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Research Article

Generalization of Hermite-Hadamard Type Inequalities via Conformable Fractional Integrals

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We establish a Hermite-Hadamard type identity and several new Hermite-Hadamard type inequalities for conformable fractional integrals and present their applications to special bivariate means.

1. Introduction

In the field of nonlinear programming and optimization theory, no one can ignore the role of convex sets and convex functions. For the class of convex functions, many inequalities have been introduced such as Jensen's, Hermite-Hadmard, and Slater's inequalities. Among those inequalities, the most famous and important inequality is the Hermite-Hadamard's inequality [1] which can be stated as follows.

Let $I \subseteq \mathbb{R}$ be an interval and $h : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on I. Then the double inequality

$$h\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} h(x)\,dx \le \frac{h(a_1) + h(a_2)}{2} \qquad (1)$$

holds for all $a_1, a_2 \in I$ with $a_1 < a_2$. Both inequalities in (1) hold in the reverse direction if the function h is concave on I.

In the last 60 years, many efforts have gone on generalizations, extensions, variants, and applications for the Hermite-Hadamard's inequality (see [2–13]). Anderson [14] and Sarikaya et al. [15] provide the important variants for the Hermite-Hadamard's inequality.

Recently, the author in [16] gave a new definition for the (conformable) fractional derivative as follows.

Let $0 < \alpha \le 1$ and $h : [0, \infty) \longrightarrow \mathbb{R}$ be a real-valued function. Then the α-order (conformable) fractional derivative of h at $s > 0$ is defined by

$$D_\alpha(h)(s) = \lim_{\epsilon \longrightarrow 0} \frac{h\left(s + \epsilon s^{1-\alpha}\right) - h(s)}{\epsilon}. \qquad (2)$$

h is said to be α-differentiable if the α-order (conformable) fractional derivative of h exists, and the α-order (conformable) fractional derivative of h at 0 is defined as $h^{\alpha}(0) = \lim_{s \longrightarrow 0^+} h^{\alpha}(s)$.

Now we discuss some theorems for the (conformable) fractional derivative.

Theorem 1. *Let $\alpha \in (0, 1]$ and h_1, h_2 be α-differentiable at $s > 0$. Then one has the following:*

(i) $(d_\alpha/d_\alpha s)(s^n) = n s^{n-\alpha}$ *for all $n \in \mathbb{R}$.*

(ii) $(d_\alpha/d_\alpha s)(c) = 0$ *for all constants $c \in \mathbb{R}$.*

(iii) $(d_\alpha/d_\alpha s)(a_1 h_1(s) + a_2 h_2(s)) = a_1 (d_\alpha/d_\alpha s)(h_1(s)) + a_2 (d_\alpha/d_\alpha s)(h_2(s))$ *for all constants $a_1, a_2 \in \mathbb{R}$.*

(iv) $(d_\alpha/d_\alpha s)(h_1(s) h_2(s)) = h_1(s)(d_\alpha/d_\alpha s)(h_2(s)) + h_2(s)(d_\alpha/d_\alpha s)(h_1(s))$.

(v) $(d_\alpha/d_\alpha s)(h_1(s)/h_2(s)) = (h_2(s)(d_\alpha/d_\alpha s)(h_1(s)) - h_1(s)(d_\alpha/d_\alpha s)(h_2(s)))/(h_2(s))^2$.

(vi) $(d_\alpha/d_\alpha s)(h_1(h_2(s))) = h_1'(h_2(s))(d_\alpha/d_\alpha s)(h_2(s))$ *if* h_1 *is differentiable at* $h_2(s)$.

In addition,

$$\frac{d_\alpha}{d_\alpha s}(h_1(s)) = s^{1-\alpha}\frac{d}{ds}(h_1(s)) \tag{3}$$

if h_1 *is differentiable.*

Definition 2 (conformable fractional integral). Let $\alpha \in (0,1]$ and $0 \le a_1 < a_2$. Then the function $h_1 : [a_1,a_2] \longrightarrow \mathbb{R}$ is said to be α-fractional integrable on $[a_1,a_2]$ if the integral

$$\int_{a_1}^{a_2} h_1(x)\, d_\alpha x := \int_{a_1}^{a_2} h_1(x)\, x^{\alpha-1} dx \tag{4}$$

exists and is finite. All α-fractional integrable functions on $[a_1,a_2]$ are indicated by $L_\alpha([a_1,a_2])$.

Remark 3.

$$I_\alpha^{a_1}(h_1)(s) = I_1^{a_1}(s^{\alpha-1}h_1) = \int_{a_1}^{s}\frac{h_1(x)}{x^{1-\alpha}}dx, \tag{5}$$

where the integral is the usual Riemann improper integral and $\alpha \in (0,1]$.

Recently, the conformable integrals and derivatives have attracted the attention of many researchers, and many remarkable properties and inequalities for the conformable integrals and derivatives can be found in the literature [17–24]. Anderson [14] found the conformable integral version of the Hermite-Hadamard inequality as follows.

Theorem 4 (see [14]). *If* $\alpha \in (0,1]$ *and* $h_1 : [a_1,a_2] \longrightarrow \mathbb{R}$ *is an* α*-fractional differentiable function such that* $D_\alpha(h)$ *is increasing, then we have the following inequality:*

$$\frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)\, d_\alpha x \le \frac{h(a_1)+h(a_2)}{2}. \tag{6}$$

Moreover if the function h is decreasing on $[a_1,a_2]$, *then we have*

$$h\left(\frac{a_1+a_2}{2}\right) \le \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)\, d_\alpha x. \tag{7}$$

If $\alpha = 1$, *then inequalities (6) and (7) reduce to the classical Hermite-Hadamard's inequalities.*

The main purpose of the article is to present the conformable fractional integrals version of the Hermite-Hadamard's inequality. We first establish an identity for the conformable fractional integrals (Lemma 5) and discuss their special cases. Then applying Jensen's inequality, power mean inequality, Hölder inequality, the convexity of the functions $x^{\alpha-1}$ and $-x^\alpha$ $(x > 0, \alpha \in (0,1])$, and the identity given by Lemma 5, we obtain inequalities for conformable fractional integrals version of the Hermite-Hadamard's inequality. At last, using particular classes of convex functions we find several new inequalities for some special bivariate means. For some related results, see [25, 26].

2. Main Results

The main results of our work can be calculated with the help of the following lemma associated with inequality (8).

Lemma 5. *Let* $a_1, a_2 \in \mathbb{R}^+$ *with* $a_1 < a_2$, $\alpha \in (0,1]$, *and* $h : [a_1,a_2] \longrightarrow \mathbb{R}$ *be an* α*-fractional differentiable function on* (a_1,a_2). *Then the identity*

$$\begin{aligned}
&\frac{(a_2^\alpha - \eta^\alpha)h(a_2) + (\eta^\alpha - a_1^\alpha)h(a_1)}{a_2^\alpha - a_1^\alpha} - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \\
&\quad\cdot \int_{a_1}^{a_2} h(s)\, d_\alpha s \\
&\quad= \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\left[\int_0^1 \left(((1-t)a_1 + t\eta)^{2\alpha-1}\right.\right. \\
&\quad - \eta^\alpha((1-t)a_1 + t\eta)^{\alpha-1}\Big) \times D_\alpha(h)((1-t)a_1 + t\eta) \\
&\quad \cdot t^{1-\alpha} d_\alpha t\Big] + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\left[\int_0^1 \left(((1-t)a_2 + t\eta)^{2\alpha-1}\right.\right. \\
&\quad - \eta^\alpha((1-t)a_2 + t\eta)^{\alpha-1}\Big) \times D_\alpha(h)((1-t)a_2 + t\eta) \\
&\quad \cdot t^{1-\alpha} d_\alpha t\Big]
\end{aligned} \tag{8}$$

holds for any $\eta \in [a_1,a_2]$ *if* $D_\alpha(h) \in L_\alpha([a_1,a_2])$.

Proof. It follows from Theorem 1, Definition 2, and integrating by parts that

$$\begin{aligned}
&\frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_1 + t\eta)^{2\alpha-1}\right. \\
&\quad - \eta^\alpha((1-t)a_1 + t\eta)^{\alpha-1}\Big) D_\alpha(h)((1-t)a_1 + t\eta)\, dt \\
&\quad + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_2 + t\eta)^{2\alpha-1}\right. \\
&\quad - \eta^\alpha((1-t)a_2 + t\eta)^{\alpha-1}\Big) D_\alpha(h)((1-t)a_2 + t\eta)\, dt \\
&\quad = \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_1 + t\eta)^\alpha - \eta^\alpha\right) h'((1-t)a_1 \\
&\quad + t\eta)\, dt + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha\right. \\
&\quad - \eta^\alpha\Big) h'((1-t)a_2 + t\eta)\, dt \\
&= \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\left[\left(((1-t)a_1 + t\eta)^\alpha - \eta^\alpha\right)\right. \\
&\quad \cdot \left.\frac{h((1-t)a_1 + t\eta)}{\eta - a_1}\right|_0^1 - \int_0^1 \alpha((1-t)a_1 + t\eta)^{\alpha-1} \\
&\quad \cdot \left.(\eta - a_1)\frac{h((1-t)a_1 + t\eta)}{\eta - a_1}dt\right]
\end{aligned} \tag{9}$$

$$+\frac{a_2-\eta}{a_2^\alpha-a_1^\alpha}\left[\left(\left((1-t)a_2+t\eta\right)^\alpha-\eta^\alpha\right)\cdot\frac{h\left((1-t)a_2+t\eta\right)}{\eta-a_2}\Bigg|_0^1-\int_0^1\alpha\left((1-t)a_2+t\eta\right)^{\alpha-1}\cdot(\eta-a_2)\frac{h\left((1-t)a_2+t\eta\right)}{\eta-a_2}dt\right]$$
$$=\frac{\eta-a_1}{a_2^\alpha-a_1^\alpha}\left[\frac{\eta^\alpha-a_1^\alpha}{\eta-a_1}h(a_1)-\frac{\alpha}{\eta-a_1}\int_{a_1}^{\eta}h(s)\,d_\alpha s\right]+\frac{a_2-\eta}{a_2^\alpha-a_1^\alpha}\left[\frac{a_2^\alpha-\eta^\alpha}{a_2-\eta}h(a_2)-\frac{\alpha}{a_2-\eta}\int_{\eta}^{a_2}h(s)\,d_\alpha s\right]$$
$$=\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha}-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\cdot\int_{a_1}^{a_2}h(s)\,d_\alpha s, \tag{10}$$

where we have used the changes of variable $s=(1-t)a_1+t\eta$ and $s=(1-t)a_2+t\eta$ to get the desired result. □

Remark 6. Let $\alpha=1$. Then identity (8) becomes

$$\frac{(a_2-\eta)h(a_2)+(\eta-a_1)h(a_1)}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(s)\,ds=\frac{(\eta-a_1)^2}{a_2-a_1}\int_0^1(t-1)h'\left((1-t)a_1+t\eta\right)dt+\frac{(a_2-\eta)^2}{a_2-a_1}\int_0^1(1-t)h'\left((1-t)a_2+t\eta\right)dt, \tag{11}$$

which was proved by Kavurmaci et al. in [2].

Theorem 7. *Let $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$, $\alpha\in(0,1]$, and $h:[a_1,a_2]\longrightarrow\mathbb{R}$ be an α-differentiable function on (a_1,a_2). Then the inequality*

$$\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha}-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)\,d_\alpha s\right|\le\frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\left[\frac{1}{2}\eta^\alpha\left|h'(a_1)\right|-\frac{1}{3}a_1^\alpha\left|h'(a_1)\right|-\frac{1}{6}\eta^\alpha\left|h'(a_1)\right|+\frac{1}{2}\eta^\alpha\left|h'(\eta)\right|-\frac{1}{6}a_1^\alpha\left|h'(\eta)\right|-\frac{1}{3}\eta^\alpha\left|h'(\eta)\right|\right]$$
$$+\frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha}\left[\frac{1}{4}a_2^\alpha\left|h'(a_2)\right|+\frac{1}{12}\eta^{\alpha-1}a_2\left|h'(a_2)\right|+\frac{1}{12}\eta a_2^{\alpha-1}\left|h'(a_2)\right|+\frac{1}{12}\eta^\alpha\left|h'(a_2)\right|-\frac{1}{2}\eta^\alpha\left|h'(a_2)\right|+\frac{1}{12}a_2^\alpha\left|h'(\eta)\right|+\frac{1}{12}\eta^{\alpha-1}a_2\left|h'(\eta)\right|+\frac{1}{12}\eta a_2^{\alpha-1}\left|h'(\eta)\right|+\frac{1}{4}\eta^\alpha\left|h'(\eta)\right|-\frac{1}{2}\eta^\alpha\left|h'(\eta)\right|\right] \tag{12}$$

holds for any $\eta\in[a_1,a_2]$ if $D_\alpha(h)\in L_\alpha([a_1,a_2])$ and $|h'|$ is convex on $[a_1,a_2]$.

Proof. Let $x>0$, $\varphi_1(x)=x^{\alpha-1}$, and $\varphi_2(x)=-x^\alpha$. Then we clearly see that both the functions $\varphi_1(x)$ and $\varphi_2(x)$ are convex. From Lemma 5 and the convexity of φ_1, φ_2, and $|h'|$, we have

$$\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha}-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\cdot\int_{a_1}^{a_2}h(s)\,d_\alpha s\right|\le\frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\eta^\alpha-\left((1-t)a_1+t\eta\right)^\alpha\right)\cdot\left|h'\left((1-t)a_1+t\eta\right)\right|dt+\frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\left((1-t)a_2+t\eta\right)^\alpha-\eta^\alpha\right)\cdot\left|h'\left((1-t)a_2+t\eta\right)\right|dt$$
$$=\frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\eta^\alpha-\left((1-t)a_1+t\eta\right)^\alpha\right)\cdot\left|h'\left((1-t)a_1+t\eta\right)\right|dt+\frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\left((1-t)a_2+t\eta\right)^{\alpha+1-1}-\eta^\alpha\right)\cdot\left|h'\left((1-t)a_2+t\eta\right)\right|dt$$
$$\le\frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\cdot\left|h'\left((1-t)a_1+t\eta\right)\right|dt+\frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\left((1-t)a_2+t\eta\right)^{\alpha-1}\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)\cdot\left|h'\left((1-t)a_2+t\eta\right)\right|dt$$
$$\le\frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)$$

$$\cdot \left|h'\left((1-t)a_1+t\eta\right)\right| dt + \frac{(a_2-\eta)}{a_2^\alpha - a_1^\alpha} \cdot \int_0^1 \left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right) \cdot \left|h'\left((1-t)a_2+t\eta\right)\right| dt \le \frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha} \cdot \int_0^1 \left(\eta^\alpha - \left((1-t)a_1^\alpha+t\eta^\alpha\right)\right) \cdot \left[(1-t)\left|h'(a_1)\right|+t\left|h'(\eta)\right|\right] dt + \frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha} \cdot \int_0^1 \left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right) \cdot \left[(1-t)\left|h'(a_2)\right|+t\left|h'(\eta)\right|\right] dt. \quad (13)$$

From the final upper bound above, we have the following:

$$\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha} - \frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2} h(s)\,d_\alpha s\right| \le \frac{(\eta-a_1)}{a_2^\alpha-a_1^\alpha}\left[\frac{1}{2}\eta^\alpha\left|h'(a_1)\right| - \frac{1}{3}a_1^\alpha\left|h'(a_1)\right| - \frac{1}{6}\eta^\alpha\left|h'(a_1)\right| + \frac{1}{2}\eta^\alpha\left|h'(\eta)\right| - \frac{1}{6}a_1^\alpha\left|h'(\eta)\right| - \frac{1}{3}\eta^\alpha\left|h'(\eta)\right|\right] + \frac{(a_2-\eta)}{a_2^\alpha-a_1^\alpha}\left[\frac{1}{4}a_2^\alpha\left|h'(a_2)\right| + \frac{1}{12}\eta^{\alpha-1}a_2\left|h'(a_2)\right| + \frac{1}{12}\eta a_2^{\alpha-1}\left|h'(a_2)\right| + \frac{1}{12}\eta^\alpha\left|h'(a_2)\right| - \frac{1}{2}\eta^\alpha\left|h'(a_2)\right| + \frac{1}{12}a_2^\alpha\left|h'(\eta)\right| + \frac{1}{12}\eta^{\alpha-1}a_2\left|h'(\eta)\right| + \frac{1}{12}\eta a_2^{\alpha-1}\left|h'(\eta)\right| + \frac{1}{4}\eta^\alpha\left|h'(\eta)\right| - \frac{1}{2}\eta^\alpha\left|h'(\eta)\right|\right]. \quad (14)$$

□

Remark 8. Let $\alpha = 1$. Then inequality (12) leads to

$$\left|\frac{(\eta-a_1)h(a_1)+(a_2-\eta)h(a_2)}{a_2-a_1} - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)\,dx\right| \le \frac{(\eta-a_1)^2}{a_2-a_1}\left[\frac{\left|h'(\eta)\right|+2\left|h'(a_1)\right|}{6}\right] + \frac{(a_2-\eta)^2}{a_2-a_1}\left[\frac{\left|h'(\eta)\right|+2\left|h'(a_2)\right|}{6}\right], \quad (15)$$

which was proved by Kavurmaci et al. in [2].

Theorem 9. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_2 > a_1$, $\alpha \in (0,1]$, $p, q > 1$ such that $1/p + 1/q = 1$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha} - \frac{\alpha}{a_2^\alpha-a_1^\alpha}\cdot\int_{a_1}^{a_2} h(s)\,d_\alpha s\right| \le \frac{\eta-a_1}{a_2^\alpha-a_1^\alpha}\left[\left(A_1(\alpha,p)\right)^{1/p}\cdot\left(\frac{\left|h'(a_1)\right|^q+\left|h'(\eta)\right|^q}{2}\right)^{1/q}\right] + \frac{a_2-\eta}{a_2^\alpha-a_1^\alpha}\left[\left(B_1(\alpha,p)\right)^{1/p}\cdot\left(\frac{\left|h'(a_2)\right|^q+\left|h'(\eta)\right|^q}{2}\right)^{1/q}\right] \quad (16)$$

holds for any $\eta \in [a_1, a_2]$ if $D_\alpha(h) \in L_\alpha([a_1, a_2])$ and $|h'|^q$ is convex on $[a_1, a_2]$, where

$$A_1(\alpha,p) = \int_0^1 \left(\eta^\alpha - \left(\left((1-t)a_1^\alpha + t\eta^\alpha\right)\right)\right)^p dt,$$
$$B_1(\alpha,p) = \int_0^1 \left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right) - \eta^\alpha\right)^p dt. \quad (17)$$

Proof. It follows from inequality (13) that

$$\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha} - \frac{\alpha}{a_2^\alpha-a_1^\alpha}\cdot\int_{a_1}^{a_2} h(s)\,d_\alpha s\right| \le \frac{\eta-a_1}{a_2^\alpha-a_1^\alpha}\cdot\int_0^1 \left(\eta^\alpha - \left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\cdot\left|h'\left((1-t)a_1+t\eta\right)\right| dt + \frac{a_2-\eta}{\eta^\alpha-a_1^\alpha}$$

$$\cdot \int_0^1 \left(\left((1-t) a_2^{\alpha-1} + t\eta^{\alpha-1} \right) ((1-t) a_2 + t\eta) - \eta^{\alpha} \right)$$
$$\cdot \left| h'((1-t) a_2 + t\eta) \right| dt. \tag{18}$$

Making use of Hölder's inequality, one has

$$\begin{aligned}
&\int_0^1 (\eta^{\alpha} - ((1-t) a_1^{\alpha} + t\eta^{\alpha})) \left| h'((1-t) a_1 + t\eta) \right| dt \\
&\le \left(\int_0^1 (\eta^{\alpha} - ((1-t) a_1^{\alpha} + t\eta^{\alpha}))^p dt \right)^{1/p} \\
&\cdot \left(\int_0^1 \left| h'((1-t) a_1 + t\eta) \right|^q dt \right)^{1/q} \\
&\le (A_1(\alpha, p))^{1/p} \\
&\cdot \left(\int_0^1 \left((1-t) \left| h'(a_1) \right|^q + t \left| h'(\eta) \right|^q \right) dt \right)^{1/q} \\
&= (A_1(\alpha, p))^{1/p} \left(\frac{\left| h'(a_1) \right|^q + \left| h'(\eta) \right|^q}{2} \right)^{1/q}.
\end{aligned} \tag{19}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 \left(\left((1-t) a_2^{\alpha-1} + t\eta^{\alpha-1} \right) ((1-t) a_2 + t\eta) - \eta^{\alpha} \right) \\
&\cdot \left| h'((1-t) a_2 + t\eta) \right| dt \\
&\le \left(\int_0^1 \left(\left((1-t) a_2^{\alpha-1} + t\eta^{\alpha-1} \right) ((1-t) a_2 + t\eta) \right.\right. \\
&\left.\left. - \eta^{\alpha} \right)^p dt \right)^{1/p} \\
&\times \left(\int_0^1 \left| h'((1-t) a_2 + t\eta) \right|^q dt \right)^{1/q} \le (B_1(\alpha, \\
&p))^{1/p} \left(\int_0^1 \left((1-t) \left| h'(a_2) \right|^q + t \left| h'(\eta) \right|^q \right) dt \right)^{1/q} \\
&\le (B_1(\alpha, p))^{1/p} \left(\frac{\left| h'(a_2) \right|^q + \left| h'(\eta) \right|^q}{2} \right)^{1/q}.
\end{aligned} \tag{20}$$

Hence, we have the result in (16). □

Remark 10. By putting $\alpha = 1$ in (16), we obtain the inequality

$$\begin{aligned}
&\left| \frac{(\eta - a_1) h(a_1) + (a_2 - \eta) h(a_2)}{a_2 - a_1} - \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} h(x) dx \right| \\
&\le \left(\frac{1}{p+1} \right)^{1/p} \left(\frac{1}{2} \right)^{1/q} \left[\frac{(\eta - a_1)^2 \left[\left| h'(\eta) \right|^q + \left| h'(a_1) \right|^q \right]^{1/q} + (a_2 - \eta)^2 \left[\left| h'(\eta) \right|^q + \left| h'(a_2) \right|^q \right]^{1/q}}{a_2 - a_1} \right],
\end{aligned} \tag{21}$$

which was proved by Kavurmaci et al. in [2].

Theorem 11. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$, $\alpha \in (0,1]$, $q > 1$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\begin{aligned}
&\left| \frac{(a_2^{\alpha} - \eta^{\alpha}) h(a_2) + (\eta^{\alpha} - a_1^{\alpha}) h(a_1)}{a_2^{\alpha} - a_1^{\alpha}} - \frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \right. \\
&\left. \cdot \int_{a_1}^{a_2} h(s) d_{\alpha}s \right| \le \frac{\eta - a_1}{a_2^{\alpha} - a_1^{\alpha}} \left[(A_1(\alpha))^{1-1/q} \right. \\
&\left. \cdot \left\{ A_2(\alpha) \left| h'(a_1) \right|^q + A_3(\alpha) \left| h'(\eta) \right|^q \right\}^{1/q} \right] \\
&+ \frac{a_2 - \eta}{a_2^{\alpha} - a_1^{\alpha}} \left[(B_1(\alpha))^{1-1/q} \right. \\
&\left. \cdot \left\{ B_2(\alpha) \left| h'(a_2) \right| + B_3(\alpha) \left| h'(\eta) \right| \right\}^{1/q} \right]
\end{aligned} \tag{22}$$

holds for any $\eta \in [a_1, a_2]$ if $D_{\alpha}(h) \in L_{\alpha}([a_1, a_2])$ and $|h'|^q$ is convex on $[a_1, a_2]$, where

$$\begin{aligned}
A_1(\alpha) &= \frac{\eta^{\alpha} - a_1^{\alpha}}{2}, \\
B_1(\alpha) &= \frac{2a_2^{\alpha} + \eta^{\alpha-1} a_2 + \eta a_2^{\alpha-1} - 4\eta^{\alpha}}{6}, \\
A_2(\alpha) &= \frac{\eta^{\alpha} - a_1^{\alpha}}{3}, \\
A_3(\alpha) &= \frac{\eta^{\alpha} - a_1^{\alpha}}{6}, \\
B_2(\alpha) &= \frac{3a_2^{\alpha} + \eta^{\alpha-1} a_2 + \eta a_2^{\alpha-1} - 5\eta^{\alpha}}{12}, \\
B_3(\alpha) &= \frac{a_2^{\alpha} + \eta^{\alpha-1} a_2 + \eta a_2^{\alpha-1} - 3\eta^{\alpha}}{12}.
\end{aligned} \tag{23}$$

Proof. It follows from inequality (13) that

$$\left| \frac{(a_2^{\alpha} - \eta^{\alpha}) h(a_2) + (\eta^{\alpha} - a_1^{\alpha}) h(a_1)}{a_2^{\alpha} - a_1^{\alpha}} - \frac{\alpha}{a_2^{\alpha} - a_1^{\alpha}} \cdot \int_{a_1}^{a_2} h(s) d_{\alpha}s \right| \le \frac{\eta - a_1}{a_2^{\alpha} - a_1^{\alpha}}$$

$$\cdot \int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) \cdot \left|h'((1-t)a_1 + t\eta)\right| dt + \frac{a_2-\eta}{\eta^\alpha - a_1^\alpha} \cdot \int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) \cdot \left|h'((1-t)a_2 + t\eta)\right| dt. \tag{24}$$

Making use of the power mean inequality, we get

$$\begin{aligned}&\int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) \left|h'((1-t)a_1 + t\eta)\right| dt \\ &\le \left(\int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) dt\right)^{1-1/q} \times \left(\int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) \cdot \left|h'((1-t)a_1 + t\eta)\right|^q dt\right)^{1/q}.\end{aligned} \tag{25}$$

Similarly, we have

$$\begin{aligned}&\int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) \cdot \left|h'((1-t)a_2 + t\eta)\right| dt \\ &\le \left(\int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) dt\right)^{1-1/q} \times \left(\int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right) \cdot ((1-t)a_2 + t\eta) - \eta^\alpha\right) \cdot \left|h'((1-t)a_2 + t\eta)\right|^q dt\right)^{1/q}.\end{aligned} \tag{26}$$

From the convexity of $|h'|^q$, we have

$$\begin{aligned}&\int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) \left|h'((1-t)a_1 + t\eta)\right|^q dt \\ &\le \int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) \cdot \left[(1-t)\left|h'(a_1)\right|^q + t\left|h'(\eta)\right|^q\right] dt = \left|h'(a_1)\right|^q \cdot \int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha))(1-t) dt + \left|h'(\eta)\right|^q \\ &\cdot \int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) t dt = \left|f'(a_1)\right|^q \left(\frac{1}{2} \cdot \eta^\alpha - \frac{1}{3}a_1^\alpha - \frac{1}{6}\eta^\alpha\right) + \left|h'(\eta)\right|^q \left(\frac{1}{2}\eta^\alpha - \frac{1}{6}a_1^\alpha - \frac{1}{3} \cdot \eta^\alpha\right)\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) \cdot \left|h'((1-t)a_2 + t\eta)\right|^q dt \\ &\le \int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) \left[(1-t)\left|h'(a_2)\right|^q + t\left|h'(\eta)\right|^q\right] dt \\ &= \left|h'(a_2)\right|^q \left(\frac{1}{4}a_2^\alpha + \frac{1}{12}\eta^{\alpha-1}a_2 + \frac{1}{12}\eta a_2^{\alpha-1} + \frac{1}{12}\eta^\alpha - \frac{1}{2}\eta^\alpha\right) + \left|h'(\eta)\right|^q \left(\frac{1}{12}a_2^\alpha + \frac{1}{12}\eta^{\alpha-1}a_2 + \frac{1}{12}\eta a_2^{\alpha-1} + \frac{1}{4}\eta^\alpha - \frac{1}{2}\eta^\alpha\right),\end{aligned} \tag{28}$$

where we have also used the facts that

$$\begin{aligned}&\int_0^1 (\eta^\alpha - ((1-t)a_1^\alpha + t\eta^\alpha)) dt = A_1(\alpha) = \eta^\alpha - \frac{1}{2}a_1^\alpha - \frac{1}{2}\eta^\alpha, \\ &\int_0^1 \left(\left((1-t)a_2^{\alpha-1} + t\eta^{\alpha-1}\right)((1-t)a_2 + t\eta) - \eta^\alpha\right) dt = B_1(\alpha) = \frac{1}{3}a_2^\alpha + \frac{1}{6}\eta^{\alpha-1}a_2 + \frac{1}{6}\eta a_2^{\alpha-1} + \frac{1}{3}\eta^\alpha - \eta^\alpha.\end{aligned} \tag{29}$$

Hence, we have the result in (22). □

Remark 12. If $\alpha = 1$, then inequality (22) becomes

$$\begin{aligned}&\left|\frac{(\eta - a_1)h(a_1) + (a_2 - \eta)h(a_2)}{a_2 - a_1} - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} h(x)\,dx\right| \\ &\le \frac{1}{2}\left(\frac{1}{3}\right)^{1/q}\left[\frac{(\eta - a_1)^2\left[\left|h'(\eta)\right|^q + 2\left|h'(a_1)\right|^q\right]^{1/q} + (a_2 - \eta)^2\left[\left|h'(\eta)\right|^q + 2\left|h'(a_2)\right|^q\right]^{1/q}}{a_2 - a_1}\right],\end{aligned} \tag{30}$$

which can be found in [2].

Theorem 13. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$, $\alpha \in (0,1]$, $q > 1$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\left|\frac{(a_2^\alpha - \eta^\alpha) h(a_2) + (\eta^\alpha - a_1^\alpha) h(a_1)}{a_2^\alpha - a_1^\alpha} - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \cdot \int_{a_1}^{a_2} h(s)\, d_\alpha s\right| \le \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha} \left[(A_1(\alpha))^{1-1/q} \cdot \left\{A_2(\alpha) \left|h'(a_1)\right|^q + A_3(\alpha)\left|h'(\eta)\right|^q\right\}^{1/q}\right] + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\left[(B_1(\alpha))^{1-1/q} \cdot \left\{B_2(\alpha)\left|h'(a_2)\right|^q + B_3(\alpha)\left|h'(\eta)\right|^q\right\}^{1/q}\right] \tag{31}$$

is valid for any $\eta \in [a_1, a_2]$ if $D_\alpha(h) \in L_\alpha([a_1, a_2])$ and $|h'|^q$ is convex on $[a_1, a_2]$, where

$$\begin{aligned}
A_1(\alpha) &= \eta^\alpha - \left[\frac{\eta^{\alpha+1} - a_1^{\alpha+1}}{(\alpha+1)(\eta - a_1)}\right],\\
B_1(\alpha) &= \left[\frac{\eta^{\alpha+1} - a_2^{\alpha+1}}{(\alpha+1)(a_2 - \eta)}\right] - \eta^\alpha,\\
A_2(\alpha) &= \frac{1}{2}\eta^\alpha + \frac{a_1^{\alpha+1}}{(\alpha+1)(\eta - a_1)}\left[\frac{(\alpha+2)(\eta - a_1) - a_1}{(\alpha+2)(\eta - a_1)}\right] - \frac{\eta^{\alpha+2}}{(\alpha+1)(\eta - a_1)^2(\alpha+2)},\\
B_2(\alpha) &= -\frac{a_2^{\alpha+1}}{(\alpha+1)(a_2 - \eta)}\left[\frac{(\alpha+2)(a_2 - \eta) + a_2}{(\alpha+2)(a_2 - \eta)}\right] + \frac{\eta^{\alpha+2}}{(\alpha+1)(a_2 - \eta)^2(\alpha+2)} - \frac{1}{2}\eta^\alpha,\\
A_3(\alpha) &= \frac{1}{2}\eta^\alpha - \frac{\eta^{\alpha+1}}{(\alpha+1)(\eta - a_1)}\left[\frac{(\alpha+2)(\eta - a_1) - \eta}{(\alpha+2)(\eta - a_1)}\right] - \frac{a_1^{\alpha+2}}{(\alpha+1)(\eta - a_1)^2(\alpha+2)},\\
B_3(\alpha) &= \frac{\eta^{\alpha+1}}{(\alpha+1)(a_2 - \eta)}\left[\frac{(\alpha+2)(a_2 - \eta) - \eta}{(\alpha+2)(a_2 - \eta)}\right] - \frac{a_2^{\alpha+2}}{(\alpha+1)(a_2 - \eta)^2(\alpha+2)} - \frac{1}{2}\eta^\alpha.
\end{aligned} \tag{32}$$

Proof. From Theorem 1, Definition 2, and Lemma 5, we get

$$\begin{aligned}
&\left|\frac{(a_2^\alpha - \eta^\alpha) h(a_2) + (\eta^\alpha - a_1^\alpha) h(a_1)}{a_2^\alpha - a_1^\alpha} - \frac{\alpha}{a_2^\alpha - a_1^\alpha} \cdot \int_{a_1}^{a_2} h(s)\, d_\alpha s\right| = \left|\frac{\eta - a_1}{a_2^\alpha - a_1^\alpha} \cdot \int_0^1 \left(((1-t)a_1 + t\eta)^{2\alpha-1} - \eta^\alpha ((1-t)a_1 + t\eta)^{\alpha-1}\right) D_\alpha(h)((1-t)a_1 + t\eta)\, dt \right.\\
&\left. + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(((1-t)a_2 + t\eta)^{2\alpha-1} - \eta^\alpha((1-t)a_2 + t\eta)^{\alpha-1}\right) D_\alpha(h)((1-t)a_2 + t\eta)\, dt\right| \le \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right)\left|h'((1-t)a_1 + t\eta)\right| dt\\
&+ \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha} \cdot \int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right)\left|h'((1-t)a_2 + t\eta)\right| dt.
\end{aligned} \tag{33}$$

Making use of power mean inequality, we get

$$\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right)\left|h'((1-t)a_1 + t\eta^\alpha)\right| dt \le \left(\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) dt\right)^{1-1/q} \times \left(\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) \cdot \left|h'((1-t)a_1 + t\eta)\right|^q dt\right)^{1/q}. \tag{34}$$

Similarly, we have

$$\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right)\left|h'((1-t)a_2 + t\eta)\right| dt \le \left(\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) dt\right)^{1-1/q} \times \left(\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) \cdot \left|h'((1-t)a_2 + t\eta)\right|^q dt\right)^{1/q}. \tag{35}$$

It follows from the convexity of $|h'|^q$ that

$$\begin{aligned}
&\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) \left|h'((1-t)a_1 + t\eta)\right|^q dt \\
&\quad \le \int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) \\
&\quad \cdot \left[(1-t)\left|h'(a_1)\right|^q + t\left|h'(\eta)\right|^q\right] dt \\
&= \left|h'(a_1)\right|^q \int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right)(1-t)\,dt \\
&\quad + \left|h'(\eta)\right|^q \int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) t\,dt \\
&= \left|h'(a)_1\right|^q \left(\frac{1}{2}\eta^\alpha\right. \\
&\quad + \frac{a_1^{\alpha+1}}{(\alpha+1)(\eta-a_1)}\left[\frac{(\alpha+2)(\eta-a_1)-a_1}{(\alpha+2)(\eta-a_1)}\right] \\
&\quad \left. - \frac{\eta^{\alpha+2}}{(\alpha+1)(\eta-a_1)^2(\alpha+2)}\right) + \left|h'(\eta)\right|^q \left(\frac{1}{2}\eta^\alpha\right. \\
&\quad - \frac{\eta^{\alpha+1}}{(\alpha+1)(\eta-a_1)}\left[\frac{(\alpha+2)(\eta-a_1)-\eta}{(\alpha+2)(\eta-a_1)}\right] \\
&\quad \left. - \frac{a_1^{\alpha+2}}{(\alpha+1)(\eta-a_1)^2(\alpha+2)}\right)
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
&\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) \left|h'((1-t)a_2 + t\eta)\right|^q dt \\
&\quad \le \int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) \\
&\quad \cdot \left[(1-t)\left|h'(a_2)\right|^q + t\left|h'(\eta)\right|^q\right] dt \\
&= \left|h'(a_2)\right|^q \int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right)(1-t)\,dt \\
&\quad + \left|h'(\eta)\right|^q \int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) t\,dt \\
&\quad = \left|h'(a_2)\right|^q \\
&\quad \cdot \left(-\frac{a_2^{\alpha+1}}{(\alpha+1)(a_2-\eta)}\left[\frac{(\alpha+2)(a_2-\eta)+a_2}{(\alpha+2)(a_2-\eta)}\right]\right. \\
&\quad \left. + \frac{\eta^{\alpha+2}}{(\alpha+1)(a_2-\eta)^2(\alpha+2)} - \frac{1}{2}\eta^\alpha\right) + \left|h'(\eta)\right|^q \\
&\quad \cdot \left(\frac{\eta^{\alpha+1}}{(\alpha+1)(a_2-\eta)}\left[\frac{(\alpha+2)(a_2-\eta)-\eta}{(\alpha+2)(a_2-\eta)}\right]\right. \\
&\quad \left. - \frac{a_2^{\alpha+2}}{(\alpha+1)(a_2-\eta)^2(\alpha+2)} - \frac{1}{2}\eta^\alpha\right),
\end{aligned} \tag{37}$$

where we have used the identities

$$\begin{aligned}
&\int_0^1 \left(\eta^\alpha - ((1-t)a_1 + t\eta)^\alpha\right) dt \\
&\quad = \eta^\alpha - \left[\frac{\eta^{\alpha+1} - a_1^{\alpha+1}}{(\alpha+1)(\eta-a_1)}\right], \\
&\int_0^1 \left(((1-t)a_2 + t\eta)^\alpha - \eta^\alpha\right) dt \\
&\quad = \left[\frac{\eta^{\alpha+1} - a_2^{\alpha+1}}{(\alpha+1)(a_2-\eta)}\right] - \eta^\alpha.
\end{aligned} \tag{38}$$

Hence, we have the result in (31). □

Theorem 14. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$, $\alpha \in (0,1]$ and $h : [a_1, a_2] \longrightarrow \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\begin{aligned}
&\left|\frac{(a_2^\alpha - \eta^\alpha)h(a_2) + (\eta^\alpha - a_1^\alpha)h(a_1)}{a_2^\alpha - a_1^\alpha}\right. \\
&\quad \left. - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(s)\,d_\alpha s\right| \\
&\quad \le \frac{\eta - a_1}{a_2^\alpha - a_1^\alpha}\left[A_1(\alpha)\left|h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right)\right|\right] \\
&\quad + \frac{a_2 - \eta}{a_2^\alpha - a_1^\alpha}\left[B_1(\alpha)\left|h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right)\right|\right]
\end{aligned} \tag{39}$$

holds for any $\eta \in [a_1, a_2]$ if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$ and $|h'|^q$ is concave on $[a_1, a_2]$ for some $q > 1$, where

$$\begin{aligned}
A_1(\alpha) &= \frac{\eta^\alpha - a_1^\alpha}{2}, \\
B_1(\alpha) &= \frac{2a_2^\alpha + \eta^{\alpha-1}a_2 + \eta a_2^{\alpha-1} - 4\eta^\alpha}{6}, \\
C_1(\alpha) &= \frac{a_1\eta^\alpha - 2a_1^{\alpha+1} + \eta^{\alpha+1}}{6}, \\
C_2(\alpha) & \\
&= \frac{3a_2^{\alpha+1} + \eta^{\alpha-1}a_2^2 + 2\eta a_2^\alpha - 4a_2\eta^\alpha + \eta^{\alpha+1}a_2^{\alpha-1} - 3\eta^{\alpha+1}}{12}.
\end{aligned} \tag{40}$$

Proof. We clearly see that $|h'|$ is concave because $|h'|^q$ is concave for some $q > 1$ (see [27]). From Theorem 1, Definition 2, Lemma 5, the concavity of $|h'|$, and Jensen's integral inequality, we have

$$
\begin{aligned}
&\left|\frac{(a_2^\alpha-\eta^\alpha)h(a_2)+(\eta^\alpha-a_1^\alpha)h(a_1)}{a_2^\alpha-a_1^\alpha}-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(s)\,d_\alpha s\right|\\
&\quad\le\frac{\eta-a_1}{a_2^\alpha-a_1^\alpha}\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\left|h'\left((1-t)a_1+t\eta\right)\right|dt\\
&\qquad+\frac{a_2-\eta}{a_2^\alpha-a_1^\alpha}\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)\left|h'\left((1-t)a_2+t\eta\right)\right|dt,\\
&\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\left|h'\left((1-t)a_1+t\eta\right)\right|dt\\
&\quad\le\left(\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)dt\right)\times\left|h'\left(\frac{\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\left((1-t)a_1+t\eta\right)dt}{\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)dt}\right)\right|\\
&\quad=A_1(\alpha)\,h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right),\\
&\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)\left|h'\left((1-t)a_2+t\eta\right)\right|dt\\
&\quad\le\left(\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)dt\right)\\
&\qquad\times\left|h'\left(\frac{\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)\left((1-t)a_2+t\eta\right)dt}{\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)dt}\right)\right|=B_1(\alpha)\,h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right),
\end{aligned}
\tag{41}
$$

where we have used the identities

$$
\begin{aligned}
&\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)dt=A_1(\alpha)\\
&\quad=\eta^\alpha-\frac{1}{2}a_1^\alpha-\frac{1}{2}\eta^\alpha,\\
&\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)dt\\
&\quad=B_1(\alpha)=\frac{1}{3}a_2^\alpha+\frac{1}{6}\eta^{\alpha-1}a_2+\frac{1}{6}\eta a_2^{\alpha-1}+\frac{1}{3}\eta^\alpha-\eta^\alpha,\\
&\int_0^1\left(\eta^\alpha-\left((1-t)a_1^\alpha+t\eta^\alpha\right)\right)\left((1-t)a_1+t\eta\right)dt\\
&\quad=C_1(\alpha)\\
&\quad=\frac{1}{2}a_1\eta^\alpha-\frac{1}{3}a_1^{\alpha+1}-\frac{1}{6}a_1\eta^\alpha+\frac{1}{2}\eta^{\alpha+1}-\frac{1}{6}\eta a_1^\alpha\\
&\qquad-\frac{1}{3}\eta^{\alpha+1}
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&\int_0^1\left(\left((1-t)a_2^{\alpha-1}+t\eta^{\alpha-1}\right)\left((1-t)a_2+t\eta\right)-\eta^\alpha\right)\\
&\quad\cdot\left((1-t)a_2+t\eta\right)dt=C_2(\alpha)=\frac{1}{4}a_2^{\alpha+1}+\frac{1}{12}\\
&\quad\cdot\eta^{\alpha-1}a_2^2+\frac{1}{6}\eta a_2^\alpha-\frac{1}{3}a_2\eta^\alpha+\frac{1}{12}\eta^{\alpha+1}a_2^{\alpha-1}-\frac{1}{4}\eta^{\alpha+1}.
\end{aligned}
\tag{43}
$$

□

Remark 15. If $\alpha=1$, then inequality (39) becomes

$$
\begin{aligned}
&\left|\frac{(\eta-a_1)h(a_1)+(a_2-\eta)h(a_2)}{a_2-a_1}-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)\,dx\right|\\
&\quad\le\frac{1}{2}\left[\frac{(\eta-a_1)^2\left|h'\left((\eta+2a_1)/3\right)\right|+(a_2-\eta)^2\left|h'\left((\eta+2a_2)/3\right)\right|}{a_2-a_1}\right].
\end{aligned}
\tag{44}
$$

3. Applications to Special Means

A bivariate function $M : (0,\infty) \times (0,\infty) \longmapsto (0,\infty)$ is said to be a mean if $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$ for all $x, y \in (0,\infty)$. Recently, the mean value theory has been the subject of intensive research, and many remarkable inequalities and properties for various bivariate means can be found in the literature [28–33].

In this section, we use the results obtained in Section 2 to give some applications to the weighted arithmetic mean

$$A(a_1, a_2; w_1, w_2) = \frac{w_1 a_1 + w_2 a_2}{w_1 + w_2} \quad (a_1, a_2 > 0) \tag{45}$$

and (α, r)-th generalized logarithmic mean

$$L_{(\alpha,r)}(a_1, a_2) = \left[\frac{\alpha(a_2^{\alpha+r} - a_1^{\alpha+r})}{(\alpha + r)(a_2^{\alpha} - a_1^{\alpha})}\right]^{1/r} \quad (a_1, a_2 > 0, a_1 \neq a_2, r \in \mathbb{R}, r \neq 0, \alpha, -\alpha \in (0,1]). \tag{46}$$

Proposition 16. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $r > 1$. Then the inequality*

$$\begin{aligned}
&\left|A(a_1^r, a_2^r; \eta^\alpha - a_1^\alpha, a_2^\alpha - \eta^\alpha) - L_{(\alpha,r)}^r(a_1, a_2)\right| \\
&\leq \frac{r(\eta - a_1)}{a_2^\alpha - a_1^\alpha}\left\{\frac{1}{2}\eta^\alpha |a_1|^{r-1} - \frac{1}{3}a_1^\alpha |a_1|^{r-1} - \frac{1}{6}\eta^\alpha |a_1|^{r-1} + \frac{1}{2}\eta^\alpha |\eta|^{r-1} - \frac{1}{6}a_1^\alpha |\eta|^{r-1} - \frac{1}{3}\eta^\alpha |\eta|^{r-1}\right\} + \frac{r(a_2 - \eta)}{a_2^\alpha - a_1^\alpha}\left\{\frac{1}{4}a_2^\alpha |a_2|^{r-1}\right. \\
&+ \frac{1}{12}\eta^{\alpha-1}a_2 |a_2|^{r-1} + \frac{1}{12}\eta a_2^{\alpha-1}|a_2|^{r-1} + \frac{1}{12}\eta^\alpha |a_2|^{r-1} - \frac{1}{2}\eta^\alpha |a_2|^{r-1} + \frac{1}{12}a_2^\alpha |\eta|^{r-1} \\
&\left.+ \frac{1}{12}\eta^{\alpha-1}a_2 |\eta|^{r-1} + \frac{1}{12}\eta a_2^{\alpha-1}|\eta|^{r-1} + \frac{1}{4}\eta^\alpha |\eta|^{r-1} - \frac{1}{2}\eta^\alpha |\eta|^{r-1}\right\}
\end{aligned} \tag{47}$$

holds for any $\alpha \in (0,1]$ and $\eta \in [a_1, a_2]$.

Proof. Let $h(x) = x^r$. Then the result follows easily from Theorem 7 and the convexity of $h(x)$ on the interval $[a_1, a_2]$. □

Proposition 17. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $r > 1$. Then the inequality*

$$\begin{aligned}
&\left|A(a_1^{-1}, a_2^{-1}; \eta^\alpha - a_1^\alpha, a_2^\alpha - \eta^\alpha) - L_{(\alpha,-1)}^r(a_1, a_2)\right| \\
&\leq \frac{(\eta - a_1)}{a_2^\alpha - a_1^\alpha}\left\{\frac{1}{2}\eta^\alpha |a_1|^{-2} - \frac{1}{3}a_1^\alpha |a_1|^{-2} - \frac{1}{6}\eta^\alpha |a_1|^{-2} + \frac{1}{2}\eta^\alpha |\eta|^{-2} - \frac{1}{6}a_1^\alpha |\eta|^{-2} - \frac{1}{3}\eta^\alpha |\eta|^{-2}\right\} \\
&+ \frac{(a_2 - \eta)}{a_2^\alpha - a_1^\alpha}\left\{\frac{1}{4}a_2^\alpha |a_2|^{-2} + \frac{1}{12}\eta^{\alpha-1}a_2 |a_2|^{-2} + \frac{1}{12}\eta a_2^{\alpha-1}|a_2|^{-2} + \frac{1}{12}\eta^\alpha |a_2|^{-2} - \frac{1}{2}\eta^\alpha |a_2|^{-2}\right. \\
&\left.+ \frac{1}{12}a_2^\alpha |\eta|^{-2} + \frac{1}{12}\eta^{\alpha-1}a_2 |\eta|^{-2} + \frac{1}{12}\eta a_2^{\alpha-1}|\eta|^{-2} + \frac{1}{4}\eta^\alpha |\eta|^{-2} - \frac{1}{2}\eta^\alpha |\eta|^{-2}\right\}
\end{aligned} \tag{48}$$

holds for any $\alpha \in (0,1]$ and $\eta \in [a_1, a_2]$.

Proof. Let $h(x) = 1/x$. Then Proposition 17 follows from Theorem 7 and the convexity of the function $h(x)$ on the interval $[a_1, a_2]$. □

Proposition 18. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $r > 1$. Then the inequality*

$$\begin{aligned}
&\left|A(a_1^r, a_2^r; \eta^\alpha - a_1^\alpha, a_2^\alpha - \eta^\alpha) - L_{(\alpha,r)}^r(a_1, a_2)\right| \\
&\leq \frac{r(\eta - a_1)}{a_2^\alpha - a_1^\alpha}\left[(A_1(\alpha))^{1-1/q} \cdot \left\{A_2(\alpha)|a_1|^{(r-1)q} + A_3(\alpha)|\eta|^{(r-1)q}\right\}^{1/q}\right] \\
&+ \frac{r(a_2 - \eta)}{a_2^\alpha - a_1^\alpha}\left[(B_1(\alpha))^{1-1/q} \cdot \left\{B_2(\alpha)|a_2|^{(r-1)q} + B_3(\alpha)|\eta|^{(r-1)q}\right\}^{1/q}\right]
\end{aligned} \tag{49}$$

holds for all $\alpha \in (0,1]$ and $\eta \in [a_1, a_2]$, where $A_i(\alpha)$ and $B_i(\alpha)$ $(i = 1,2,3)$ are defined as in Theorem 11.

Proof. Let $h(x) = x^r$. Then Proposition 18 follows from Theorem 11 and the convexity of $h(x)$ on $[a_1, a_2]$ immediately. □

Proposition 19. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $r > 1$. Then the inequality*

$$\begin{aligned}
&\left|A(a_1^{-1}, a_2^{-1}; \eta^\alpha - a_1^\alpha, a_2^\alpha - \eta^\alpha) - L_{(\alpha,-1)}^r(a_1, a_2)\right| \\
&\leq \frac{(\eta - a_1)}{a_2^\alpha - a_1^\alpha}\left[(A_1(\alpha))^{1-1/q} \cdot \left\{A_2(\alpha)|a_1|^{-2q} + A_3(\alpha)|\eta|^{-2q}\right\}^{1/q}\right] \\
&+ \frac{(a_2 - \eta)}{a_2^\alpha - a_1^\alpha}\left[(B_1(\alpha))^{1-1/q} \cdot \left\{B_2(\alpha)|a_2|^{-2q} + B_3(\alpha)|\eta|^{-2q}\right\}^{1/q}\right]
\end{aligned} \tag{50}$$

holds for all $\alpha \in (0,1]$ and $\eta \in [a_1, a_2]$, where $A_i(\alpha)$ and $B_i(\alpha)$ $(i = 1,2,3)$ are defined as in Theorem 11.

Proof. Let $h(x) = 1/x$. Then the result follows easily from Theorem 11 and the convexity of $h(x)$ on the interval $[a_1, a_2]$. □

4. Conclusion

In the article, we establish an identity and several new inequalities of Hermite-Hadamard type for conformable fractional integrals by use of the convexity theory and Jesen's inequality, Hölder inequality, and power mean inequality and present their applications to special bivariate means. The given Hermite-Hadamard type inequalities for conformable fractional integrals are the generalizations of the corresponding results established by Kavurmaci, Avci, and Özdemir in [2], and the idea may stimulate further research in the theory of Hermite-Hadamard's inequalities, conformable fractional integrals, and generalized convex functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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