Generalization of results about the Bohr radius for power series

by

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Abstract. The Bohr radius for power series of holomorphic functions mapping Reinhardt domains $\mathcal{D} \subset \mathbb{C}^n$ into a convex domain $G \subset \mathbb{C}$ is independent of the domain G.

1. Preliminaries. Let us recall the theorem of H. Bohr [13] of 1914.

Theorem 1.1. *If a power series*

$$
(1.1) \qquad f(z_1) = \sum_{k=0}^{\infty} c_k z_1^k
$$

converges in the unit disk U_1 and its sum has modulus less than 1, then

$$
(1.2) \qquad \sum_{k=0}^{\infty} |c_k z_1^k| < 1
$$

for $|z_1| < 1/3$. Moreover, the constant $1/3$ cannot be improved.

For convenience we write the inequality (1.2) in the following equivalent form:

$$
(1.3) \qquad \sum_{k=1}^{\infty} |c_k z_1^k| < 1 - |c_0|.
$$

Several generalizations of this result have been obtained.

1° ([24]). If the sum of the series (1.1) is such that $|\Re f(z_1)| < 1$ in U_1 and $c_0 > 0$, then the inequality (1.3) holds for $|z_1| < 1/3$.

2° ([23], [22]). If $\Re f(z_1) < 1$ in U_1 and $c_0 > 0$, then (1.3) holds for $|z_1| < 1/3$.

3° ([20]). If $\Re\{[\exp(-i \arg f(0))]f(z_1)\} < 1$ in U_1 (here we assume that $\arg f(0) = 0$ if $f(0) = 0$) then (1.3) holds for $|z_1| < 1/3$.

Formulations of Bohr's theorem in several complex variables appeared very recently. We recall some of them.

Given a complete Reinhardt domain $\mathcal{D}$, we denote by $R_1(\mathcal{D})$ the largest nonnegative number r with the property that if the power series

$$(1.4) \qquad f(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha, \qquad z \in \mathcal{D},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$ and all α_i are nonnegative integers, converges in $\mathcal{D}$ and the modulus of its sum is less than 1, then

$$\sum_{|\alpha| \geq 1} |c_\alpha z^\alpha| < 1 - |c_0|$$

in the homothetic domain $\mathcal{D}_r = r\mathcal{D}$. Here $c_0 = c_{0,\ldots,0}$. If $\mathcal{D}$ is moreover bounded, we define $R_2(\mathcal{D})$ similarly, with the last inequality replaced by

$$\sum_{|\alpha| \geq 1} \sup_{\mathcal{D}_r} |c_\alpha z^\alpha| < 1 - |c_0|.$$

We call $R_1(\mathcal{D})$ (resp. $R_2(\mathcal{D})$) the *first* (resp. *second*) *Bohr radius* of $\mathcal{D}$.

Let

$$\mathcal{D}_p^n = \{z \in \mathbb{C}^n : |z_1|^p + \cdots + |z_n|^p < 1\},$$

where $0 < p \leq \infty$. The domain $\mathcal{D}_\infty^n$ is the poly-disk $\{z \in \mathbb{C}^n : |z_j| < 1, \, j = 1, \ldots, n\}$.

THEOREM 1.2 ([12], see also [18]). *For $n > 1$ one has*

$$(1.5) \qquad \frac{1}{3\sqrt{n}} < R_1(\mathcal{D}_\infty^n) < \frac{2\sqrt{\log n}}{\sqrt{n}}.$$

THEOREM 1.3 ([1]). *For $n > 1$ one has*

$$(1.6) \qquad \frac{1}{3\sqrt[3]{e}} < R_1(\mathcal{D}_1^n) \leq \frac{1}{3}.$$

The estimates (1.5) and (1.6) were generalized to $R_1(\mathcal{D}_p^n)$ for $1 \leq p < \infty$ in [11] and for $0 < p \leq 1$ in [3]. We point out the following new remarkable result which improves the lower estimate in (1.5).

THEOREM 1.4 ([15]). *For $n > 1$ one has*

$$C\sqrt{\frac{\log n}{n \log \log n}} < R_1(\mathcal{D}_\infty^n),$$

where the constant C is independent of n.

Both Bohr radii coincide for a polydisk, and in the case $n = 1$ they coincide with the classical Bohr radius $1/3$. If $\mathcal{D}$ is not a polydisk, then naturally $R_2(\mathcal{D}) < R_1(\mathcal{D})$.

THEOREM 1.5 ([1]). *For every complete bounded Reinhardt domain* $\mathcal{D}$,

$$1 - \sqrt[n]{2/3} < R_2(\mathcal{D}).$$

THEOREM 1.6 ([1]). *One has*

$$R_2(\mathcal{D}_1^n) < \frac{0.44663}{n}.$$

The radius $R_2(\mathcal{D})$ was investigated in [11], [17]. Other results about the Bohr radius for holomorphic functions can be found in [2], [5], [7]–[10]. We would like to call a special attention to the paper [16] in which the relation between the Bohr radius and the Banach–Mazur distance between Banach spaces was discovered.

2. Generalized Bohr radii. One of the proofs of Bohr's theorem (Theorem 1.1) is based on the Landau inequality [21]: if the function (1.1) satisfies in U_1 the inequality $|f(z_1)| < 1$, then $|c_k| \leq 2(1 - |c_0|)$ for every $k \geq 1$. This inequality can be obtained as a simple consequence of the Carathéodory inequality [14]: if the function (1.1) satisfies in U_1 the inequality $\Re f(z_1) > 0$, then $|c_k| \leq 2\Re c_0$ for every $k \geq 1$. Both inequalities are particular cases of a more general assertion.

Let $\widetilde{G}$ be the convex hull of G.

PROPOSITION 2.1 ([3]). *If* $f(U_1) \subset G$, *then*

$$(2.1) \qquad |c_k| \leq 2 \operatorname{dist}(c_0, \partial\widetilde{G}) \quad \text{for all } k \geq 1.$$

Now it is not difficult to prove a generalization of Theorem 1.1. Let $G \subset \mathbb{C}$ be any domain. A point $p \in \partial G$ is called a *point of convexity* if $p \in \partial\widetilde{G}$. A point of convexity p is called *regular* if there exists a disk $U \subset G$ so that $p \in \partial U$.

THEOREM 2.1. *If the function* (1.1) *is such that* $f(U_1) \subset G$, *with* $\widetilde{G} \neq \mathbb{C}$, *then*

$$(2.2) \qquad \sum_{k=1}^{\infty} |c_k z_1^k| < \operatorname{dist}(c_0, \partial\widetilde{G})$$

for $|z_1| < 1/3$. *The constant* $1/3$ *cannot be improved if* ∂G *contains at least one regular point of convexity.*

Proof. 1) If $|z_1| < 1/3$ then (2.1) yields

$$\sum_{k=1}^{\infty} |c_k z_1^k| < 2 \operatorname{dist}(c_0, \partial\widetilde{G}) \sum_{k=1}^{\infty} \frac{1}{3^k} = \operatorname{dist}(c_0, \partial\widetilde{G}).$$

2) We now prove the exactness of the constant $1/3$ in case the boundary contains at least one regular point of convexity. In the classical case

of Bohr's Theorem 1.1 this is obtained by considering the family of functions ([21])

$$(2.3) \qquad f(z_1) = \frac{\alpha - z_1}{1 - \alpha z_1}, \qquad 0 < \alpha < 1.$$

Here

$$\sum_{k=1}^{\infty} |c_k z_1^k| = 1 \quad \text{if and only if} \quad |z_1| = \frac{1}{1 + 2\alpha}.$$

Letting $\alpha \to 1$, we obtain the desired result. Note that instead of (2.3) one can use the family $e^{i\phi} f(z_1)$, where $f(z_1)$ is taken from (2.3). In this case $c_0 = e^{i\phi}\alpha$, and when $\alpha \to 1$ we see that c_0 tends to ∂U_1 along the radius of argument ϕ. If G is an arbitrary disk U, then, remarking that (2.2) does not change under homotheties and translations, we deduce the exactness of $1/3$ for any disk. Let ζ be a regular point of convexity. Then there exists a disk $U \subset G$ such that $\zeta \in \partial U \cap \partial G$. Consider the functions f in (1.1) such that $f(U_1) \subset U$. For suitable c_0 (see above) we have $\mathrm{dist}(c_0, \partial U) = \mathrm{dist}(c_0, \partial G) = \mathrm{dist}(c_0, \partial \widetilde{G})$. Therefore, in the inequality (2.2) one cannot take $|z_1| < r$, where $r > 1/3$. ∎

We remark that Theorem 1.1, assertion 3º, as well as generalizations of 1º and 2º are contained in Theorem 2.1. For example, in 1º there is no need to assume $c_0 > 0$, and instead of (1.3) one gets

$$\sum_{k=1}^{\infty} |c_k z_1^k| < 1 - |\Re c_0|.$$

Similarly in 2º there is no need to assume $c_0 > 0$, and instead of (1.3) one gets

$$\sum_{k=1}^{\infty} |c_k z_1^k| < 1 - \Re c_0.$$

Let us recall another fact, known earlier:

4º ([4]). If $\Re f(z_1) > 0$ in U_1 and $c_0 > 0$, then one has

$$(2.4) \qquad \sum_{k=1}^{\infty} |c_k z_1^k| < c_0$$

for $|z_1| < 1/3$, and the constant $1/3$ cannot be improved.

I thought before that Theorem 1.1 and 4º were two different facts involving the same Bohr radius. Now I know that both results are particular cases of Theorem 2.1. Namely, in the case of 4º, without the assumption $c_0 > 0$, we

get

$$\sum_{k=1}^{\infty} |c_k z_1^k| < \Re c_0$$

instead of (2.4).

Theorem 2.1 motivates the following generalization of the first and second Bohr radii. Denote by $R_1(\mathcal{D}, G)$ (resp. $R_2(\mathcal{D}, G)$), where $G \subset \mathbb{C}$, $\widetilde{G} \neq \mathbb{C}$, and $\mathcal{D}$ is a complete Reinhardt domain (resp. a bounded complete Reinhardt domain) in $\mathbb{C}^n$, the largest $r \geq 0$ such that if the function (1.4) is holomorphic in $\mathcal{D}$ and $f(\mathcal{D}) \subset G$ then

$$\sum_{|\alpha| \geq 1}^{\infty} |c_\alpha z^\alpha| < \text{dist}(c_0, \partial \widetilde{G})$$

in a homothety $\mathcal{D}_r$ (respectively

$$\sum_{|\alpha| \geq 1}^{\infty} \sup_{\mathcal{D}_r} |c_\alpha z^\alpha| < \text{dist}(c_0, \partial \widetilde{G})).$$

Theorem 2.1 and the result from [6] about the Rogosinski radius allow one to hope that the two Bohr radii $R_1(\mathcal{D}, G)$ and $R_2(\mathcal{D}, G)$ are independent of the convex domain G. The main result of the present paper is the proof of this more general assertion.

3. The main result. Let M be a complex manifold, and let $\mathcal{H}(M)$ be the space of functions holomorphic on M, equipped with the natural topology of uniform convergence on compact subsets of M.

Let $\| \cdot \|_r$, $r \in (0, 1)$, be a one-parameter family of seminorms in $\mathcal{H}(M)$ that are continuous with respect to the topology of $\mathcal{H}(M)$. In what follows we always assume that

(a) $\| \cdot \|_{r_1} \leq \| \cdot \|_{r_2}$ if $r_1 \leq r_2$.
(b) $\|f \cdot g\|_r \leq \|f\|_r \cdot \|g\|_r$ for all $r \in (0, 1)$,

and there exists a point $z_0 \in M$ such that

(c) $\|f\|_r \to |f(z_0)|$ as $r \to 0$, for all $f \in \mathcal{H}(M)$,
(d) $\|f\|_r = |f(z_0)| + \|f - f(z_0)\|_r$, for all $f \in \mathcal{H}(M)$.

Denote by $B(\{\| \cdot \|_r\}, G)$ the largest $r \geq 0$ such that for each $f \in \mathcal{H}(M)$ with $f(M) \subset G$ one has

(3.1) $$\|f - f(z_0)\|_r < \text{dist}(f(z_0), \partial \widetilde{G}),$$

where $\widetilde{G}$ is the convex hull of the domain $G \subset \mathbb{C}$.

PROPOSITION 3.1. *If U is any disk and Π is any half-plane, then*

$$(3.2) \qquad B(\{\|\cdot\|_r\}, \Pi) = B(\{\|\cdot\|_r\}, U).$$

Proof. Let $\Pi_1 = \{z_1 : \Re z_1 > 0\}$. Then ([4, Theorem 7])

$$B(\{\|\cdot\|_r\}, U_1) = B'(\{\|\cdot\|_r\}, \Pi_1),$$

where B' is defined in the same way as B but with the additional assumption $f(z_0) > 0$. This assumption can be removed as follows. If $\Re f(z_0) > 0$ in M then $\Re f_1(z_0) > 0$, where $f_1(z) = f(z) - \Im f(z_0)$. But $f_1(z_0) > 0$, hence

$$B(\{\|\cdot\|_r\}, U_1) = B(\{\|\cdot\|_r\}, \Pi_1).$$

We remark that (3.1) does not change under homotheties, translations and rotations of the domain G. Therefore (3.2) holds. ∎

THEOREM 3.1. *If $\widetilde{G} \neq \mathbb{C}$, then $B(\{\|\cdot\|_r\}, G)$ is not smaller than (3.2). If ∂G contains at least one regular point of convexity, then $B(\{\|\cdot\|_r\}, G)$ is equal to (3.2).*

Proof. Let $\widetilde{G} \neq \mathbb{C}$ and $f(M) \subset G$. Fix any $f(z_0) \in G$. On the boundary $\partial\widetilde{G}$ there exists a point ζ so that $\operatorname{dist}(f(z_0), \partial\widetilde{G}) = \operatorname{dist}(f(z_0), \zeta)$. Through the point ζ passes the line of support of $\widetilde{G}$ which defines a half-plane $\Pi_0 \supseteq G$. Then

$$\operatorname{dist}(f(z_0), \partial\widetilde{G}) = \operatorname{dist}(f(z_0), \partial\Pi_0).$$

Therefore $B(\{\|\cdot\|_r\}, G) \geq B(\{\|\cdot\|_r\}, \Pi_0)$, since $\{f \in \mathcal{H}(M) : f(M) \subset G\} \subset \{f \in \mathcal{H}(M) : f(M) \subset \Pi_0\}$.

Assume now that there is a regular point of convexity in ∂G. Then the proof repeats part 2) of the proof of Theorem 2.1. Note that there we did not use the concrete form of the family (2.3), but rather the fact that c_0 can lie on any radius emanating from the center of the disk U to its boundary. So, assume that $U \subset G$ and $\zeta \in \partial U \cap \partial G \cap \partial\widetilde{G}$. Consider $f(z_0)$ lying on the radius from the center of U to ζ. Then $\operatorname{dist}(f(z_0), \zeta) = \operatorname{dist}(f(z_0), \partial U) = \operatorname{dist}(f(z_0), \partial\widetilde{G})$, hence $B(\{\|\cdot\|_r\}, G) \leq B(\{\|\cdot\|_r\}, U)$, since $\{f \in \mathcal{H}(M) : f(M) \subset U\} \subset \{f \in \mathcal{H}(M) : f(M) \subset G\}$. ∎

COROLLARY 3.1. *If the domain G is convex and $G \neq \mathbb{C}$, then $B(\{\|\cdot\|_r\}, G)$ is independent of the choice of G.*

Proof. There exists a disk $U \subset G$ such that $\partial U \cap \partial G \neq \emptyset$. Therefore there exist regular points of convexity on ∂G. ∎

COROLLARY 3.2. *The first Bohr radius $R_1(\mathcal{D}, G)$ and the second Bohr radius $R_2(\mathcal{D}, G)$ are independent of the choice of the convex domain G, $G \neq \mathbb{C}$.*

In particular, the assertions of Theorems 1.2–1.4 are valid for $R_1(\mathcal{D}, G)$ while those of Theorems 1.5 and 1.6 are valid for $R_2(\mathcal{D}, G)$ for every convex domain $G \neq \mathbb{C}$.

Some concluding remarks. If the family of seminorms $\|\cdot\|_r$ does not satisfy some of the conditions (a)–(d), then the assertion of Theorem 3.1 is not valid anymore. Examples can be found in [4]. If $\widetilde{G} = \mathbb{C}$, then the right-hand side of (3.1) is equal to ∞, therefore in this case

$$B(\{\|\cdot\|_r\}, G) = 1.$$

One can also consider different realizations of $B(\{\|\cdot\|_r\}, G)$ than the first and second Bohr radii $R_1(\mathcal{D}, G)$ and $R_2(\mathcal{D}, G)$.

We conclude the present article with an *open problem*: if $\widetilde{G} \neq \mathbb{C}$, is it always true that $B(\{\|\cdot\|_r\}, G)$ is equal to (3.2)? The same question makes sense for the first and second Bohr radii $R_1(\mathcal{D}, G)$ and $R_2(\mathcal{D}, G)$.

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