# GENERALIZATION OF MYERS' THEOREM ON A CONTACT MANIFOLD 

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## 1. Introduction

In 1941, Myers [4] proved that a complete Riemannian manifold for which Ric $\geq \delta>0$, is compact. In 1981, Hasegawa and Seino [3] generalized Myers' theorem for a Sasakian manifold by proving that a complete Sasakian (normal contact metric) manifold for which Ric $\geq-\delta>-2$, is compact. Actually their proof uses only that the structure is $K$-contact and not the full strength of the Sasakian condition. A $K$-contact structure is a contact metric structure such that the characteristic vector field of the contact structure is Killing.

Now a contact metric structure is $K$-contact if and only if all sectional curvatures of plane sections containing the characteristic vector field are equal to 1 (see e.g. [1], p. 65) and hence there is a lot of positive curvature involved in the problem from the outset. The question then arises for a general contact metric structure: Can we relax the condition that the sectional curvature $K(\xi, X)$ of any plane section containing the characteristic vector field $\xi$ be equal to 1 ; even if we must increase $-\delta$ from near -2 to near 0 to compensate? In general, the notion of a contact metric structure is quite weak; in fact, the set of all such structures associated to a given contact structure is infinite dimensional. So we seemingly must assume some condition generalizing the $K$-contact structure, then we can study $K(X, \xi) \geq \varepsilon>$ $\delta^{\prime} \geq 0$ and Ric $\geq-\delta>-2$ where $\delta^{\prime}$ is a function of $\delta$.

Let $M$ denote a $(2 n+1)$-dimensional contact metric manifold with structure tensors $(\varphi, \xi, \eta, g)$; i.e., $\eta$ is a globally defined contact form

$$
\left(\eta \wedge(d \eta)^{n} \neq 0\right)
$$

$\xi$ its characteristic vector field $(d \eta(\xi, X)=0, \eta(\xi)=1), g$ a Riemannian metric, and $\varphi$ a skew-symmetric field of endomorphisms satisfying

$$
\varphi^{2}=-I+\eta \otimes \xi, \quad \eta(X)=g(X, \xi), \quad(d \eta)(X, Y)=g(X, \varphi Y)
$$

[^0]Following [1] we denote the operator $\frac{1}{2} \mathscr{L}_{\xi} \varphi$ by $h$ where $\mathscr{L}$ denotes Lie differentiation. It is well known [1] that $M$ is $K$-contact (i.e., $\xi$ is Killing) if and only if $h=0$. We also define the strain tensor $\tau$ of $M$ along $\xi$ by

$$
g(\tau X, Y)=\left(\mathscr{L}_{\xi} g\right)(X, Y)
$$

Then using the relation ([1], p. 66)

$$
\begin{equation*}
\nabla_{X} \xi=-\varphi X-\varphi h X \tag{1.1}
\end{equation*}
$$

one obtains $\tau=2 h \varphi$. As a generalization of the $K$-contact condition we suppose that $\operatorname{div} \tau=\sigma \eta$. Recall that for a contact metric structure

$$
\begin{gather*}
\nabla_{t} \varphi_{j}^{t}=-2 n \eta_{j}  \tag{1.2}\\
R_{r}^{j} \xi^{r}=\nabla^{r} \nabla_{r} \xi^{j}+4 n \xi^{j} \tag{1.3}
\end{gather*}
$$

The first identity can be found in Olszak [5] and the second one in Tanno [7]. Thus, using equations (1.1) and (1.2) in (1.3) we obtain

$$
R_{r}^{j} \xi^{r}=\nabla_{r}\left(h_{m}^{r} \varphi^{m j}\right)+2 n \xi^{j}
$$

and hence if $\operatorname{div} \tau=\sigma \eta, \xi$ is an eigenvector of the Ricci operator.
We also note the following example. Consider $\mathbf{R}^{3}$ with the contact structure

$$
\eta=\frac{1}{2}\left(\cos x^{3} d x^{1}+\sin x^{3} d x^{2}\right)
$$

and the associated metric $g_{i j}=\frac{1}{4} \delta_{i j}$. Since $\eta$ is invariant by the translations in the coordinate directions by $2 \pi$, the torus $T^{3}$ is a compact manifold also carrying this structure. For this contact metric structure, div $\tau=2 \eta$. Thus there are both compact and non-compact contact metric manifolds satisfying $\operatorname{div} \tau=\sigma \eta$.

We present two theorems generalizing Myers' theorem for contact metric manifolds as follows:

Theorem 1. Let $M$ be $a(2 n+1)$-dimensional complete contact metric manifold with $\operatorname{div} \tau=\sigma \eta$. If $\operatorname{Ric} \geq-\delta>-2$ and the sectional curvatures of plane sections containing $\xi$ are $\geq \varepsilon>\delta^{\prime} \geq 0$ where

$$
\delta^{\prime}=2 \sqrt{n(\delta-2 \sqrt{2 \delta}+n+2)}-(\delta-2 \sqrt{2 \delta}+1+2 n)
$$

then $M$ is compact.

In dimension 3 we can obtain a better estimate for $\delta^{\prime}$ and have the following result.

Theorem 2. Let $M$ be a 3-dimensional complete contact metric manifold with $\operatorname{div} \tau=\sigma \eta$. If Ric $\geq-\delta>-2$ and the sectional curvatures of plane sections containing $\xi$ are $\geq \varepsilon>\delta^{\prime} \geq 0$ where

$$
\delta^{\prime}=-\frac{\delta^{2}}{4}+\sqrt{2} \delta^{3 / 2}-3 \delta+2 \sqrt{2} \delta^{1 / 2}
$$

then $M$ is compact.
Before turning to the proofs let us review a few properties of the tensor field $h$ on a contact metric manifold $M$ :
(1) $h$ is a symmetric and trace-free field of endomorphisms such that

$$
h \varphi+\varphi h=0 \quad \text { and } \quad h \xi=0
$$

If $\lambda$ is an eigenvalue of $h$, so is $-\lambda$ and hence in dimension $3(n=1)$ the eigenvalues of $h$ are $0, \lambda,-\lambda$ and we adopt the convention that $\lambda$ will denote the non-negative eigenvalue.
(2) $\operatorname{Ric}(\xi)=2 n-\operatorname{tr} h^{2}$ (see [1], p. 67), and therefore in dimension 3 (i.e., $n=1$ ) we have

$$
\operatorname{Ric}(\xi)=2\left(1-\lambda^{2}\right)
$$

(3) In dimension $3, h^{2}$ acts on the contact subbundle $\{\eta=0\}$ as $\lambda^{2} I$, thus for a unit vector $X$ orthogonal to $\xi$,

$$
|h X|^{2}=\lambda^{2}
$$

Also for any such $X$ we have

$$
g(h X, X) \leq \lambda
$$

where, again by our convention, $\lambda$ denotes the non-negative eigenvalue of $h$.
The main idea of the proofs of our theorems is to use a $D$-homothetic deformation of the structure. This technique was introduced by Tanno [6] and used by Goldberg and Toth [2] as well as by Hasegawa and Seino [3]. Given a contact metric structure $(\varphi, \xi, \eta, g)$ let

$$
\bar{\eta}=\alpha \eta, \quad \bar{\xi}=\frac{1}{\alpha} \xi, \quad \bar{\varphi}=\varphi \quad \text { and } \quad \bar{g}=\alpha g+\alpha(\alpha-1) \eta \otimes \eta
$$

for some positive constant $\alpha$. Then ( $\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g}$ ) is again a contact metric
structure. Such a change of structure is called a D-homothetic deformation. A $D$-homothetic deformation preserves many basic properties like being $K$-contact or Sasakian, but most notably completeness for our purpose [3], [6]. Computing the Ricci tensor $\bar{R}_{j k}$ of $\bar{g}_{j k}$ on a contact metric manifold $M$ we have

$$
\begin{aligned}
\bar{R}_{j k}= & R_{j k}-\frac{(2 \alpha-1)(\alpha-1)}{\alpha} g_{j k}+\left[2 n\left(\alpha^{2}-1\right)+\frac{(2 \alpha-1)(\alpha-1)}{\alpha}\right] \eta_{j} \eta_{k} \\
& +\frac{\alpha-1}{\alpha}\left(2 h_{j k}-h_{j m} h_{k}^{m}-R_{j l i k} \xi^{l} \xi^{i}\right)
\end{aligned}
$$

## 2. Proof of Theorem 1

We now turn to the proof of Theorem 1 , which is to seek a number $\alpha$ ( $0<\alpha<1$ ) such that for the structure obtained by the $D$-homothetic deformation, the new associated metric tensor has its Ricci curvature bounded below by a positive constant and hence by Myers' theorem $M$ must be compact.

Let $(\varphi, \xi, \eta, g)$ denote the contact metric structure satisfying the hypothesis. Let $X$ be a unit vector and decompose $X$ into the form $a X_{D}+b \xi$ where $X_{D}$ is a unit vector orthogonal to $\xi$ and, of course, $a^{2}+b^{2}=1$. We now expand $\operatorname{Ric}(X)$ in terms of this decomposition. Since $\operatorname{div} \tau=\sigma \eta, \xi$ is an eigenvector of the Ricci operator as noted in section 1 and hence the coefficient of $a b$ is 0 .

From property (2), we have

$$
\begin{equation*}
\operatorname{Ric}(\xi)=2\left(n-\sum_{i=1}^{n} \lambda_{i}^{2}\right) \geq 2 n \varepsilon \tag{2.1}
\end{equation*}
$$

where the $\lambda_{i}$ 's are the non-negative eigenvalues of $h$, and hence for the coefficient of $b^{2}$,

$$
\operatorname{Ric}(\xi)+2 n\left(\alpha^{2}-1\right) \geq 2 n\left(\alpha^{2}-(1-\varepsilon)\right)
$$

Thus, one of the requirements on the number $\alpha$ that we seek is

$$
\alpha>\sqrt{1-\varepsilon}
$$

It follows immediately from (2.1) that $\sum_{i=1}^{n} \lambda_{i}^{2} \leq n(1-\varepsilon)$ and hence

$$
\begin{equation*}
\lambda_{i} \leq \sqrt{n} \sqrt{1-\varepsilon}<\sqrt{n} \alpha \tag{2.2}
\end{equation*}
$$

Finally, the coefficient of $a^{2}$ :

$$
\begin{aligned}
& \operatorname{Ric}\left(X_{D}\right)+2-2 \alpha+\frac{1-\alpha}{\alpha}\left[-1-2 g\left(h X_{D}, X_{D}\right)+\left|h X_{D}\right|^{2}+K\left(\xi, X_{D}\right)\right] \\
& \quad \geq-\delta+2-2 \alpha+\frac{1-\alpha}{\alpha}(-1-2 \lambda+\varepsilon)
\end{aligned}
$$

where $\lambda$ stands for the maximum of the $\lambda_{i}$ 's over $i=1,2, \ldots, n$. As per our requirements we must seek $\alpha$ such that

$$
\frac{\alpha}{1-\alpha}(2-\delta-2 \alpha)+\delta^{\prime}-2 \lambda-1>0
$$

where $\delta^{\prime}<\varepsilon$. Hence

$$
\begin{equation*}
\lambda<\frac{1}{2}\left(\delta^{\prime}-1+\frac{\alpha}{1-\alpha}(2-\delta-2 \alpha)\right) \tag{2.3}
\end{equation*}
$$

Consider the following curve in the $x y$-plane, thinking of $x$ as corresponding to $\alpha$ and $y$ to $\lambda$ :

$$
y=\frac{1}{2}\left(\delta^{\prime}-1+\frac{x}{1-x}(2-\delta-2 x)\right)
$$

for $0<x<1$. $y$ has a positive maximum at $x=1-\sqrt{\delta / 2}$, viz.,

$$
y(1-\sqrt{\delta / 2})=\frac{1}{2}\left(1-2 \sqrt{2 \delta}+\delta+\delta^{\prime}\right)
$$

Thus, since $\lambda<\sqrt{n} \sqrt{1-\varepsilon}<\sqrt{n} \sqrt{1-\delta^{\prime}}$ (from (2.2)), $\alpha=1-\sqrt{\delta / 2}$ gives (2.3) if

$$
\sqrt{n} \sqrt{1-\delta^{\prime}}=\frac{1}{2}\left(\delta^{\prime}-1+\frac{\alpha}{1-\alpha}(2-\delta-2 \alpha)\right)
$$

that is, if

$$
\delta^{\prime}=2 \sqrt{n(\delta-2 \sqrt{2 \delta}+n+2)}-(\delta-2 \sqrt{2 \delta}+1+2 n)
$$

for then

$$
\frac{1}{2}\left(1-2 \sqrt{2 \delta}+\delta+\delta^{\prime}\right)<1-\sqrt{\delta / 2}
$$

This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

Here $n=1$ and $h$ has only 3 eigenvalues $0, \lambda$ and $-\lambda$ each of multiplicity 1. For the coefficient of $b^{2}$, we have

$$
\begin{equation*}
\alpha>\sqrt{1-\varepsilon} \tag{3.1}
\end{equation*}
$$

as before.
For the coefficient of $a^{2}$ we have by property (3) that

$$
\begin{aligned}
& \operatorname{Ric}\left(X_{D}\right)+2-2 \alpha+\frac{1-\alpha}{\alpha}\left[-1-2 g\left(h X_{D}, X_{D}\right)+\left|h X_{D}\right|^{2}+K\left(\xi, X_{D}\right)\right] \\
& \quad \geq-\delta+2-2 \alpha+\frac{1-\alpha}{\alpha}\left(-1-2 \lambda+\lambda^{2}+\varepsilon\right)
\end{aligned}
$$

and we seek $\alpha(0<\alpha<1)$ such that, in addition to (3.1), we have this last expression bounded below by a positive constant. In particular, we study the inequality

$$
\frac{\alpha}{1-\alpha}(2-\delta-2 \alpha)+\delta^{\prime}-2+(\lambda-1)^{2}>0
$$

where $\delta^{\prime}<\varepsilon$. Solving for $\lambda$, the non-negative eigenvalue of $h$, we observe

$$
\begin{equation*}
\lambda<1-\sqrt{2-\delta^{\prime}-\frac{\alpha}{1-\alpha}(2-\delta-2 \alpha)} . \tag{3.2}
\end{equation*}
$$

Now consider the following curve in the $x y$-plane taking $x$ as $\alpha$ and $y$ as $\lambda$ :

$$
y=1-\sqrt{2-\delta^{\prime}-\frac{x}{1-x}(2-\delta-2 x)}
$$

on $0<x<1$. $y$ has a positive maximum at $x=1-\sqrt{\delta / 2}$, viz.,

$$
y\left(1-\sqrt{\frac{\delta}{2}}\right)=1-\sqrt{2 \sqrt{2 \delta}-\delta^{\prime}-\delta}
$$

Thus, since $1-\lambda^{2} \geq \varepsilon$ so that $\lambda \leq \sqrt{1-\varepsilon}<\sqrt{1-\delta^{\prime}}, \alpha=1-\sqrt{\delta / 2}$ gives (3.2) if

$$
\sqrt{1-\delta^{\prime}}=1-\sqrt{2 \sqrt{2 \delta}-\delta^{\prime}-\delta}
$$

that is, if

$$
\delta^{\prime}=-\frac{\delta^{2}}{4}+\sqrt{2} \delta^{3 / 2}-3 \delta+2 \sqrt{2} \delta^{1 / 2}
$$



Fig. 1


Fig. 2
for then

$$
1-\sqrt{2 \sqrt{2 \delta}-\delta^{\prime}-\delta}<1-\sqrt{\delta / 2}
$$

This completes the proof of Theorem 2.
Remark. In Theorem $2(n=1)$ we observe that when $\delta=0, \delta^{\prime}=0$ and when $\delta=2, \delta^{\prime}=1$. In Theorem $1(n \geq 1)$ we observe that when $\delta=0$, $\delta^{\prime}=2 \sqrt{n(n+2)}-(2 n+1)$ which approaches 1 as $n$ tends to $\infty$, and when $\delta=2, \delta^{\prime}=1$. Thus we have a better estimate for $\delta^{\prime}$ in Theorem 2 (for $n=1$ ) than that provided by Theorem 1 (for $n=1$ ). Figure 1 below shows the graph of $\delta^{\prime}$ in Theorem 1 for $n=1$ and the graph of $\delta^{\prime}$ in Theorem 2. Figure 2 shows the graphs of $\delta^{\prime}$ in Theorem 1 for $n=1,2,3,4$.

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