D}{\partial\alpha}\right)_{st}\\
&=\left(D^T\frac{\partial A}{\partial\alpha}-C^T\frac{\partial B}{\partial\alpha}\right)_{ts}\,,\\
C_{N,st}&=\left(D^T\frac{\partial C}{\partial\alpha}-C^T\frac{\partial D}{\partial\alpha}\right)_{st}\,,
\end{aligned}\right\} \tag{3·25a}$$

$$\left.\begin{aligned}
B_{st,N}&=\left(B^T\frac{\partial A}{\partial\pi}-A^T\frac{\partial B}{\partial\pi}\right)_{st}\,,\\
C_{s,tN}&=\left(B^T\frac{\partial C}{\partial\pi}-A^T\frac{\partial D}{\partial\pi}\right)_{st}\\
&=\left(D^T\frac{\partial A}{\partial\pi}-C^T\frac{\partial B}{\partial\pi}\right)_{ts}\,,\\
D_{stN}&=\left(D^T\frac{\partial C}{\partial\pi}-C^T\frac{\partial D}{\partial\pi}\right)_{st}\,.
\end{aligned}\right\} \tag{3·25b}$$

With the aid of the relations (3·23), the (N, N)-elements in the matrix relations (3·9b) are reduced to

$$\sum_r\left(\frac{\partial Q_r}{\partial\alpha}\frac{\partial P_r}{\partial\pi}-\frac{\partial Q_r}{\partial\pi}\frac{\partial P_r}{\partial\alpha}\right)=1\,. \tag{3·26}$$

Also, the (N, l)-elements are as follows:

$$\left.\begin{aligned}
&\sum_r\left(D_{rl}\frac{\partial Q_r}{\partial\alpha}-C_{rl}\frac{\partial P_r}{\partial\alpha}\right)=0\,,\quad \sum_r\left(D_{rl}\frac{\partial Q_r}{\partial\pi}-C_{rl}\frac{\partial P_r}{\partial\pi}\right)=0\,,\\
&\sum_r\left(B_{rl}\frac{\partial Q_r}{\partial\alpha}-A_{rl}\frac{\partial P_r}{\partial\alpha}\right)=0\,,\quad \sum_r\left(B_{rl}\frac{\partial Q_r}{\partial\pi}-A_{rl}\frac{\partial P_r}{\partial\pi}\right)=0\,,
\end{aligned}\right\} \tag{3·27}$$

where l runs from 1 to $N-1$. The (m, n)-elements are not related with $\partial Q_r/\partial \alpha$, etc.;

$$\left.\begin{array}{ll}(A^TD-B^TC)_{mn}=\delta_{mn}\,, & (D^TA-C^TB)_{mn}=\delta_{mn}\,,\\ (A^TB-B^TA)_{mn}=0\,, & (D^TC-C^TD)_{mn}=0\,.\end{array}\right\} \tag{3·28}$$

$$(m,\ n=1,\ 2,\ \cdots,\ N-1)$$

Afterward, we will see that the above three relations (3·26), (3·27) and (3·28) are basic in our treatment.

Thus, in the framework of the expansions up to the quadratic order, the solutions of Eqs. (2·3a) are formally given by

$$\left.\begin{array}{l}Q_r{}^0=Q_r(\alpha\pi)+\tilde{Q}_r{}^{(1)}(\alpha\pi)+\tilde{Q}_r{}^{(2)}(\alpha\pi),\\ P_r{}^0=P_r(\alpha\pi)+\tilde{P}_r{}^{(1)}(\alpha\pi)+\tilde{P}_r{}^{(2)}(\alpha\pi),\end{array}\right\} \tag{3·29}$$

$$\left.\begin{array}{l}\tilde{Q}_r{}^{(1)}(\alpha\pi)=\sum\limits_{l=1}^{N-1}(A_{rl}q_l+C_{rl}p_l),\\ \tilde{P}_r{}^{(1)}(\alpha\pi)=\sum\limits_{l=1}^{N-1}(B_{rl}q_l+D_{rl}p_l),\end{array}\right\} \tag{3·29a}$$

$$\left.\begin{array}{rl}\tilde{Q}_r{}^{(2)}(\alpha\pi)=\dfrac{1}{2}\sum\limits_{mn=1}^{N-1}\Big[&\sum\limits_s(A_{rs}B_{mn,s}-C_{rs}A_{smn})q_mq_n\\ &+2\sum\limits_s(A_{rs}C_{m,ns}-C_{rs}B_{sm,n})q_mp_n\\ &+\sum\limits_s(A_{rs}D_{mns}-C_{rs}C_{s,mn})p_mp_n\Big],\\ \tilde{P}_r{}^{(2)}(\alpha\pi)=\dfrac{1}{2}\sum\limits_{mn=1}^{N-1}\Big[&\sum\limits_s(B_{rs}B_{mn,s}-D_{rs}A_{smn})q_mq_n\\ &+2\sum\limits_s(B_{rs}C_{m,ns}-D_{rs}B_{sm,n})q_mp_n\\ &+\sum\limits_s(B_{rs}D_{mns}-D_{rs}C_{s,mn})p_mp_n\Big].\end{array}\right\} \tag{3·29b}$$

Needless to say, we obtained the above results by putting $q_N=p_N=0$ in Eqs. (3·8) and (3·22). It should be noted that at the present stage the parameters α and π are regarded as dynamical variables. The function W^0 is given as

$$W^0=W(\alpha\pi)+\tilde{W}^{(1)}(\alpha\pi)+\tilde{W}^{(2)}(\alpha\pi), \tag{3·30}$$

$$\tilde{W}^{(1)}(\alpha\pi)=\sum_{r=1}^{N}P_r(\alpha\pi)\tilde{Q}_r{}^{(1)}(\alpha\pi), \tag{3·30a}$$

$$\tilde{W}^{(2)}(\alpha\pi)=\sum_{r=1}^{N}P_r(\alpha\pi)\tilde{Q}_r{}^{(2)}(\alpha\pi)+\frac{1}{2}\Big(\sum_{r=1}^{N}\tilde{Q}_r{}^{(1)}(\alpha\pi)\tilde{P}_r{}^{(1)}(\alpha\pi)-\sum_{l=1}^{N-1}q_lp_l\Big). \tag{3·30b}$$

The equations for $W(\alpha\pi)$ will be shown explicitly in Eqs. (6·5). As was already mentioned, our system is treated with the use of the canonical variables $(\alpha\pi)$ and $(q_lp_l$: $l=1, 2, \cdots, N-1)$. In this treatment, one degree of freedom described by α and π is clearly discriminated from the others, because the expansion in terms of α and π is not performed. Therefore, the canonical transformation (3·29) is applicable to the system in which one degree of freedom is of large amplitude and the remaining are of small ones. In this sense, we can expect that the variables α and π are for collective motion and intrinsic degrees of freedom are described by the variables q_l and p_l $(l=1, 2, \cdots, N-1)$.

§ 4. Unique separation of the whole system into collective and intrinsic degrees of freedom

We showed that many-fermion system is treated classically in terms of two types of the degrees of freedom. The separation into the two gives us the expectation that the variables α and π are for collective motion and the others can be related with the intrinsic degrees of freedom. However, it should be performed uniquely. If it depends on the adopted coordinate system, the concept of the collective and the intrinsic degrees of freedom becomes meaningless or the principle of which coordinate system should be adopted must be given. In this section, we will give the conditions for the unique separation and show that they lead us to a generalization of equation of collective submanifold.

Let us investigate the problem above in the framework of the Hamiltonian approximated up to the quadratic order for q_l and p_l $(l=1, 2, \cdots, N-1)$. This approximation is consistent with the expansions of Q_r^0, P_r^0 and W^0 shown in Eqs. (3·29) and (3·30) and it may be valid if the intrinsic motions are of the small amplitudes. However, there does not exist any restriction in the amplitude of the collective degrees of freedom. In this sense, the approximation is applicable to the large amplitude collective motion. The Hamiltonian H^0 is a function of Q_r^0 and P_r^0 $(r=1, 2, \cdots, N)$:

$$H^0=H^0(Q^0, P^0)=H^0(Q+\tilde{Q}^{(1)}+\tilde{Q}^{(2)}, P+\tilde{P}^{(1)}+\tilde{P}^{(2)}). \tag{4·1}$$

First, we expand H^0 in terms of q_l and p_l $(l=1, 2, \cdots, N-1)$ in the following form:

$$H^0=H+\tilde{H}^{(1)}+\tilde{H}^{(2)}+H^{(2)}, \tag{4·2}$$

$$H=H^0(Q, P), \tag{4·3}$$

$$\tilde{H}^{(1)}=\sum_{r=1}^{N}\left(\frac{\partial H}{\partial Q_r}\cdot\tilde{Q}_r^{(1)}+\frac{\partial H}{\partial P_r}\cdot\tilde{P}_r^{(1)}\right), \tag{4·4a}$$

$$\tilde{H}^{(2)}=\sum_{r=1}^{N}\left(\frac{\partial H}{\partial Q_r}\cdot\tilde{Q}_r^{(2)}+\frac{\partial H}{\partial P_r}\cdot\tilde{P}_r^{(2)}\right), \tag{4·4b}$$

$$H^{(2)}=\frac{1}{2}\sum_{rs=1}^{N}\left(\frac{\partial^2 H}{\partial Q_r\partial Q_s}\cdot\tilde{Q}_r^{(1)}\tilde{Q}_s^{(1)}+\frac{\partial^2 H}{\partial P_r\partial P_s}\cdot\tilde{P}_r^{(1)}\tilde{P}_s^{(1)}+2\cdot\frac{\partial^2 H}{\partial Q_r\partial P_s}\cdot\tilde{Q}_r^{(1)}\tilde{P}_s^{(1)}\right). \tag{4·5}$$

Our starting point is the following requirement: $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$ should vanish for any values of q_l and p_l $(l=1, 2, \cdots, N-1)$, that is,

$$\tilde{H}^{(1)}=\sum_{r=1}^{N}\left(\frac{\partial H}{\partial Q_r}\cdot\tilde{Q}_r^{(1)}+\frac{\partial H}{\partial P_r}\cdot\tilde{P}_r^{(1)}\right)=0, \tag{4·6a}$$

$$\tilde{H}^{(2)}=\sum_{r=1}^{N}\left(\frac{\partial H}{\partial Q_r}\cdot\tilde{Q}_r^{(2)}+\frac{\partial H}{\partial P_r}\cdot\tilde{P}_r^{(2)}\right)=0. \tag{4·6b}$$

Let two coordinate systems $(\alpha\pi,\ q_l\ p_l)$ and $(\alpha'\pi',\ q_l'p_l')$, which obey the conditions (4·6a) and (4·6b), exist. We can see from the conditions (4·6) that both the coordinate systems should be connected as

$$\begin{bmatrix}\alpha'\\ \pi'\end{bmatrix}=\begin{bmatrix}f(\alpha\pi)\\ g(\alpha\pi)\end{bmatrix}+\begin{bmatrix}\text{higher than the quadratic}\\ \text{for } q \text{ and } p\end{bmatrix}, \tag{4·7a}$$

$$\begin{bmatrix}q'\\ p'\end{bmatrix}=s\begin{bmatrix}q\\ p\end{bmatrix}+\begin{bmatrix}\text{higher than the quadratic}\\ \text{for } q \text{ and } p\end{bmatrix}. \tag{4·7b}$$

Here, $f(\alpha\pi)$ and $g(\alpha\pi)$ are any functions of α and π which obey

$$\frac{\partial f}{\partial\alpha}\cdot\frac{\partial g}{\partial\pi}-\frac{\partial f}{\partial\pi}\cdot\frac{\partial g}{\partial\alpha}=1\,. \tag{4·8}$$

The symbol s denotes $2(N-1)\times2(N-1)$ matrix with the properties

$$s^TJs=J\,,\qquad \frac{\partial s}{\partial\alpha}=\frac{\partial s}{\partial\pi}=0\,. \tag{4·9}$$

The terms, which are higher than the quadratic for q and p, do not give any change in the Hamiltonian H^0 in the quadratic order $(H^0=H+H^{(2)})$. Therefore, in our present approximation, we are not necessary to take into account the higher than the quadratic terms:

$$\begin{bmatrix}\alpha'\\ \pi'\end{bmatrix}=\begin{bmatrix}f(\alpha\pi)\\ g(\alpha\pi)\end{bmatrix}, \tag{4·10a}$$

$$\begin{bmatrix}q'\\ p'\end{bmatrix}=s\begin{bmatrix}q\\ p\end{bmatrix}. \tag{4·10b}$$

The properties (4·8) and (4·9) tell us that the transformations (4·10) are canonical. The both degrees of freedom are independently transformed. This means that the separation is uniquely performed. In this sense, the requirement shown in Eqs. (4·6) is the condition for the unique separation. It should be noted that the condition is valid in the framework of the quadratic approximation.

Now, let us investigate the condition (4·6a). The term $\tilde{H}^{(1)}$ given in Eq. (4·4a) can be written as

$$\tilde{H}^{(1)}=\sum_{l=1}^{N-1}\left[\sum_{r=1}^{N}\left(A_{rl}\frac{\partial H}{\partial Q_r}+B_{rl}\frac{\partial H}{\partial P_r}\right)\cdot q_l+\sum_{r=1}^{N}\left(C_{rl}\frac{\partial H}{\partial Q_r}+D_{rl}\frac{\partial H}{\partial P_r}\right)\cdot p_l\right]. \tag{4·11}$$

Here, we used Eqs. (3·29a). Then, the condition (4·6a) gives us

$$\left.\begin{aligned}&\sum_{r=1}^{N}\left(A_{rl}\frac{\partial H}{\partial Q_r}+B_{rl}\frac{\partial H}{\partial P_r}\right)=0\,,\\ &\sum_{r=1}^{N}\left(C_{rl}\frac{\partial H}{\partial Q_r}+D_{rl}\frac{\partial H}{\partial P_r}\right)=0\,.\end{aligned}\right\}\quad (l=1,2,\cdots,N-1) \tag{4·12}$$

The (l, N)-elements $(l=1, 2, \cdots, N-1)$ in the matrices (3·9b) are explicitly given by

$$\left.\begin{aligned}&\sum_{r=1}^{N}[A_{rl}D_{rN}+B_{rl}(-C_{rN})]=0\,,\\&\sum_{r=1}^{N}[A_{rl}B_{rN}+B_{rl}(-A_{rN})]=0\,,\end{aligned}\right\} \tag{4·13a}$$

$$\left.\begin{aligned}&\sum_{r=1}^{N}[C_{rl}D_{rN}+D_{rl}(-C_{rN})]=0\,,\\&\sum_{r=1}^{N}[C_{rl}B_{rN}+D_{rl}(-A_{rN})]=0\,.\end{aligned}\right\} \tag{4·13b}$$

Therefore, combining Eqs. (4·12) with Eqs. (4·13), we obtain the following linear combinations:

$$\left.\begin{aligned}\lambda_\alpha D_{rN}-\lambda_\pi B_{rN}&=\frac{\partial H}{\partial Q_r}\,,\\-\lambda_\alpha C_{rN}+\lambda_\pi A_{rN}&=\frac{\partial H}{\partial P_r}\,.\end{aligned}\right\} \tag{4·14}$$

Here, λ_α and λ_π are, at the present stage, any function of α and π. Noticing the relations (3·23), we have

$$\left.\begin{aligned}\lambda_\alpha\frac{\partial P_r}{\partial\pi}-\lambda_\pi\frac{\partial P_r}{\partial\alpha}&=\frac{\partial H}{\partial Q_r}\,,\\-\lambda_\alpha\frac{\partial Q_r}{\partial\pi}+\lambda_\pi\frac{\partial Q_r}{\partial\alpha}&=\frac{\partial H}{\partial P_r}\,.\end{aligned}\right\} \tag{4·15}$$

The above is just well-known equation of collective submanifold. In this way, we could prepare Eqs. (4·15) and (3·26) for determining Q_r and P_r as functions of α and π. Later, we will give equations to determine linear dependence for the variables q_l and p_l, i.e., $\tilde{Q}_r^{(1)}$ and $\tilde{P}_r^{(1)}$. In this sense, we call the set of Eqs. (4·15) the first equation of collective submanifold.

Next, we will investigate the condition (4·6b). Substituting $\partial H/\partial Q_r$ and $\partial H/\partial P_r$ given in Eqs. (4·15) into Eq. (4·4b), we have

$$\begin{aligned}\tilde{H}^{(2)}=\lambda_\alpha\cdot\sum_{r=1}^{N}\left(\frac{\partial P_r}{\partial\pi}\cdot\tilde{Q}_r^{(2)}-\frac{\partial Q_r}{\partial\pi}\cdot\tilde{P}_r^{(2)}\right)\\+\lambda_\pi\cdot\sum_{r=1}^{N}\left(\frac{\partial Q_r}{\partial\alpha}\cdot\tilde{P}_r^{(2)}-\frac{\partial P_r}{\partial\alpha}\cdot\tilde{Q}_r^{(2)}\right).\end{aligned} \tag{4·16}$$

The above $\tilde{H}^{(2)}$ is quadratic for q_l and p_l $(l=1, 2, \cdots, N-1)$ and the condition (4·6b) leads us to the relations

$$\left.\begin{aligned}\lambda_\alpha B_{mn,N}-\lambda_\pi A_{Nmn}&=0\,,\\\lambda_\alpha C_{m,nN}-\lambda_\pi B_{Nm,n}&=0\,,\\\lambda_\alpha D_{mnN}-\lambda_\pi C_{N,mn}&=0\,.\end{aligned}\right\} \tag{4·17}$$

Substituting A_{Nmn} etc. given by Eqs. (3·25) into Eqs. (4·17), we can get

$$\left.\begin{aligned}
&\left[D^T\left(\lambda_\alpha\cdot\frac{\partial A}{\partial\pi}-\lambda_\pi\cdot\frac{\partial A}{\partial\alpha}\right)-C^T\left(\lambda_\alpha\cdot\frac{\partial B}{\partial\pi}-\lambda_\pi\cdot\frac{\partial B}{\partial\alpha}\right)\right]_{mn}=0\,,\\
&\left[D^T\left(\lambda_\alpha\cdot\frac{\partial C}{\partial\pi}-\lambda_\pi\cdot\frac{\partial C}{\partial\alpha}\right)-C^T\left(\lambda_\alpha\cdot\frac{\partial D}{\partial\pi}-\lambda_\pi\cdot\frac{\partial D}{\partial\alpha}\right)\right]_{mn}=0\,,\\
&\left[-B^T\left(\lambda_\alpha\cdot\frac{\partial A}{\partial\pi}-\lambda_\pi\cdot\frac{\partial A}{\partial\alpha}\right)+A^T\left(\lambda_\alpha\cdot\frac{\partial B}{\partial\pi}-\lambda_\pi\cdot\frac{\partial B}{\partial\alpha}\right)\right]_{mn}=0\,,\\
&\left[-B^T\left(\lambda_\alpha\cdot\frac{\partial C}{\partial\pi}-\lambda_\pi\cdot\frac{\partial C}{\partial\alpha}\right)+A^T\left(\lambda_\alpha\cdot\frac{\partial D}{\partial\pi}-\lambda_\pi\cdot\frac{\partial D}{\partial\alpha}\right)\right]_{mn}=0\,.
\end{aligned}\right\}\qquad(4\cdot18)$$

$$(m,\ n=1,\ 2,\ \cdots,\ N-1)$$

Thus, we could prepare Eqs. (4·18) and (3·27) in order to determine A_{rl}, B_{rl}, C_{rl} and D_{rl} for $l=1,\ 2,\ \cdots,\ N-1$. If we determine these quantities as functions of α and π, $\tilde{Q}_r^{(1)}$ and $\tilde{P}_r^{(1)}$ can be fixed. We called the set of Eqs. (4·15) the first equation of collective submanifold. In a similar meaning, we call the set of Eqs. (4·18) the second equation of collective submanifold. We should note that we have no equations for A_{lmn} etc. However, as far as the Hamiltonian in the quadratic approximation ($H^0=H+H^{(2)}$) is concerned, it is not necessary to determine A_{lmn} etc. Thus, we could generalize equation of collective submanifold which has been widely investigated. With the aid of the generalization, we can treat coupling between collective and intrinsic degrees of freedom in the first order.

§ 5. Canonical invariance of equations of collective submanifold

In §4, we have prepared the basic equations for determining Q_r, P_r, A_{rl}, B_{rl}, C_{rl} and D_{rl} ($l=1,\ 2,\ \cdots,\ N-1$). Then, our problem is reduced to solving the equations. In this section, we will discuss some general properties, especially, canonical invariance of the equations of collective submanifold, which will help us to solve the equations.

First, we note that λ_α and λ_π in Eqs. (4·15) and (4·18) can be expressed as

$$\lambda_\alpha=\frac{\partial H}{\partial\alpha}\,,\qquad \lambda_\pi=\frac{\partial H}{\partial\pi}\,.\qquad(5\cdot1)$$

This can be proved with the help of Eqs. (4·3) and (4·15) with Eq. (3·26). Then, the first equation, together with Eq. (3·26), can be rewritten as

$$\left.\begin{aligned}
&\left(\frac{\partial H}{\partial\alpha},\ \frac{\partial H}{\partial\pi}\right)J\begin{bmatrix}\dfrac{\partial P_r}{\partial\alpha}\\[2mm] \dfrac{\partial P_r}{\partial\pi}\end{bmatrix}=\frac{\partial H}{\partial Q_r}\,,\\
&-\left(\frac{\partial H}{\partial\alpha},\ \frac{\partial H}{\partial\pi}\right)J\begin{bmatrix}\dfrac{\partial Q_r}{\partial\alpha}\\[2mm] \dfrac{\partial P_r}{\partial\pi}\end{bmatrix}=\frac{\partial H}{\partial P_r}\,,
\end{aligned}\right\}\qquad(5\cdot2)$$

$$\sum_{r=1}^{N}\left(\frac{\partial Q_r}{\partial \alpha}, \frac{\partial Q_r}{\partial \pi}\right) J \begin{bmatrix} \dfrac{\partial P_r}{\partial \alpha} \\ \dfrac{\partial P_r}{\partial \pi} \end{bmatrix} = 1 . \tag{5·3}$$

Here, J is the 2×2 matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5·4}$$

Since the transformation (4·10a) is canonical, the following 2×2 matrix is symplectic:

$$\mathcal{S} = \begin{bmatrix} \dfrac{\partial \alpha}{\partial \alpha'} & \dfrac{\partial \alpha}{\partial \pi'} \\ \dfrac{\partial \pi}{\partial \alpha'} & \dfrac{\partial \pi}{\partial \pi'} \end{bmatrix}, \qquad \mathcal{S}^T J \mathcal{S} = J . \tag{5·5}$$

Then, we have

$$\begin{bmatrix} \dfrac{\partial}{\partial \alpha'} \\ \dfrac{\partial}{\partial \pi'} \end{bmatrix} = \mathcal{S} \begin{bmatrix} \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial \pi} \end{bmatrix} . \tag{5·6}$$

With the use of the relation (5·6) with Eqs. (5·5), we can prove that Eqs. (5·2) and (5·3) are invariant under the transformation (4·10a). Therefore, we can see an important fact: In the framework of Eqs. (5·2) and (5·3), we cannot specify the collective coordinate. Hence, we cannot solve Eqs. (5·2) and (5·3). Some additional conditions are necessary. This fact has been already stressed by the present authors.[9]

Our next task is to prove canonical invariance of the second equations (4·18), together with Eqs. (3·27). This case is rather tedious. First, we introduce the $2N\times 2(N-1)$ matrix $\tilde{S}$ defined in the vector form of the linear combinations (3·29a):

$$\begin{bmatrix} \tilde{Q}^{(1)} \\ \tilde{P}^{(1)} \end{bmatrix} = \tilde{S} \begin{bmatrix} q \\ p \end{bmatrix}, \qquad \tilde{S} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}. \tag{5·7}$$

The above relations (5·7) should not be confused with Eqs. (3·8a). The properties of $\tilde{S}$ are as follows:

$$\tilde{S}^{-1} = \begin{bmatrix} D^T & -C^T \\ -B^T & A^T \end{bmatrix}, \quad \tilde{S}^{-1}\cdot\tilde{S} = 1, \quad \tilde{S}\cdot\tilde{S}^{-1} = \Gamma . \tag{5·8}$$

Clearly, $\tilde{S}^{-1}$ is the $2(N-1)\times 2N$ matrix. The symbol Γ denotes the $2N\times 2N$ idempotent matrix, which satisfies

$$\Gamma^2 = \Gamma , \qquad \mathrm{Tr}[\Gamma] = 2(N-1). \tag{5·9}$$

The element is given as

$$\Gamma=\begin{bmatrix} K & L \\ M & K^T \end{bmatrix}, \qquad (L^T=-L,\ M^T=-M) \tag{5·10}$$

$$K_{rs}=\delta_{rs}-\left(\frac{\partial Q_r}{\partial\alpha}\frac{\partial Q_r}{\partial\pi}\right)J\begin{bmatrix} \dfrac{\partial P_s}{\partial\alpha} \\ \dfrac{\partial P_s}{\partial\pi} \end{bmatrix}, \tag{5·10a}$$

$$L_{rs}=-\left(\frac{\partial Q_r}{\partial\alpha}\frac{\partial Q_r}{\partial\pi}\right)J\begin{bmatrix} \dfrac{\partial Q_s}{\partial\alpha} \\ \dfrac{\partial Q_s}{\partial\pi} \end{bmatrix}, \tag{5·10b}$$

$$M_{rs}=-\left(\frac{\partial P_r}{\partial\alpha}\frac{\partial P_r}{\partial\pi}\right)J\begin{bmatrix} \dfrac{\partial P_s}{\partial\alpha} \\ \dfrac{\partial P_s}{\partial\pi} \end{bmatrix}. \tag{5·10c}$$

In the same way as in the previous case, we can prove that Γ is canonically invariant:

$$\Gamma'=\Gamma\,. \tag{5·11}$$

With the use of the matrix $\tilde{S}^{-1}$, Eqs. (4·18) can be rewritten as

$$\tilde{S}^{-1}\cdot\Lambda=0\,, \tag{5·12}$$

where Λ is given by

$$\Lambda=\begin{bmatrix} \lambda_\alpha\cdot\dfrac{\partial A}{\partial\pi}-\lambda_\pi\cdot\dfrac{\partial A}{\partial\alpha} & \lambda_\alpha\cdot\dfrac{\partial C}{\partial\pi}-\lambda_\pi\cdot\dfrac{\partial C}{\partial\alpha} \\ \lambda_\alpha\cdot\dfrac{\partial B}{\partial\pi}-\lambda_\pi\cdot\dfrac{\partial B}{\partial\alpha} & \lambda_\alpha\cdot\dfrac{\partial D}{\partial\pi}-\lambda_\pi\cdot\dfrac{\partial D}{\partial\alpha} \end{bmatrix}. \tag{5·12a}$$

The matrix Λ is of the form $2N\times2(N-1)$. In the same way as in the previous case, Λ can be proved to be canonically invariant:

$$\Lambda'=\Lambda\,. \tag{5·13}$$

Under the above preparation, we can prove that if $\tilde{S}^{-1}\cdot\Lambda=0$, then $\tilde{S}'^{-1}\cdot\Lambda'=0$ in the following procedure:

$$\begin{aligned}\tilde{S}'^{-1}\cdot\Lambda'&=\tilde{S}'^{-1}\Gamma'\cdot\Lambda'=\tilde{S}'^{-1}\Gamma\cdot\Lambda=\tilde{S}'^{-1}\tilde{S}\tilde{S}^{-1}\cdot\Lambda\\ &=\tilde{S}'^{-1}\tilde{S}\times\tilde{S}^{-1}\cdot\Lambda\,.\end{aligned} \tag{5·14}$$

Next, we consider Eqs. (3·27). These can be expressed as

$$\tilde{S}^{-1}\cdot\Lambda_0=0\,, \tag{5·15}$$

$$\Lambda_0=\begin{bmatrix} \dfrac{\partial Q}{\partial\alpha} & \dfrac{\partial Q}{\partial\pi} \\ \dfrac{\partial P}{\partial\alpha} & \dfrac{\partial P}{\partial\pi} \end{bmatrix}, \tag{5·15a}$$

where Λ_0 is $2N\times 2$ matrix. This matrix satisfies

$$\Lambda_0'=\Lambda_0\,. \tag{5·16}$$

Then, we can prove that if $\tilde{S}^{-1}\cdot\Lambda_0=0$, then $\tilde{S}'^{-1}\cdot\Lambda_0'=0$ in the following way:

$$\begin{aligned}\tilde{S}'^{-1}\cdot\Lambda_0'&=\tilde{S}'^{-1}\Gamma'\cdot\Lambda_0'=\tilde{S}'^{-1}\Gamma\cdot\Lambda_0\\&=\tilde{S}'^{-1}\tilde{S}\tilde{S}^{-1}\cdot\Lambda_0\\&=\tilde{S}'^{-1}\tilde{S}\times\tilde{S}^{-1}\cdot\Lambda_0\,.\end{aligned} \tag{5·17}$$

Thus, we could prove that Eqs. (3·27) are canonically invariant under the transformation (4·10a).

The transformation (4·10b) is also interesting. Clearly, Eqs. (5·2) and (5·3) are invariant, because they have no connection with the transformation (4·10b). Then, let us investigate Eqs. (5·12) and (5·15). With the use of the matrix S in the transformation (4·10b), the relation (5·7) can be given by

$$\begin{bmatrix}\tilde{Q}^{(1)}\\ \tilde{P}^{(1)}\end{bmatrix}=\tilde{S}''\begin{bmatrix}q'\\ p'\end{bmatrix}=\tilde{S}''s\begin{bmatrix}q\\ p\end{bmatrix}=\tilde{S}\begin{bmatrix}q\\ p\end{bmatrix}. \tag{5·18}$$

Therefore, we have

$$\tilde{S}''^{-1}=s\cdot\tilde{S}^{-1}\,. \tag{5·19}$$

The matrix Λ given in Eq. (5·12a) is transformed as

$$\Lambda''=\Lambda\cdot s^{-1}\,. \tag{5·20}$$

With the use of Eqs. (5·19) and (5·20), we have

$$\tilde{S}''^{-1}\cdot\Lambda''=s\times\tilde{S}^{-1}\cdot\Lambda\times s^{-1}=0\,. \tag{5·21}$$

The matrix Λ_0 given in Eq. (5·15a) is invariant:

$$\Lambda_0''=\Lambda_0\,. \tag{5·22}$$

Therefore, we have

$$\tilde{S}''^{-1}\cdot\Lambda_0''=s\times\tilde{S}^{-1}\Lambda_0=0\,. \tag{5·23}$$

The relations (5·21) and (5·23) tell us that Eqs. (5·12) and (5·15) are invariant under the transformation (4·10b). Therefore, in the framework of Eqs. (5·12) and (5·15), we cannot specify intrinsic coordinate system, even if collective coordinate system is fixed. This means that some additional conditions may be also necessary.

Finally, we will give an important remark concerning Eqs. (3·28) which, up to the present stage, we have not made contact with: A possible solution of Eqs. (4·18) satisfies Eqs. (3·28). Summing the first equation and the transposed of the fourth in Eqs. (4·18), we can obtain

$$\left(\frac{\partial H}{\partial\alpha}\frac{\partial}{\partial\pi}-\frac{\partial H}{\partial\pi}\frac{\partial}{\partial\alpha}\right)(D^TA-C^TB)_{mn}=0\,. \tag{5·24}$$

General solution of the above partial differential equation is a function of α and π through arbitrary function of H. Therefore, as a possible solution, we can choose $(D^TA-C^TB)_{mn}=\delta_{mn}$. The other cases are also treated in the way similar to the above case.

§ 6. Specification of coordinate system

In the previous section, we have shown that the equations of collective submanifold are canonically invariant. This fact is quite natural, because the collective submanifold does not depend on the choice of its coordinate system. However, it is necessary to fix the coordinate system in one form in order to express the Hamiltonian in a concrete form. The main aim of this section is to discuss this problem.

First, we note H given in Eq. (4·3). This is a function of only α and π through $Q_r(\alpha\pi)$ and $P_r(\alpha\pi)$. Therefore, we can call H the collective Hamiltonian and, hereafter, we denote it as H_{coll}. Our important task is to obtain H_{coll} concretely. Practically, we have to obtain it successively from the lower order to the higher in the form of the power series expansion. Needless to say, we have to stop the expansion at a finite order. As for the expansion, we know the following two forms:

$$H_{\text{coll}}=\pi^2/2M_N+K_N\alpha^2/2+k_N\alpha\pi+\sum_{n=3}^{\infty}\sum_{l=0}^{n}h_{nl}\alpha^l\pi^{n-l}, \tag{6·1}$$

$$H_{\text{coll}}=V(\alpha)+\pi^2/2M(\alpha)+\sum_{n=3}^{\infty}h_n\pi^n. \tag{6·2}$$

Clearly, the expression (6·1) is based on the Taylor expansion for the two variables α and π. The expression (6·2) is written down as the Taylor expansion for only π. Therefore, the coeficients depend on α. In principle, there exists third possibility, i.e., the expansion in terms of α. However, physically, the third is not so interesting as the other two cases and we will not discuss this one. In the above expansions, we assumed that H_{coll} is stationary at the point $\alpha=\pi=0$. Let us investigate the structures of the expansions (6·1) and (6·2) from the viewpoint of canonical transformation. As was already mentioned, we have to stop the expansion at a finite power. Therefore, it is undesirable that the power, at which the expansion stops, changes if we view from another coordinate system. The expansion (6·1) does not change the power of α and π $(\alpha^l\pi^{n-l})$ under the symplectic transformation:

$$\begin{bmatrix}\alpha'\\ \pi'\end{bmatrix}=\begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}\alpha\\ \pi\end{bmatrix},\qquad (ad-bc=1) \tag{6·3}$$

where a, b, c and d are constants. On the other hand, the expansion (6·2) does not change the power of π (π^n) under the point transformation:

$$\alpha'=f(\alpha),\qquad \pi'=\pi\bigg/\frac{df(\alpha)}{d\alpha}, \tag{6·4}$$

where $f(\alpha)$ is an arbitrary function of α. From the above argument, it may be interesting to find methods, which are suitable for the two forms above, for solving the equations of collective submanifold.

Under the above preparation, let us investigate the first equation. The equation is canonically invariant and, then, some additional conditions are necessary in order to fix the collective coordinate system. With the aim of finding the conditions, we introduce the following relations:

$$\left.\begin{aligned}\sum_r P_r\frac{\partial Q_r}{\partial\alpha}-\frac{\partial W}{\partial\alpha}&=\pi\,,\\ \sum_r P_r\frac{\partial Q_r}{\partial\pi}-\frac{\partial W}{\partial\pi}&=0\,.\end{aligned}\right\}\tag{6·5}$$

The above relations come from the zero-th order of Eqs. (2·3a) for q_l and p_l ($l=1, 2, \cdots, N-1$). Of course, we used the correspondence (2·11). It is interesting to see that the relations (6·5) are reduced to Eq. (3·26) if W is eliminated. However, the inverse does not hold. This means that once W is fixed in one form, the resultant form is not always invariant for any canonical transformation. Therefore, we can fix a certain collective coordinate system by choosing a proper form of W a priori. Let W be of the following form:

$$W=\frac{1}{2}\left(\sum_r Q_rP_r-\alpha\pi\right).\tag{6·6}$$

Then, the relations (6·5) are reduced to

$$\left.\begin{aligned}\sum_r\left(P_r\frac{\partial Q_r}{\partial\alpha}-Q_r\frac{\partial P_r}{\partial\alpha}\right)&=\pi\,,\\ \sum_r\left(Q_r\frac{\partial P_r}{\partial\pi}-P_r\frac{\partial Q_r}{\partial\pi}\right)&=\alpha\,.\end{aligned}\right\}\tag{6·7}$$

We can prove that the relations (6·7) are invariant under the symplectic transformation (6·3). Therefore, if we obtain H_{coll} in the form (6·1), we should start from the relations (6·7). However, there still exists one ambiguity; the choice of the constants a, b, c and d (the three are free) in the transformation (6·3). In order to fix the ambiguity, we give the additional condition for the Hamiltonian (6·1):

$$M_N=1,\ K_N=\Omega_N,\quad k_N=0\,.\tag{6·8}$$

Here, Ω_N is a certain real number given later. Next, we consider the case

$$W=0\,.\tag{6·9}$$

In this case, the relations (6·5) become

$$\sum_r P_r\frac{\partial Q_r}{\partial\alpha}=\pi,\qquad \sum_r P_r\frac{\partial Q_r}{\partial\pi}=0\,.\tag{6·10}$$

The above relations are still invariant under the point transformation (6·4). Therefore, if we obtain H_{coll} given in Eq. (6·2), the condition (6·9) should be adopted. Here, there exists an ambiguity; the choice of $f(\alpha)$ in the point transformation (6·4). In order to fix the ambiguity, we give the following additional condition in the Hamiltonian (6·2):

$$M(\alpha)=1\,.\tag{6·11}$$

Thus, for the first equation of collective submanifold, we could prepare the conditions

(6·7) and (6·8) for the symplectic transformation and the conditions (6·10) and (6·11) for the point transformation. Both are suitable for $H_{\rm coll}$ given by Eqs. (6·1) and (6·2), respectively.

As was already discussed, the second equation of collective submanifold is invariant under the symplectic transformation (4·10b) in addition to the same canonical invariance as that for the first equation. Therefore, even if the collective coordinate system is fixed, there still exists ambiguity for the intrinsic coordinate system. In order to fix the coordinate system, let us consider $H^{(2)}$ given in Eq. (4·5). With the use of Eqs. (3·29a), $H^{(2)}$ can be written down as

$$H^{(2)}=\frac{1}{2}\sum_{mn=1}^{N-1}[u_{mn}(\alpha\pi)p_m p_n+v_{mn}(\alpha\pi)q_m q_n+2w_{mn}(\alpha\pi)q_m p_n]. \tag{6·12}$$

This can be decomposed into

$$H^{(2)}=H_{\rm intr}+H_{\rm coupl}, \tag{6·13}$$

$$H_{\rm intr}=\frac{1}{2}\sum_{mn=1}^{N-1}[u_{mn}(00)p_m p_n+v_{mn}(00)q_m q_n+2w_{mn}(00)q_m p_m], \tag{6·13a}$$

$$H_{\rm coupl}=\frac{1}{2}\sum_{mn=1}^{N-1}[\{u_{mn}(\alpha\pi)-u_{mn}(00)\}p_m p_n + \{v_{mn}(\alpha\pi)-v_{mn}(00)\}q_m q_n+2\{w_{mn}(\alpha\pi)-w_{mn}(00)\}q_m p_n]. \tag{6·13b}$$

The above $H_{\rm intr}$ is invariant under the symplectic transformation (4·10b). Therefore, there exists one ambiguity: The $2(N-1)^2+(N-1)$ parameters are free. Then, we set up the following conditions for fixing the intrinsic coordinate system:

$$u_{mn}(00)=\delta_{mn},\quad v_{mn}(00)=\Omega_n\delta_{mn},\quad w_{mn}(00)=0. \tag{6·14}$$

$$(m,\ n=1,\ 2,\ \cdots,\ N-1)$$

Therefore, $H_{\rm intr}$ can be expressed as

$$H_{\rm intr}=\frac{1}{2}\sum_{n=1}^{N-1}(p_n{}^2+\Omega_n q_n{}^2). \tag{6·15}$$

Here, Ω_n are certain real numbers, which will be given later. The reason why we used the indices, intr and coupl, may be self-evident. Thus, with the use of the conditions (6·14), we can solve the second equation.

The equations of collective submanifold are partial differential equations. Therefore, for solving them, the boundary conditions are necessary. As for the conditions, we consider the small amplitude limit. We adopt the following Hamiltonian approximated in the bilinear form of $Q_r{}^0$ and $P_r{}^0$:

$$H^0=\frac{1}{2}\sum_{rs=1}^{N}(U_{rs}P_r{}^0P_s{}^0+V_{rs}Q_r{}^0Q_s{}^0). \tag{6·16}$$

On the other hand, $H_{\rm coll}$ may be given by

$$H_{\rm coll}=\pi^2/2+\Omega_N\alpha^2/2. \tag{6·17}$$

Then, the first equation can be written as

$$\left.\begin{aligned}\Omega_N\alpha\cdot\frac{\partial P_r}{\partial\pi}-\pi\cdot\frac{\partial P_r}{\partial\alpha}=\sum_s V_{rs}Q_s\,,\\ -\Omega_N\alpha\cdot\frac{\partial Q_r}{\partial\pi}+\pi\cdot\frac{\partial Q_r}{\partial\alpha}=\sum_s U_{rs}P_s\,.\end{aligned}\right\}\tag{6·18}$$

The above equation is reduced to the following, together with Eqs. (6·7) and (6·10):

$$\left.\begin{aligned}\Omega_N\psi_r=\sum_s V_{rs}\phi_s\,,\\ \phi_r=\sum_s U_{rs}\psi_s\,,\end{aligned}\right\}\tag{6·19}$$

$$\sum_r\phi_r\psi_r=1\,.\tag{6·20}$$

Here, ϕ_r and ψ_r are given by

$$Q_r=\phi_r\cdot\alpha\,,\qquad P_r=\psi_r\cdot\pi\,.\tag{6·21}$$

It should be noted that Eqs. (6·7) and (6·10) are reduced to the common relation (6·20). Equation (6·19) is nothing but the equation of RPA and Eq. (6·20) is the normalization condition for the amplitudes. Clearly, if Ω_N is positive, it is the square of the frequency. In our treatment, Ω_N is not necessarily positive. We pick up a solution from those of Eq. (6·19), which we denote as Ω_N. This is the boundary condition for the first equation. Then, we can understand that Ω_N given in Eqs. (6·8) is nothing but the solution of RPA. In the case of the point transformation, $V(\alpha)$ is reduced to $\Omega_N\alpha^2/2$ for the small α.

Next, we will investigate the second equation. The boundary condition of the first is a solution of the following RPA equation which is copied from Eq. (6·19):

$$\left.\begin{aligned}\Omega\Psi_r=\sum_s V_{rs}\Phi_s\,,\\ \Phi_r=\sum_s U_{rs}\Psi_s\,.\end{aligned}\right\}\tag{6·22}$$

The above equation has N independent solutions, which we discriminate by indices n ($n=1, 2, \cdots, N-1, N$) such as $\Phi_r^{(n)}$. The index N corresponds to the boundary condition for the first equation. From Eq. (6·22), we can get

$$\sum_r\Phi_r^{(n)}\Psi_r^{(m)}=0\,.\qquad(n\neq m)\tag{6·23}$$

With the use of Eqs. (6·21), the relations (3·27) are reduced to

$$\left.\begin{aligned}\sum_r D_{rl}\phi_r=\sum_r D_{rl}\Phi_r^{(N)}=0,\quad \sum_r C_{rl}\psi_r=\sum_r C_{rl}\Psi_r^{(N)}=0\,,\\ \sum_r B_{rl}\phi_r=\sum_r B_{rl}\Phi_r^{(N)}=0\,,\quad \sum_r A_{rl}\psi_r=\sum_r A_{rl}\Psi_r^{(N)}=0\,.\end{aligned}\right\}\tag{6·24}$$

$$(l=1, 2, \cdots, N-1)$$

From the comparison of Eq. (6·23) with Eqs. (6·24), we have

$$A_{rl}=\Phi_r^{(l)},\quad B_{rl}=C_{rl}=0\,,\quad D_{rl}=\Psi_r^{(l)}\,. \tag{6·25}$$

$$(l=1,\,2,\,\cdots,\,N-1)$$

Clearly, we should adopt the solution of Eq. (6·22) as Ω_n appearing in Eqs. (6·14). Since A_{rl} etc. in the RPA limit do not depend on α and π, they satisfy the second equation identically. Thus, we could prepare the boundary condition. Judging from the principle of our approximation, Ω_n $(n\neq N)$ may be positive. Further, we should not forget the conditions mentioned in the final part of §5.

Finally, we will give some concluding remarks. In this paper, we developed a possible method, with the help of which we can describe large amplitude collective motion and its coupling with intrinsic degrees of freedom. The essence is to solve two kinds of equations of collective submanifold. After solving the equations, we can obtain the Hamiltonian expressed in terms of the collective and the intrinsic variables. The other any physical quantity X^0 as a function of Q_r^0 and P_r^0 $(X^0=X^0(Q^0,\,P^0))$ can be expressed as

$$X^0=X+\tilde{X}^{(1)}\,, \tag{6·26}$$

$$X=X^0(Q,\,P), \tag{6·26a}$$

$$\tilde{X}^{(1)}=\sum_{l=1}^{N-1}\left[\sum_{r=1}^{N}\left(\frac{\partial X}{\partial Q_r}\cdot A_{rl}+\frac{\partial X}{\partial P_r}\cdot B_{rl}\right)\cdot q_l+\sum_{r=1}^{N}\left(\frac{\partial X}{\partial Q_r}\cdot C_{rl}+\frac{\partial X}{\partial P_r}\cdot D_{rl}\right)\cdot p_l\right]. \tag{6·26b}$$

It should be noted that the higher order than the linear for q_l and p_l loses its meaning under the present approximation. In this paper, we treated the case of one collective degree of freedom and, further, the coupling between the collective and the intrinsic degrees of freedom in the first order. The generalization may be straightforward. The present theory aimed at the description of many-fermion system. However, as is clear from the treatment, we can describe the cases of many-boson system and the system of classical particles with the use of the present theory.

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