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Generalization of symmetric α-stable Lévy distributions for $q>1$

Sabir Umarov,[1,a)] Constantino Tsallis,[2,3,b)] Murray Gell-Mann,[3,c)] and Stanly Steinberg[4,d)]

[1]*Department of Mathematics, Tufts University, Medford, Massachusetts 02155, USA*
[2]*Centro Brasileiro de Pesquisas Fisicas and National Institute of Science and Technology for Complex Systems, Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, Brazil*
[3]*Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA*
[4]*Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA*

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The α-stable distributions introduced by Lévy play an important role in probabilistic theoretical studies and their various applications, e.g., in statistical physics, life sciences, and economics. In the present paper we study sequences of long-range dependent random variables whose distributions have asymptotic power-law decay, and which are called (q,α)-stable distributions. These sequences are generalizations of independent and identically distributed α-stable distributions and have not been previously studied. Long-range dependent (q,α)-stable distributions might arise in the description of anomalous processes in nonextensive statistical mechanics, cell biology, finance. The parameter q controls dependence. If $q=1$ then they are classical independent and identically distributed with α-stable Lévy distributions. In the present paper we establish basic properties of (q,α)-stable distributions and generalize the result of Umarov *et al.* [Milan J. Math. **76**, 307 (2008)], where the particular case $\alpha=2, q\in[1,3)$ was considered, to the whole range of stability and nonextensivity parameters $\alpha\in(0,2]$ and $q\in[1,3)$, respectively. We also discuss possible further extensions of the results that we obtain and formulate some conjectures. © *2010 American Institute of Physics.* [doi:10.1063/1.3305292]

I. INTRODUCTION

The central limit theorem (CLT) and α-stable distributions have rich applications in various fields including the Boltzmann–Gibbs (BG) statistical mechanics. The nonextensive statistical mechanics[1–6] characterized by the nonextensivity index q (which recovers the BG theory in the case $q=1$) studies, in particular, strongly correlated random states, mathematical models of which can be represented by specific long-range dependent random variables. The q-CLT consistent with nonextensive statistical mechanics was established in Ref. 7. The main objective of Ref. 7 was to study the scaling limits (attractors) of sums of q-independent random variables with a finite $(2q-1)$-variance. The mapping

$$F_q:\mathcal{G}_q[2]\to\mathcal{G}_{z(q)}[2], \tag{1}$$

where F_q is the q-Fourier transform (see Sec. II) $z(s)=(1+s)/(3-s)$, and $\mathcal{G}_q[2]$ is the set of q-Gaussians up to a constant factor (see, e.g., Refs. 2 and 3), was essentially used in the descrip-

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a)Electronic mail: sabir.umarov@tufts.edu.
b)Electronic mail: tsallis@cbpf.br.
c)Electronic mail: mgm@santafe.edu.
d)Electronic mail: stanly@math.unm.edu.

tion of attractors. The number 2 in the notation will soon become transparent in the context of the current paper.

In the present work we study a q-analog of the α-stable Lévy distributions. In this sense, the present paper is a conceptual continuation of Ref. 7. The classic theory of α-stable distributions was originated by Lévy and developed by Lévy, Gnedenko, Feller and others; for details and history see, for instance, Refs. 8–14 and references therein. Distributions with asymptotic power-law decay (α-stables, and particularly q-Gaussians, in the first place) found a huge number of applications in various practical studies (see, e.g., Refs. 14–23 just to mention a few), confirming the frequent nature of these distributions. As it will become clear later on, (q,α)-stable distributions unify both of them. Indeed, $(1,\alpha)$-stable distributions correspond to the α-stable ones, and the $(q,2)$-stable distributions correspond to the q-Gaussian ones. All (q,α)-stable distributions, except Gaussians [(1,2)-distributions], exhibit asymptotic power laws. In practice the researcher is often interested in identification of a correct attractor of correlated states, which plays a major role in the adequate modeling of physical phenomenon itself. This motivates the study of sequences of (q,α)-stable distributions and their attractors, as focused in the present paper.

For simplicity we will consider only symmetric (q,α)-stable distributions in the one-dimensional case (see Ref. 8, for the multivariate q-CLT). We denote the class of random variables with (q,α)-stable distributions by $\mathcal{L}_q[\alpha]$. A random variable $X\in\mathcal{L}_q[\alpha]$ has a symmetric density $f(x)$ with asymptotics $f\sim C|x|^{-(1+\alpha)/(1+\alpha(q-1))}$, $|x|\to\infty$, where $1\le q<2$, $0<\alpha<2$, and C is a positive constant. Hereafter $g(x)\sim h(x)$, $x\to a$, means that $\lim_{x\to a} g(x)/h(x)=1$. Linear combinations and properly scaling limits of sequences of q-independent random variables with (q,α)-stable distributions are again random variables with (q,α)-stable distributions, justifying that $\mathcal{L}_q[\alpha]$ form a class of "stable" distributions. To this end, we note that $\mathcal{L}_q[\alpha]$ shares the same asymptotic behavior with the set $\mathcal{L}_{sym}(\gamma)$ of symmetric Lévy distributions centered at 0, where

$$\gamma=\gamma(q,\alpha)=\frac{\alpha(2-q)}{1+\alpha(q-1)}.$$

However, there is an essential difference between (q,α)-stability of q-independent random variables and the classic stability of α-stable distributions. Namely, q-independence exhibits a special long-range correlation between random variables (see the exact definition in Sec. III). In practice this notion reflects physical states (arising e.g., in nonextensive statistical mechanics), which are strongly correlated. The term "global correlation" instead of "strong correlation" is also used widely in physics literature; see, e.g., Refs. 3 and 4. Examples of such systems include earthquakes,[24] cold atoms in optical dissipative lattices,[25] and dusty plasma.[26] A decomposition of nonextensive processes with strong correlation into independent states cannot adequately reflect their evolution. Likewise, (q,α)-stable distributions cannot be captured by the existing theory of α-stable distributions, which is heavily based on the concept of independence (or weak dependence). This distinction ends up with essential implication: attractors of (q,α)-stable distributions are different from the attractors of α-stable distributions, unless $q=1$. If $q=1$ then correlation disappears, that is, q-independence becomes usual probabilistic independence, and $\gamma(1,\alpha)=\alpha$, implying $\mathcal{L}_1[\alpha]=\mathcal{L}_{\mathrm{sym}}[\alpha]$.

Following the method established in Ref. 7, we will apply F_q-transform for the study of sequences of q-independent (q,α)-stable distributions. Parameter q controls correlation. We will classify (q,α)-stable distributions depending on parameters $1\le q<2$ (or equivalently $1\le Q<3$, $Q=2q-1$) and $0<\alpha\le 2$. We establish the mapping

$$F_q:\mathcal{G}_{q^L}[2]\to\mathcal{G}_q[\alpha], \tag{2}$$

where $\mathcal{G}_q[\alpha]$ is the set of functions $\{be_q^{-\beta|\xi|^\alpha}, b>0, \beta>0\}$, and

$$q^L = \frac{3+Q\alpha}{1+\alpha}, \quad Q = 2q-1,$$

i.e.,

$$\frac{2}{q^L-1} = \frac{1+\alpha}{1+\alpha(q-1)}.$$

The particular case $q=Q=1$ recovers $q^L=(3+\alpha)/(1+\alpha)$, already known in the literature.[2] Denote $\mathcal{Q}_1=\{(Q,\alpha):1\le Q<3,\alpha=2\}$, $\mathcal{Q}_2=\{(Q,\alpha):1\le Q<3,0<\alpha<2\}$, and $\mathcal{Q}=\mathcal{Q}_1\cup\mathcal{Q}_2$. Note that the case $(Q,\alpha)\in\mathcal{Q}_1$ for q-independent random variables with a finite Q-variance was studied in Ref. 7. For $(Q,\alpha)\in\mathcal{Q}_2$ the Q-variance is infinite. We will focus our analysis, namely, on the latter case. Note that the case $\alpha=2$, in the framework of this classification like the classic α-stable distributions, becomes peculiar.

In the scope of second classification we study the attractors of scaled sums and expand the results of Ref. 7 to the region $\mathcal{Q}$ generalizing the mapping (1) to

$$F_{\zeta_\alpha(q)}:\mathcal{G}_q[\alpha]\to\mathcal{G}_{z_\alpha(q)}[\alpha], \quad 1\le q<2, \quad 0<\alpha\le 2, \tag{3}$$

where

$$\zeta_\alpha(s) = \frac{\alpha-2(1-q)}{\alpha} \quad \text{and} \quad z_\alpha(s) = \frac{\alpha q+1-q}{\alpha+1-q}.$$

Note that, if $\alpha=2$, then $\zeta_2(q)=q$ and $z_2(q)=(1+q)/(3-q)$, thus recovering the mapping (1), and consequently, the result of Ref. 7.

These two classifications of (q,α)-stable distributions based on mappings (2) and (3), respectively, can be unified to the scheme

$$\mathcal{L}_q[\alpha]\overset{F_q}{\to}\mathcal{G}_q[\alpha]\overset{F_{q_{\zeta(\alpha)}}}{\leftrightarrow}\mathcal{G}_{q_{\zeta(\alpha)}}[2],$$

$$\updownarrow F_q$$

$$\mathcal{G}_{q^L}[2], \tag{4}$$

which gives the full picture of interrelations of (q,α)-stable distributions with parameters $q\in[1,2)$ and $\alpha\in(0,2)$ (see details in Sec. VIII).

II. PRELIMINARIES AND AUXILIARY RESULTS

A. Basic operations of *q*-algebra

In this section we briefly recall the basic operations of q-algebra. Indeed, the analysis we will conduct is entirely based on the q-structure of nonextensive statistical mechanics (for more details, see Refs. 3–5 and references therein). To this end, we recall the well-known fact that the classical BG entropy $S_{\mathrm{BG}}=-\Sigma_i p_i \ln p_i$ satisfies the additivity property. Namely, if A and B are two independent subsystems, then $S_{\mathrm{BG}}(A+B)=S_{\mathrm{BG}}(A)+S_{\mathrm{BG}}(B)$. However, the q-generalization of the classic entropy introduced in Ref. 1 and given by $S_q=(1-\Sigma_i p_i^q)/(q-1)$ with $q\in\mathcal{R}$ and $S_1=S_{\mathrm{BG}}$, does not possess this property if $q\neq 1$. Instead, it satisfies the *pseudoadditivity* (or q *-additivity*),[1,2,4]

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B).$$

Inherited from the right hand side of this equality, the q *-sum* of two given real numbers, x and y, is defined as $x\oplus_q y=x+y+(1-q)xy$. The q-sum is commutative, associative, recovers the usual summing operation if $q=1$ (i.e., $x\oplus_1 y=x+y$), and preserves 0 as the neutral element (i.e.,

$x\oplus_q 0=x$). By inversion, we can define the *q -subtraction* as $x\ominus_q y=(x-y)/(1+(1-q)y)$. The *q -product* for x,y is defined by the binary relation $x\otimes_q y=[x^{1-q}+y^{1-q}-1]_+^{1/(1-q)}$. Here the symbol $[x]_+$ means that $[x]_+=x$ if $x\geq 0$, and $[x]_+=0$ if $x<0$. This operation also commutative, associative, recovers the usual product when $q=1$, and preserves 1 as the unity. The q-product is defined if $x^{1-q}+y^{1-q}\geq 1$. Again by inversion, it can be defined the *q -division*: $x\oslash_q y=(x^{1-q}-y^{1-q}+1)^{1/(1-q)}$.

B. *q*-generalization of the exponential and cyclic functions

Now let us recall the main properties of two functions, q-exponential and q-logarithm, which will be essentially used in this paper. Let e_q^x and $\ln_q x$ denote, respectively, the functions

$$e_q^x=[1+(1-q)x]_+^{1/(1-q)} \quad \text{and} \quad \ln_q x=\frac{x^{1-q}-1}{1-q} \quad (x>0).$$

The entropy S_q then can be conveniently rewritten in the form $S_q=\Sigma_i p_i \ln_q 1/p_i$. For the q-exponential the relations $e_q^{x\oplus_q y}=e_q^x e_q^y$ and $e_q^{x+y}=e_q^x\otimes_q e_q^y$ hold true. These relations can be written equivalently as follows: $\ln_q(x\otimes_q y)=\ln_q x+\ln_q y$ and $\ln_q(xy)=(\ln_q x)\oplus_q(\ln_q y)$. The q-exponential and q-logarithm have the asymptotics,

$$e_q^x=1+x+\frac{q}{2}x^2+o(x^2), \quad x\to 0, \tag{5}$$

and

$$\ln_q(1+x)=x-\frac{q}{2}x^2+o(x^2), \quad x\to 0, \tag{6}$$

respectively. The q-product and q-exponential can be extended to complex numbers $z=x+iy$ (see Refs. 7, 27, and 28). In addition, for $q\neq 1$ the function e_q^z can be analytically extended to the complex plain except the point $z_0=-1/(1-q)$ and defined as the principal value along the cut $(-\infty,z_0)$. If $q<1$, then, for real y, $|e_q^{iy}|\geq 1$ and $|e_q^{iy}|\sim K_q(1+y^2)^{1/2(1-q)}$, $y\to\infty$, with $K_q=(1-q)^{1/(1-q)}$. Similarly, if $q>1$, then $0<|e_q^{iy}|\leq 1$ and $|e_q^{iy}|\to 0$ if $|y|\to\infty$. For complex z it is not hard to verify the power series representation,

$$e_q^z=1+z+z^2\sum_{n=0}^{\infty}\frac{A_n(q)}{(n+2)!}z^n, \quad |z|<\frac{1}{|1-q|}, \tag{7}$$

where $A_n(q)=\Pi_{k=0}^n a_k(q)$, $a_k(q)=q-k(1-q)$. Let $I_q=(-1/|1-q|,1/|1-q|)$. Then it follows from (7) that for arbitrary real number $x\in I_q$ the equation

$$e_q^{ix}=\left\{1-x^2\sum_{n=0}^{\infty}\frac{(-1)^n A_{2n}(q)}{(2n+2)!}x^{2n}\right\}+i\left\{x-x^2\sum_{n=0}^{\infty}\frac{(-1)^n A_{2n+1}(q)}{(2n+3)!}x^{2n+1}\right\}$$

holds. Define for $x\in I_q$ the functions q-cos and q-sin by formulas

$$\cos_q(x)=1-x^2\sum_{n=0}^{\infty}\frac{(-1)^n A_{2n}(q)}{(2n+2)!}x^{2n} \tag{8}$$

and

$$\sin_q(x) = x - x^2 \sum_{n=0}^{\infty} \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}. \tag{9}$$

In fact, $\cos_q(x)$ and $\sin_q(x)$ can be defined for all real x by using appropriate power series expansions. Properties of q-sin, q-cos, and corresponding q-hyperbolic functions, were studied in Ref. 29. Here we note that the q-analogs of Euler's formulas read

$$e_q^{ix} = \cos_q(x) + i \sin_q(x)$$

and

$$\cos_q(x) = \frac{e_q^{ix} + e_q^{-ix}}{2}, \quad \sin_q(x) = \frac{e_q^{ix} - e_q^{-ix}}{2i}.$$

It follows from the definitions of $\cos_q(x)$ and $\sin_q(x)$, and from the equality $(e_q^x)^2 = e_{(1+q)/2}^{2x}$ (see Lemma 2.1 in Ref. 7), that

$$\cos_q(2x) = e_{2q-1}^{2(1-q)x^2} - 2 \sin_{2q-1}^2(x). \tag{10}$$

Denote $\Psi_q(x) = \cos_q 2x - 1$. Then Eq. (10) implies

$$\Psi_q(x) = (e_{2q-1}^{2(1-q)x^2} - 1) - 2 \sin_{2q-1}^2(x). \tag{11}$$

The following two properties of Ψ_q will be used later on.

Proposition II.1: Let $q \geq 1$. Then

(1) $-2 \leq \Psi_q(x) \leq 0$;
(2) $\Psi_q(x) = -2qx^2 + o(x^3)$, $x \to 0$.

Proof: Assume $q \geq 1$. Since $e_{2q-1}^{-2(q-1)x^2} \leq 1$, then (11) immediately implies that $\Psi_q(x) \leq 0$. Further, $\sin_q(x)$ can be written in the form (see Ref. 29) $\sin_q(x) = \rho_q(x) \sin[\varphi_q(x)]$, where $\rho_q(x) = (e_q^{(1-q)x^2})^{1/2}$ and $\varphi_q(x) = (\arctan(1-q)x)/(1-q)$. A simple calculation yields $\Psi_q(x) \geq -2$. Using asymptotic relation (5), we get

$$e_{2q-1}^{2(1-q)x^2} - 1 = 2(1-q)x^2 + o(x^3), \quad x \to 0. \tag{12}$$

In turn, it follows from (9) that

$$-2 \sin_{2q-1}^2(x) = -2x^2 + o(x^3), \quad x \to 0. \tag{13}$$

Now (11)–(13) imply the second part of the statement. ■

Representation (7) shows the behavior of q-exponential near the origin. It is not hard to verify that in the case $q > 1$ for $x > (q-1)^{-1}$ the representation

$$e_q^{-x} = [(q-1)x]^{-1/(q-1)} \left\{ 1 - \frac{1}{(1-q)^2 x} + \frac{1}{(1-q)^4 x^2} \sum_{n=0}^{\infty} \frac{(-1)^n A_n(q)}{(n+2)!(q-1)^{2n}} \left(\frac{1}{x} \right)^n \right\}$$

holds.

C. *q*-Fourier transform for symmetric densities

The *q -Fourier transform* for $q \geq 1$ was introduced in Ref. 7 and used as a basic tool in establishing the q-analog of the standard CLT. Formally the q-Fourier transform for a given function $f(x)$ is defined by

$$F_q[f](\xi) = \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx. \tag{14}$$

For discrete functions $f_k, k=0, \pm 1, \ldots,$ this definition takes the form

$$F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e_q^{ik\xi} \otimes_q f(k). \tag{15}$$

In the future we use the same notation in both cases. We also call (14) or (15) the *q -characteristic function* of a given random variable X with an associated density $f(x)$, using the notations $F_q[X]$ or $F_q[f]$ equivalently.

It should be noted that, if in the formal definition (14) f is compactly supported then integration has to be taken over this support, although, in contrast with the usual analysis, the function $e_q^{ix\xi} \otimes_q f(x)$ under the integral does not vanish outside the support of f. This is an effect of the q-product.

The q-Fourier transform for non-negative $f(x)$ can be written in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) e_q^{ix\xi(f(x))^{q-1}} dx. \tag{16}$$

We note that, if the q-Fourier transform of $f(x)$ defined by (14) exists, then it coincides with (16). The q-Fourier transform determined by formula (16) has an advantage to compare to the formal definition: it does not use the q-product, which is, as we noticed above, restrictive in use (for $q<1$).

Proposition II.2: Let $f(x)$ be an even function. Then its q -Fourier transform can be written in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \cos_q(x\xi[f(x)]^{q-1}) dx. \tag{17}$$

Proof: Notice that, because of the symmetry of f,

$$\int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x) dx = \int_{-\infty}^{\infty} e_q^{-ix\xi} \otimes_q f(x) dx.$$

Taking this into account, we have

$$F_q[f](\xi) = \frac{1}{2} \int_{-\infty}^{\infty} (e_q^{ix\xi} \otimes_q f(x) + e_q^{-ix\xi} \otimes_q f(x)) dx.$$

Now due to (16) we obtain

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \frac{e_q^{ix\xi[f(x)]^{q-1}} + e_q^{-ix\xi[f(x)]^{q-1}}}{2} dx,$$

which coincides with (17). ■

Further, denote

$$H_{q,\alpha} = \{f \in L_1 : f(x) \sim C|x|^{-(1+\alpha)/(1+\alpha(q-1))}, |x| \to \infty\}.$$

For a given $f \in H_{q,\alpha}$ the constant $C=C_f$ is defined uniquely by f. It is readily seen that $\phi(q,\alpha)=(\alpha+1)/(1+\alpha(q-1))>1$ for all $\alpha \in (0,2)$ and $q \in [1,2)$. Moreover, $\phi(q,\alpha)(2q-1)<3$ for all $\alpha \in (0,2)$ and $q \in [1,2)$, which implies $\sigma_{2q-1}^2(f)=\infty$. Notice also $\phi(q,\alpha)=1+\alpha^*(q,\alpha)$, where

$$\alpha^* = \alpha^*(q,\alpha) = \frac{\alpha(2-q)}{1+\alpha(q-1)}.$$

On the other hand, the density g of any α^*-stable Lévy distributions has the asymptotic behavior $g(x) \sim C/|x|^{1+\alpha^*}$, $|x| \to \infty$. Hence, for a fixed $q \in [1,2)$ $H_{q,\alpha}$ is asymptotically equivalent to the set of densities of α^*-stable Lévy distributions.

The following proposition plays a key role in our further analysis.

Proposition II.3: Let $f(x)$, $x \in R$ *, be a symmetric probability density function. Further, let either*

(i) *the* $(2q-1)$*-variance* $\sigma^2_{2q-1}(f) < \infty$ *(associated with* $\alpha=2$ *and* $1 \le q < 2$*) or*
(ii) $f(x) \in H_{q,\alpha}$*, where* $(2q-1,\alpha) \in \mathcal{Q}_2$.

Then, for the q-Fourier transform of $f(x)$ *, the following asymptotic relation holds true:*

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^{\alpha} + o(|\xi|^{\alpha}), \quad \xi \to 0, \tag{18}$$

where

$$\mu_{q,\alpha} = \begin{cases} \dfrac{q}{2}\sigma^2_{2q-1}\nu_{2q-1}, & \text{if } \alpha = 2, \\ \dfrac{2^{2-\alpha}(1+\alpha(q-1))C_f}{2-q}\displaystyle\int_0^{\infty}\frac{-\Psi_q(y)}{y^{\alpha+1}}dy, & \text{if } (2q-1,\alpha) \in \mathcal{Q}_2, \end{cases} \tag{19}$$

with $\nu_{2q-1}(f) = \int_{-\infty}^{\infty}[f(x)]^{2q-1}dx$.

Proof: First, assume that $\alpha=2$. By Proposition II.2,

$$F_q[f](\xi) = \int_{-\infty}^{\infty}(e_q^{ix\xi}) \otimes_q f(x)dx = \int_{-\infty}^{\infty} f(x)\cos_q(x\xi[f(x)]^{q-1})dx. \tag{20}$$

Making use of the asymptotic expansion (5) we can rewrite the right hand side of (20), in the form

$$\begin{aligned} F_q[f](\xi) &= \int_{-\infty}^{\infty} f(x)\{1 + ix\xi[f(x)]^{q-1} - q/2x^2\xi^2[f(x)]^{2(q-1)}\}dx + o(\xi^3) \\ &= 1 - (q/2)\xi^2\sigma^2_{2q-1}\nu_{2q-1} + o(\xi^3), \quad \xi \to 0, \end{aligned} \tag{21}$$

from which the first part of proposition follows.

Now, we assume $(2q-1,\alpha) \in \mathcal{Q}_2$. We apply Proposition II.2 to obtain

$$\begin{aligned} F_q[f](\xi) - 1 &= \int_{-\infty}^{\infty} f(x)[\cos_q(x\xi[f(x)]^{q-1}) - 1]dx = 2\int_0^N f(x)\Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right)dx \\ &\quad + 2\int_N^{\infty} f(x)\Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right)dx, \end{aligned}$$

where N is a sufficiently large finite number. In the first integral we use the asymptotic relation $\Psi(x/2) = -(q/2)x^2 + o(x^3)$, $x \to 0$, which follows from Proposition II.1, and get

$$2\int_0^N f(x)\Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right)dx = -q\xi^2\int_0^N x^2 f^{2q-1}(x)dx + o(\xi^3), \quad \xi \to 0, \tag{22}$$

that is, a quantity of order $o(|\xi|^{\delta})$, $\xi \to 0$, for any $\delta < 2$. In the second integral taking into account the hypothesis of the proposition with respect to $f(x)$, we have

$$2\int_N^{\infty} f(x)\Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right)dx = 2C_f\int_N^{\infty}\frac{1}{x^{(\alpha+1)/(1+\alpha(1-q))}}\Psi_q\left(\frac{x^{1-(\alpha+1)(q-1)/1+\alpha(q-1)}\xi}{2C_f^{1-q}}\right)dx. \quad (23)$$

We use the substitution

$$x^{(2-q)/(1+\alpha(q-1))} = \frac{2y}{C_f^{q-1}\xi} \quad (24)$$

in the last integral and obtain

$$2\int_N^{\infty} f(x)\Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right)dx = \mu_{q,\alpha}|\xi|^{\alpha} + o(|\xi|^{\alpha}), \quad \xi \to 0, \quad (25)$$

where

$$\mu_{q,\alpha} = -\frac{2^{2-\alpha}(1+\alpha(q-1))C_f}{2-q}\int_0^{\infty}\frac{\Psi_q(y)}{y^{\alpha+1}}dy.$$

Hence, the obtained asymptotic relations (22) [we take $\delta \in (\alpha, 2)$] and (25) complete the proof. ■

For stable distributions $\mu_{q,\alpha}$ must be positive. We have seen (Proposition II.1) that if $q \geq 1$, then $\Psi_q(x) \leq 0$ (not being identically zero), which yields $\mu_{q,\alpha} > 0$. Note also that the condition for $f(x)$ to be symmetric was not required in Ref. 7 if $\sigma_{2q-1}(f) < \infty$.

III. WEAK CONVERGENCE OF CORRELATED RANDOM VARIABLES

Let us start this section by introducing the notion of *q -independence*. We will also introduce two types of convergence, namely, *q -convergence* and *weak q-convergence* and establish their equivalence.

By definition, two random variables X and Y are said to be (q', q, q'') *-independent* if

$$F_{q'}[X+Y](\xi) = F_q[X](\xi) \otimes_{q''} F_q[Y](\xi). \quad (26)$$

In terms of densities, Eq. (26) can be rewritten as follows. Let f_X and f_Y be densities of X and Y, respectively, and let f_{X+Y} be the density of $X+Y$. Then

$$\int_{-\infty}^{\infty} e_{q'}^{ix\xi} \otimes_{q'} f_{X+Y}(x)dx = F_q[f_X](\xi) \otimes_{q''} F_q[f_Y](\xi). \quad (27)$$

If all three parameters q', q, and q'' coincide, i.e., $q=q'=q''$, then we call simply q-independent. For $q=1$ the condition (26) turns into the well-known relation

$$F[f_X * f_Y] = F[f_X] \cdot F[f_Y]$$

between the convolution (noted $*$) of two densities and the multiplication of their (classical) characteristic functions and holds for independent X and Y. If $q \neq 1$, then (q', q, q'')-independence describes a specific class of correlations.

Remark III.1: It is worth to mention at this point that q-independence appears to be relevant to the notion of *scale invariance*.[30] To be more specific, it might well be that q-independence implies scale invariance, i.e., scale invariance is necessary for q-independence, although it is by now clear that it is not sufficient. Indeed, scale-invariant probabilistic models exist in the literature. Some of them presumably involve q-independence since their $N \to \infty$ limits are q-Gaussians;[30,31] others do not involve q-independence[32–34] (if they did involve, their $N \to \infty$ limits would have to be q-Gaussians and they are not). See also Ref. 35 which discusses limit distributions in general setting within the exchangeability concept.

Let X_N be a sequence of identically distributed random variables. Denote $Y_N = X_1 + \cdots + X_N$. By definition, X_N *is said to be* (q',q,q'')-independent [or (q',q,q'')-independent and identically distributed) if the relations

$$F_{q'}[Y_N](\xi) = F_q[X_1](\xi) \otimes_{q''} \cdots \otimes_{q''} F_q[X_N](\xi) \tag{28}$$

hold for all $N=2,3,\ldots$.

For $q=q'=q''=1$ the condition (28) turns into the condition for the sequence X_N to be usual independent and identically distributed. If $q=q'=q''$ then we call the sequence X_N simply a q- independent and identically distributed. Consider example of an (q',q,q)- independent and identically distributed sequence of random variables, where $q \in (1,3)$ and $q'=(3q-1)/(q+1)$. Assume X_N is the sequence of identically distributed random variables with the associated Gaussian density,

$$G_{q'}(\beta,x) = \frac{\sqrt{\beta}}{C_{q'}} e_{q'}^{-\beta x^2},$$

where $C_{q'}$ is the normalizing constant (see, e.g., Ref. 7). Further, assume the sums $X_1+\cdots+X_N$, $N=2,3,\ldots$, are distributed according to the density $G_{q'}(\alpha,x)$, where $\alpha=N^{-1/2-q'}\beta$. Then the sequence X_N satisfies (28) for all $N=2,3,\ldots$, with $q=q''$, thus being (q',q,q)-independent identically distributed sequence of random variables.

For the sake of simplicity in this paper we will consider only q- independent and identically distributed random variables.

By definition, a sequence of random variables X_N is said to be *q-convergent* to a random variable X_∞ if $\lim_{N\to\infty} F_q[X_N](\xi)=F_q[X_\infty](\xi)$ locally uniformly in ξ.

Evidently, this definition is equivalent to the weak convergence (denoted by "$\Rightarrow$") of random variables if $q=1$. For $q\neq 1$ denote by W_q the set of continuous functions ϕ satisfying the condition $|\phi(x)| \le C(1+|x|)^{-q/(q-1)}$, $x\in R$.

A sequence of random variables X_N with the density f_N is called *weakly q-convergent* to a random variable X_∞ with the density f if $\int_R f_N(x)dm_q \to \int_{R^d} f(x)dm_q$ for arbitrary measure m_q defined as $dm_q(x)=\phi_q(x)dx$, where $\phi_q\in W_q$. We denote the weak q-convergence by the symbol $\overset{q}{\Rightarrow}$.

Proposition III.2: Let $q>1$. *Then* $X_N\Rightarrow X_0$ *yields* $X_N \overset{q}{\Rightarrow} X_0$.

The proof of this statement immediately follows from the obvious fact that W_q is a subset of the set of bounded continuous functions. Recall that a sequence of probability measures μ_N is called *tight* if, for an arbitrary $\epsilon>0$, there is a compact K_ϵ and an integer N_ϵ^*, such that $\mu_N(R^d\setminus K_\epsilon)<\epsilon$ for all $N\ge N_\epsilon^*$.

Proposition III.3: Let $1<q<2$. *Assume a sequence of random variables* X_N*, defined on a probability space with a probability measure P, and associated densities* f_N*, is q -convergent to a random variable X* with an associated density f. *Then the sequence of associated probability measures* $\mu_N=P(X_N^{-1})$ *is tight.*

Proof: Assume that $1<q<2$ and X_N is a q-convergent sequence of random variables with associated densities f_N and associated probability measures μ_N. We have

$$\frac{1}{R}\int_{-R}^{R}(1-F_q[f_N](\xi))d\xi = \frac{1}{R}\int_{-R}^{R}\left(1-\int_R f_N e_q^{ix\xi f_N^{q-1}}dx\right)d\xi = \int_R\left(\frac{1}{R}\int_{-R}^{R}(1-e_q^{ix\xi f_N^{q-1}})d\xi\right)d\mu_N(x). \tag{29}$$

It is not hard to verify that

$$\frac{1}{R}\int_{-R}^{R} e_q^{ix\xi t} d\xi = \frac{2\,\sin_{1/(2-q)}(Rx(2-q)t)}{Rx(2-q)t}. \tag{30}$$

It follows from (29) and (30) that

$$\frac{1}{R}\int_{-R}^{R} (1 - F_q[f_N](\xi)) d\xi = 2\int_{-\infty}^{\infty} \left(1 - \frac{\sin_{1/2-q}(x(2-q)Rf_N^{q-1})}{Rx(2-q)f_N^{q-1}}\right) d\mu_N(x). \tag{31}$$

Since $1<q<2$ by assumption, $1/(2-q)>1$ as well. It is known[29,36,37] that for any $q'>1$ the properties $\sin_{q'}(x)\le 1$ and $(\sin_{q'}(x))/x \to 1$, $x\to 0$ hold. Moreover, $(\sin_{q'}(x))/x \le 1$, $\forall x \in R$. Suppose, $\lim_{|x|\to\infty}|x|f_N^{q-1}=L_N$, $N\ge 1$. Divide the set $\{N\ge N_0\}$ into two subsets $A=\{N_j\ge N_0 : L_{N_j}>1\}$ and $B=\{N_k\ge N_0 : L_{N_k}\le 1\}$. If $N\in A$, since $\sin_{1/(2-q)}\le 1$, there is a number $a>0$, such that

$$\frac{1}{R}\int_{-R}^{R} (1 - F_q[f_N](\xi)) d\xi \ge 2\int_{|x|\ge a} \left(1 - \frac{1}{R|x|(2-q)f_N^{q-1}}\right) d\mu_N(x) \ge C\mu_N(|x|\ge a), \quad C>0,$$
$$\forall\, N \in A,$$

for R small enough. Now taking into account the q-convergence of X_N to X and, if necessary, taking R smaller, for any $\epsilon>0$, we obtain

$$\mu_N(|x|\ge a) \le \frac{1}{CR}\int_{-R}^{R} (1 - F_q[f_0](\xi)) d\xi < \epsilon, \quad \forall\, N \in A.$$

If $N\in B$, then there exist constants $b>0$, $\delta>0$, such that

$$f_N(x) \le \frac{L_N+\delta}{|x|^{1/(q-1)}} \le \frac{1+\delta}{|x|^{1/(q-1)}}, \quad |x|\ge b, \quad \forall\, N \in B.$$

Hence, we have

$$\mu_N(|x|>b) = \int_{|x|>b} f_N(x)dx \le (1+\delta)\int_{|x|>b} \frac{dx}{|x|^{1/(q-1)}}, \quad N \in B.$$

Since, $1/(q-1)>1$, for any $\epsilon>0$ we can select a number $b_\epsilon \ge b$, such that $\mu_N(|x|>b_\epsilon)<\epsilon$, $N\in B$. As far as $A\cup B=\{N\ge N_0\}$ the proof of the statement is complete. ■

Further, we introduce the function

$$D_q(t) = D_q(t;a) = te_q^{iat^{q-1}} = t(1+i(1-q)at^{q-1})^{-1/(q-1)}, \tag{32}$$

defined on $[0,1]$, where $1<q<2$ and a is a fixed real number. Obviously, $D_q(t)$ is continuous on $[0,1]$ and differentiable in the interval $(0,1)$. In accordance with the classical Lagrange average theorem for any $t_1, t_2, 0\le t_1<t_2\le 1$ there exists a number $t_*, t_1<t_*<t_2$, such that

$$D_q(t_1) - D_q(t_2) = D_q'(t_*)(t_1 - t_2), \tag{33}$$

where D_q' means the derivative of $D_q(t)$ with respect to t.

Consider the following Cauchy problem for the Bernoulli equation:

$$y' - \frac{1}{t}y = \frac{ia(q-1)}{t}y^q, \quad y(0)=0. \tag{34}$$

It is not hard to verify that $y(t)=D_q(t)$ is a solution to problem (34).

Proposition III.4: For $D_q'(t)$ the estimate

$$|D_q'(t;a)| \le C(1+|a|)^{-q/(q-1)}, \quad t \in (0,1], \quad a \in R^1, \tag{35}$$

holds, where constant C does not depend on t.

Proof: It follows from (32) and (34) that

$$|y'(t)| \leq t^{-1}|y + ia(q-1)y^q| = |e_q^{iat^{q-1}} + ia(q-1)t^{q-1}(e_q^{iat^{q-1}})^q| = |1 + ia(1-q)t^{q-1}|^{-q/(q-1)}$$
$$\leq C(1+|a|)^{-q/(q-1)}, \quad t \in (0,1].$$

■

Now we are in a position to formulate the following two theorems on the relationship between q-convergence and weak q-convergence.

Theorem III.5: *Let $1<q<2$ and a sequence of random variables X_N be weakly q -convergent to a random vector X. Then X_N is q -convergent to X.*

Proof: Assume X_N, with associated densities f_N, is weakly q-convergent to a X, with an associated density f. The difference $\mathcal{F}_q[f_N](\xi)-\mathcal{F}_q[f](\xi)$ can be written in the form

$$\mathcal{F}_q[f_N](\xi) - \mathcal{F}_q[f](\xi) = \int_R (D_q(f_N(x)) - D_q(f(x)))dx, \tag{36}$$

where $D_q(t)=D_q(t;a)$ is defined in (32) with $a=x\xi$. It follows from (33) and (35) that

$$|\mathcal{F}_q[f_N](\xi) - \mathcal{F}_q[f](\xi)| \leq C\int_R |(1+|x|)^{-q/(q-1)}(f_N(x) - f(x))|dx,$$

which yields $\mathcal{F}_q[f_N](\xi)\to\mathcal{F}_q[f](\xi)$ for all $\xi\in R$. ■

Theorem III.6: *Let $1<q<2$ and a sequence of random variables X_N with the associated densities f_N is q -convergent to a random variable X with the associated density f and $\mathcal{F}_q[f](\xi)$ is continuous at $\xi=0$. Then X_N weakly q -converges to X.*

Proof: Suppose that f_N converges to f in the sense of q-convergence. It follows from Proposition III.3 that the corresponding sequence of induced probability measures $\mu_N=P(X_N^{-1})$ is tight. This yields relatively weak compactness of μ_N. Theorem III.5 implies that each weakly convergent subsequence $\{\mu_{N_j}\}$ of μ_N converges to $\mu=P(X^{-1})$. Hence, $\mu_N\Rightarrow\mu$, or the same, $X_N\Rightarrow X$. Now applying Proposition III.2 we complete the proof. ■

IV. SYMMETRIC (q,α)-STABLE DISTRIBUTIONS AND THEIR PROPERTIES

In this section we introduce the symmetric (q,α)-stable distributions and classify them on the base of mapping (2). In this classification q takes any value in $[1,2)$, however, we distinguish the cases $\alpha=2$ and $0<\alpha<2$.

Definition IV.1: A random variable X is said to have a (q,α) -stable distribution if its q -Fourier transform is represented in the form $e_q^{-\beta|\xi|^\alpha}$, with $\beta>0$. We denote the set of random variables with (q,α) -stable distributions by $\mathcal{L}_q[\alpha]$.

Denote $\mathcal{G}_q[\alpha]=\{be_q^{-\beta|\xi|^\alpha}, b>0, \beta>0\}$. In other words $X\in\mathcal{L}_q[\alpha]$, if $F_q[X]\in\mathcal{G}_q[\alpha]$ with $b=1$. Note that if $\alpha=2$, then $\mathcal{G}_q[2]$ represents the set of q-Gaussians and $\mathcal{L}_q[2]$—the set of random variables whose densities are q_*-Gaussians, where $q_*=(3q-1)/(1+q)$. Further, from the asymptotic relation (5) we have $e_q^{-\beta|\xi|^\alpha}=1-\beta|\xi|^\alpha+o(|\xi|^\alpha)$. This and Proposition II.3 imply that the associated density of any (q,α)-stable distribution belongs to $H_{q,\alpha}$.

Proposition IV.2: Let q-independent random variables $X_j\in\mathcal{L}_q[\alpha]$, $j=1,..,m$. Then for constants $a_1,\ldots,a_m$,

$$\sum_{j=1}^{m} a_j X_j \in \mathcal{L}_q[\alpha].$$

Proof: Let

$$F_q[X_j](\xi) = e_q^{-\beta_j}|\xi|^\alpha, \quad j = 1, \ldots, m.$$

Using the properties $e_q^x \otimes_q e_q^y = e_q^{x+y}$ and $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$, it follows from the definition of the q-independence that

$$F_q\left[\sum_{j=1}^m a_j X_j\right] = e_q^{-\beta|\xi|^\alpha}, \quad \beta = \sum_{j=1}^m \beta_j |a|^{\alpha(2-q)} > 0.$$

■

Proposition IV.2 justifies the stability of distributions in $\mathcal{L}_q[\alpha]$. Recall that if $q=1$ then q-independent random variables are independent in the usual sense. Thus, if $q=1$, $0<\alpha<2$, then $\mathcal{L}_1[\alpha] \equiv \mathcal{L}_{\mathrm{sym}}[\alpha]$, where $\mathcal{L}_{\mathrm{sym}}[\alpha]$ is the set of α-stable Lévy distributions.

Moreover, the appropriately scaling limit of sequences of q-independent random variables with (q,α)-stable distributions has again a (q,α)-stable distribution. To this end consider the sum

$$Z_N = \frac{1}{s_N(q,\alpha)}(X_1 + \cdots + X_N), \quad N = 1,2,\ldots$$

where $s_N(q,\alpha)$ is a scaling parameter specified below. First we prove a general result.

Theorem IV.3: *Assume* $(2q-1,\alpha) \in \mathcal{Q}_2$. *Let* X_N *be symmetric* q *-independent random variables all having the same probability density function* $f(x) \in H_{q,\alpha}$. *Then* Z_N, *with* $s_N(q,\alpha) = (\mu_{q,\alpha} N)^{1/\alpha(2-q)}$ *, is* q *-convergent to a* (q,α) *-stable distribution, as* $N \to \infty$.

Proof: Assume $(Q,\alpha) \in \mathcal{Q}_2$. Let f be the density associated with X_1. First we evaluate $F_q[X_1] = F_q[f(x)]$. Using Proposition II.3 we have

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^\alpha + o(|\xi|^\alpha), \quad \xi \to 0. \tag{37}$$

Denote $Y_j = N^{-1/\alpha} X_j$, $j=1,2,\ldots$. Then $Z_N = Y_1 + \cdots + Y_N$. Further, it is readily seen that for a given random variable X and real $a>0$, the equality $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$ holds. It follows from this relation that $F_q[Y_j] = F_q[f](\xi/(\mu_{q,\alpha}N)^{1/\alpha})$, $j=1,2,\ldots$. Moreover, it follows from the q-independence of $X_1, X_2, \ldots$, and the associativity of the q-product that

$$F_q[Z_N](\xi) = F_q[f]\left(\frac{\xi}{(\mu_{q,\alpha}N)^{1/\alpha}}\right) \underbrace{\otimes_q \cdots \otimes_q}_{N \text{ factors}} F_q[f]\left(\left(\frac{\xi}{(\mu_{q,\alpha}N)^{1/\alpha}}\right)\right). \tag{38}$$

Further, making use of the expansion (6) for the q-logarithm, Eq. (38) implies

$$\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f]((\mu_{q,\alpha}N)^{-1/\alpha}\xi) = N \ln_q\left(1 - \frac{|\xi|^\alpha}{N} + o\left(\frac{|\xi|^\alpha}{N}\right)\right) = -|\xi|^\alpha + o(1), \quad N \to \infty, \tag{39}$$

locally uniformly by ξ. Hence, locally uniformly by ξ,

$$\lim_{N\to\infty} F_q[Z_N] = e_q^{-|\xi|^\alpha} \in \mathcal{G}_q[\alpha]. \tag{40}$$

Thus, Z_N is q-convergent to a random variable with (q,α)-stable distribution, as $N \to \infty$. ■

Since the density of $X \in \mathcal{L}_q[\alpha]$ is in $H_q[\alpha]$ it follows immediately the following Corollary from Theorem IV.3.

Corollary IV.4: Assume $(2q-1,\alpha) \in \mathcal{Q}_2$. *Let* X_N *be a sequence of symmetric* q *-independent* (q,α) *-stable random variables. Then* Z_N, *with the same* $s_N(q,\alpha)$ *in Theorem IV.3,* q *weakly converges to a* (q,α) *-stable distribution.*

Note that $\alpha=2$ is not included to $\mathcal{Q}_2$ in Theorem IV.3. The case $\alpha=2$, in accordance with the first part of Proposition II.3, coincides with Theorem 2 of Ref. 7. Recall that in this case $\mathcal{L}_q[2]$ consists of random variables whose densities are in $\mathcal{G}_{q^*}[2]$, where $q^* = (3q-1)/(q+1)$.

Theorem IV.3 also allows to establish a connection between the classic Lévy distributions and q_α^L-Gaussians. Indeed, for a $X \in \mathcal{L}_q[\alpha]$, its density function f has asymptotics

$$f \sim C_f/|x|^{(\alpha+1)/(1+\alpha(q-1))}, \quad |x| \to \infty.$$

It is not hard to verify that there exists a q_α^L-Gaussian, which is asymptotically equivalent to f. Let us now find q_α^L. Any q_α^L-Gaussian behaves asymptotically $C_1/|x|^\eta = C_2/|x|^{2/(q_\alpha^L-1)}$, $C_j=\text{const}$, $j=1,2$, i.e., $\eta=2/(q_\alpha^L-1)$. Hence, we obtain the relation

$$\frac{\alpha+1}{1+\alpha(q-1)} = \frac{2}{q_\alpha^L-1}. \tag{41}$$

Solving this equation with respect to q_α^L, we have

$$q_\alpha^L = \frac{3+Q\alpha}{\alpha+1}, \quad Q=2q-1, \tag{42}$$

linking three parameters: α, the parameter of the α-stable Lévy distributions, q, the parameter of correlation, and q_α^L, the parameter of attractors in terms of q_α^L-Gaussians. Equation (42) identifies all (Q,α)-stable distributions with the same index of attractor $G_{q_\alpha^L}$, proving the following proposition.

Proposition IV.5: Let $1 \le Q < 3$ $(Q=2q-1)$, $0<\alpha<2$ *, and*

$$\frac{3+Q\alpha}{\alpha+1} = q_\alpha^L. \tag{43}$$

Then the density of $X \in \mathcal{L}_q[\alpha]$ *is asymptotically equivalent to* q_α^L *-Gaussian.*

In the particular case $Q=1$, we recover the known connection between the classical Lévy distributions $(q=Q=1)$ and corresponding q_α^L-Gaussians. In fact, putting $Q=1$ in Eq. (42), we obtain

$$q_\alpha^L = \frac{3+\alpha}{1+\alpha}, \quad 0<\alpha<2. \tag{44}$$

When α increases between 0 and 2 (i.e., $0<\alpha<2$), q_α^L decreases between 3 and 5/3 (i.e., $5/3<q_\alpha^L<3$).

It is useful to find the relationship between $\eta=2/(q_\alpha^L-1)$, which corresponds to the asymptotic behavior of the attractor depending on (α,Q). Using formula (41), we obtain

$$\eta = \frac{2(\alpha+1)}{2+\alpha(Q-1)}. \tag{45}$$

Proposition IV.6: Let $X \in \mathcal{L}_Q[\alpha]$, $1 \le Q < 3$, $0<\alpha<2$ *. Then the associated density function* f_X *has asymptotics* $f_X(x) \sim |x|^\eta$, $|x| \to \infty$ *, where* $\eta=\eta(Q,\alpha)$ *is defined in (45).*

If $Q=1$ (classic Lévy distributions), then (45) implies the well-known fact $\eta=\alpha+1$.

Analogous relationships can be obtained for other values of Q. We call, for convenience, a (Q,α)-stable distribution a Q-Cauchy distribution, if $\alpha=1$. We obtain the classic Cauchy–Poisson distribution if $Q=1$. For Q-Cauchy distributions (43) and (45) imply

$$q_1^L(Q) = \frac{3+Q}{2} \quad \text{and} \quad \eta = \frac{4}{Q+1}, \tag{46}$$

respectively.

V. SCALING LIMITS OF SUMS OF (q,α)-STABLE DISTRIBUTIONS

In this section we generalize the q-CLT established in Ref. 7 for q-Gaussians, that is, in the case of $\alpha=2$, to symmetrical (q,α)-stables with any $\alpha \in (0,2]$.

Let $1<q<2$ and $f \in \mathcal{G}_q[\alpha]$, $0<\alpha\leq 2$. It follows from the definition of the q-exponential that $f \sim C_f|x|^{-\alpha/q-1}$, $C_f>0$, as $|x|\to\infty$. Analogously, if $g\in\mathcal{G}_q[2]$, then $g\sim C_g|x|^{-2/(q-1)}$, $C_g>0$, as $|x|\to\infty$. Comparing orders of asymptotics we can easily verify that for a fixed $\alpha\in(0,2]$ and for any $q\in(1,2)$ there exists a one-to-one mapping,

$$\mathcal{M}_{q,q^*}:\mathcal{G}_q[\alpha]\to\mathcal{G}_{q^*}[2], \quad q^*=\frac{\alpha+2(q-1)}{\alpha},$$

such that the image of a density $f\in\mathcal{G}_q[\alpha]$ is again density. Analogously, there is a one-to-one mapping,

$$\mathcal{K}_{q,q^*}:\mathcal{G}_q[\alpha]\to\mathcal{G}_{q^*}[2],$$

with the same q^*, such that it maps $f(x)=e_q^{-\beta|x|^\alpha}$, an element of $\mathcal{G}_q[\alpha]$ with the coefficient $b=1$ onto the element $g(x)=e_{q^*}^{-(\alpha\beta/2)|x|^2}$ with the same coefficient $b=1$. We notice that if $\alpha=2$, then $q^*=q$ and both operators coincide with the identity operator.

Let $\mathcal{F}_q$ be an operator defined as $\mathcal{F}_q=\mathcal{K}^{-1}_{z(q^*),q_*}F_{q^*}\mathcal{M}_{q,q^*}$, where $z(q^*)=(1+q^*)/(3-q^*)$. It is readily seen that in the particular case $\alpha=2$ it coincides with the q-Fourier transform, $\mathcal{F}_q=F_q$. We call $\mathcal{F}_q$ a generalized q-Fourier transform.

Proposition V.1: Assume $0<\alpha\leq 2$ *and let the numbers* q^*, q_*, *and* q *be connected through the relationships*

$$q^*=\frac{\alpha-2(q-1)}{\alpha} \quad \text{and} \quad q_*=\frac{\alpha q+(q-1)}{\alpha+(q-q)}. \tag{47}$$

Then the mapping

$$\mathcal{F}_q:\mathcal{G}_q[\alpha]\to\mathcal{G}_{q_*} \tag{48}$$

holds.

Proof: We use the scheme

$$\begin{array}{ccc} \mathcal{G}_q[\alpha] & \stackrel{\mathcal{F}_q}{\to} & \mathcal{G}_{q_*}[\alpha], \\ \mathcal{M}_{q,q^*}\downarrow & & \uparrow\mathcal{K}^{-1}_{z(q^*),q_*} \\ \mathcal{G}_{q^*}[2] & \stackrel{F_{q^*}}{\to} & \mathcal{G}_{z(q^*)}[2] \end{array} \tag{49}$$

for the proof. Let a density $f\in\mathcal{G}_q[\alpha]$, i.e., asymptotically $f(x)\sim C_f|x|^{-\alpha/(q-1)}$, $x\to\infty$ with some $C_f>0$. Its image $\mathcal{M}_{q,q^*}[f](x)$, a q^*-Gaussian $G_{q^*}(\beta;x)$, in order to be asymptotically equivalent to f, necessarily

$$G_{q^*}(\beta;x)\sim\frac{C_1}{|x|^{2/(q^*-1)}}\sim\frac{C_f}{|x|^{\alpha/(q-1)}}, \quad |x|\to\infty.$$

Hence,

$$q^*=\frac{\alpha+2(q-1)}{\alpha}=1+\frac{2(q-1)}{\alpha}.$$

Further, it follows from Corollary 2.10 of Ref. 7 that

$$F_{q^*}:\mathcal{G}_{q^*}[2] \to \mathcal{G}_{q_1}[2],$$

where

$$q_1 = \frac{1+q^*}{3-q^*} = \frac{\alpha+(q-1)}{\alpha-(q-1)}.$$

Now taking into account the asymptotic equality [the right vertical line in (49)]

$$G_{q_1}(\beta_1;x) \sim \frac{C_2}{|x|^{2/(q_1-1)}} \sim \frac{C_3}{|x|^{\alpha/(q_*-1)}}, \quad |x| \to \infty,$$

we obtain

$$q_* = \frac{\alpha q-(q-1)}{\alpha-(q-1)} = 1 + \frac{\alpha(q-1)}{\alpha-(q-1)}.$$

Thus, the mapping (48) holds with q^* and q_* in Eq. (47). ■

Let us now introduce two functions that are important for our further analysis,

$$z_\alpha(s) = \frac{\alpha s-(s-1)}{\alpha-(s-1)} = 1 + \frac{\alpha(s-1)}{\alpha-(s-1)}, \tag{50}$$

where $0<\alpha\leq 2$, $s<\alpha+1$, and

$$\zeta_\alpha(s) = \frac{\alpha+2(s-1)}{\alpha} = 1 + \frac{2(s-1)}{\alpha}, \quad 0<\alpha\leq 2. \tag{51}$$

It can be easily verified that $\zeta_\alpha(s)=s$ if $\alpha=2$.

The inverse, $z_\alpha^{-1}(t)$, $t\in(1-\alpha,\infty)$, of the first function reads

$$z_\alpha^{-1}(t) = \frac{\alpha t+(t-1)}{\alpha+(t-1)} = 1 + \frac{\alpha(t-1)}{\alpha+(t-1)}. \tag{52}$$

The function $z(s)$ possesses the properties $z_\alpha(1/z_\alpha(s))=1/s$ and $z_\alpha(1/s)=1/z_\alpha^{-1}(s)$. If we denote $q_{\alpha,1}=z_\alpha(q)$ and $q_{\alpha,-1}=z_\alpha^{-1}(q)$, then

$$z_\alpha\left(\frac{1}{q_{\alpha,1}}\right) = \frac{1}{q} \quad \text{and} \quad z_\alpha\left(\frac{1}{q}\right) = \frac{1}{q_{\alpha,-1}}. \tag{53}$$

Proposition V.1 implies that for $0<\alpha\leq 2$ and $1\leq q<\min\{2,1+\alpha\}$ the following mappings hold:

(i) $\mathcal{F}_q:\mathcal{G}_q[\alpha]\to\mathcal{G}_{z_\alpha(q)}[\alpha]$,
(ii) $\mathcal{F}_q^{-1}:\mathcal{G}_{z_\alpha(q)}[\alpha]\to\mathcal{G}_q[\alpha]$,

where $\mathcal{F}_q^{-1}$ is the inverse to $\mathcal{F}_q$.

It should be noted that as Hilhorst[38] noticed q-Fourier transform, in general, is not one to one in the space of densities. In Ref. 39 the invertibility of F_q in the set of q-Gaussians is established. Since mappings $\mathcal{M}_{q,q^*}$ and $\mathcal{K}_{q,q^*}$ are one to one, relationship (49) yields invertibility of $\mathcal{F}_q$ in $\mathcal{G}_{z_\alpha(q)}[\alpha]$ and validity of property (ii).

Further, we introduce the sequence $q_{\alpha,n}=z_{\alpha,n}(q)=z(z_{\alpha,n-1}(q))$, $n=1,2,\ldots$, with a given $q=z_0(q)$, $q<1+\alpha$. We can extend the sequence $q_{\alpha,n}$ for negative integers $n=-1,-2,\ldots$ as well, setting $q_{\alpha,-n}=z_{\alpha,-n}(q)=z_\alpha^{-1}(z_{\alpha,1-n}(q))$, $n=1,2,\ldots$. It is not hard to verify that

$$q_{\alpha,n} = 1 + \frac{\alpha(q-1)}{\alpha - n(q-1)} = \frac{\alpha q - n(q-1)}{\alpha - n(q-1)} \tag{54}$$

for all integer n satisfying $-\infty < n \le [\alpha/(q-1)]$. The restriction $n \le [\alpha/(q-1)]$ implies the necessary condition $q_{\alpha,n} > 1$, since q-Fourier transform is defined for $q \ge 1$. Note that $q_{\alpha,n}$ is a function of $(q, n/\alpha)$, that $q_{\alpha,n} \equiv 1$ for all $n = 0, \pm 1, \pm 2, \ldots$, if $q = 1$, and that $\lim_{n\to\pm\infty} z_{\alpha,n}(q) = 1$ for all $q \ne 1$. Equation (54) can be rewritten as follows:

$$\frac{\alpha}{q_{\alpha,n} - 1} - n = \frac{\alpha}{q-1}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{55}$$

We note that the latter coincides with Eq. (13) of Ref. 40, once we identify α with the quantity z therein defined, which was obtained through a quite different approach (related to the renormalization of the index q emerging from summing a specific expression over one degree of freedom).

We also note an interesting property of $q_{\alpha,n}$. If we have a q-Gaussian in the variable $|x|^{\alpha/2}$ ($q \ge 1$), i.e., a q-exponential in the variable $|x|^{\alpha}$, its successive derivatives, and integrations with respect to $|x|^{\alpha}$ precisely correspond to $q_{\alpha,n}$-exponentials in the same variable $|x|^{\alpha}$.

Further, we introduce the sequence $q^*_{\alpha,n} = \zeta(q_{\alpha,n})$, which can be written in the form

$$q^*_{\alpha,n} = 1 + \frac{2(q-1)}{\alpha - n(q-1)} = \frac{\alpha + (n-2)(1-q)}{\alpha - n(q-1)} \tag{56}$$

for $n = 0, \pm 1, \ldots$, or, equivalently,

$$\frac{2}{q^*_{\alpha,n} - 1} + n = \frac{\alpha}{q-1}, \quad n = 0, \pm 1, \ldots . \tag{57}$$

It follows from Proposition V.1 and definitions of sequences $q_{\alpha,n}$ and $q^*_{\alpha,n}$ that

$$\mathcal{F}_{q_{\alpha,n}} : \mathcal{G}_{q_{\alpha,n}}[\alpha] \to \mathcal{G}_{q_{\alpha,n+1}}, \quad -\infty < n \le \left[\frac{\alpha}{q-1}\right]. \tag{58}$$

Proposition V.2: For all $n = 0, \pm 1, \pm 2, \ldots$ *the following relations*

$$q^*_{\alpha,n-1} + \frac{1}{q^*_{\alpha,n+1}} = 2, \tag{59}$$

$$q^*_{2,n} = q_{2,n} \tag{60}$$

hold.

Proof: We notice that

$$\frac{1}{q^*_{\alpha,n+1}} = 1 - \frac{2(q-1)}{\alpha - (n-1)(q-1)}.$$

On the other hand, by (56)

$$-\frac{2(q-1)}{\alpha - (n-1)(q-1)} = 1 - q^*_{\alpha,n-1},$$

which implies (59) immediately. The relation (60) can be checked easily. ■

The property $q^*_{2,n} = q_{2,n}$ shows that the sequences (54) and (56) coincide if $\alpha = 2$. Hence, the mapping (58) takes the form $F_{q_{2,n}} : \mathcal{G}_{q_{2,n}}(2) \to \mathcal{G}_{q_{2,n+1}}(2)$, recovering Lemma 2.16 of Ref. 7. Moreover, in this case the duality (59) holds for the sequence $q_{\alpha,n}$ as well. If $\alpha < 2$ then the values of $q^*_{\alpha,n}$ are distinct from the values of $q_{\alpha,n}$. The difference is given by

$$q_{\alpha,n} - q^*_{\alpha,n} = \frac{(2-\alpha)(1-q)}{\alpha + n(1-q)},$$

vanishing for $\alpha=2, \forall q$, or for $q=1, \forall \alpha$. In the latter case $q_{\alpha,n}=q^*_{\alpha,n}\equiv 1$.

Further, we define for $n=0,\pm 1,\ldots$, $k=1,2,\ldots$, $n+k\leq[\alpha/(q-1)]+1$, the operators

$$\mathcal{F}^k_n(f) = \mathcal{F}_{q_{\alpha,n+k-1}} \circ \cdots \circ \mathcal{F}_{q_{\alpha,n}}[f] = \mathcal{F}_{q_{\alpha,n+k-1}}[\cdots \mathcal{F}_{q_{\alpha,n+1}}[\mathcal{F}_{q_{\alpha,n}}[f]]\cdots]$$

and

$$\mathcal{F}^{-k}_n(f) = \mathcal{F}^{-1}_{q_{\alpha,n-k}} \circ \cdots \circ \mathcal{F}^{-1}_{q_{\alpha,n-1}}[f] = \mathcal{F}^{-1}_{q_{\alpha,n-k}}[\cdots \mathcal{F}^{-1}_{q_{\alpha,n-2}}[\mathcal{F}^{-1}_{q_{\alpha,n-1}}[f]]\cdots].$$

In addition, we assume that $\mathcal{F}^k_q[f]=f$, if $k=0$ for any appropriate q. Summarizing the above mentioned relationships, we obtain the following assertions.

Proposition V.3: The following mappings hold:

(1) $\mathcal{F}_{q_{\alpha,n}}:\mathcal{G}_{q_{\alpha,n}}[\alpha]\to\mathcal{G}_{q_{\alpha,n+1}}[\alpha]$, $-\infty<n\leq[\alpha/(q-1)]$;
(2) $\mathcal{F}^k_n:\mathcal{G}_{q_{\alpha,n}}[\alpha]\to\mathcal{G}_{q_{\alpha,k+n}}[\alpha]$, $k=1,2,\ldots$, $n=0,\pm 1,\ldots$, $-\infty<n+k\leq[\alpha/(q-1)]+1$;
(3) $\lim_{k\to-\infty}\mathcal{F}^k_n\mathcal{G}_q[\alpha]=\mathcal{G}[\alpha]$, $n=0,\pm 1,\ldots$,

where $\mathcal{G}(\alpha)$ is the set of densities of classic symmetric α-stable Lévy distributions.

Theorem V.4: *Assume $0<\alpha\leq 2$ and a sequence $q_{\alpha,n},-\infty<n\leq[\alpha/(q-1)]$, is given as in (54) with $q_0=q\in[1,\min\{2,1+\alpha\})$. Let X_N be a symmetric $q_{\alpha,k}$ -independent (for some $-\infty<k\leq[\alpha/(q-1)]$ and $\alpha\in(0,2]$)) random variables all having the same probability density function $f(x)$ satisfying the conditions of Proposition II.3.*

Then the sequence

$$Z_N = \frac{X_1 + \cdots + X_N}{(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha(2-q_{\alpha,k})}},$$

is $q_{\alpha,k}$-convergent to a $(q_{\alpha,k-1},\alpha)$ -stable distribution, as $N\to\infty$.

Proof: The case $\alpha=2$ coincides with Theorem 1 of Ref. 7. For $k=0$, the first part of theorem (q-convergence) is proven in Sec. V of the present paper. The same method can be applied for $k\neq 1$. For the readers convenience we proceed the proof of the first part also in the general case, namely, for arbitrary k. Suppose that $0<\alpha<2$. We evaluate $F_{q_{\alpha,k}}(Z_N)$. Denote $Y_j=X_j/s_N(q_{\alpha,k})$, $j=1,2,\ldots$, where $s_N(q_{\alpha,k})=(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha(2-q_{\alpha,k})}$. Then $Z_N=Y_1+\cdots+Y_N$. Again using the relationship $F_q[aX](\xi)=F_q[X](a^{2-q}\xi)$, we obtain $F_{q_{\alpha,k}}(Y_1)=F_{q_{\alpha,k}}[f](\xi/(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha})$. Further, it follows from $q_{\alpha,k}$-independence of $X_1,X_2,\ldots$ and the associativity property of the q-product that

$$F_{q_{\alpha,k}}[Z_N](\xi) = \otimes^N_{q_{\alpha,k}} F_{q_{\alpha,k}}[f]\left(\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha}}\right), \tag{61}$$

the right hand side of which exhibits the $q_{\alpha,k}$-product of N identical factors $F_{q_{\alpha,k}}[f]\left(\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha}}\right)$. Hence, making use of the properties of the q-logarithm, from (61) we obtain

$$\begin{aligned}\ln_{q_{\alpha,k}} F_{q_{\alpha,k}}[Z_N](\xi) &= N \ln_{q_{\alpha,k}} F_{q_{\alpha,k}}[f]\left(\frac{\xi}{(\mu_{q_{\alpha,k},\alpha}N)^{1/\alpha}}\right) = N \ln_{q_{\alpha,k}}\left(1 - \frac{|\xi|^\alpha}{N} + o\left(\frac{|\xi|^\alpha}{N}\right)\right)\\ &= -|\xi|^\alpha + o(1), \quad N\to\infty,\end{aligned} \tag{62}$$

locally uniformly by ξ. Consequently, locally uniformly by ξ,

$$\lim_{N\to\infty} F_{q_{\alpha,k}}(Z_N) = e_{q_{\alpha,k}}^{-|\xi|^\alpha} \in \mathcal{G}_{q_{\alpha,k}}(\alpha). \tag{63}$$

Thus, Z_N is $q_{\alpha,k}$-convergent.

To show the second part of Theorem we use Proposition V.3. In accordance with this lemma there exists a density $f(x) \in \mathcal{G}_{q_{\alpha,k-1}}[\alpha]$, such that $\mathcal{F}_{\alpha,k-1}[f]=e_{q_{\alpha,k}}^{-|\xi|^\alpha}$. Hence, Z_N is $q_{\alpha,k}$-convergent to a $(q_{\alpha,k-1},\alpha)$-stable distribution, as $N\to\infty$. ■

VI. SCALING RATE ANALYSIS

In Ref. 7 the formula

$$\beta_k = \left(\frac{3-q_{k-1}}{4q_k C_{q_{k-1}}^{2q_{k-1}-2}}\right)^{1/(2-q_{k-1})} \tag{64}$$

was obtained for the q-Gaussian parameter β of the attractor. It follows from this formula that the scaling rate in the case $\alpha=2$ is

$$\delta = \frac{1}{2-q_{k-1}} = q_{k+1}, \tag{65}$$

where q_{k-1} is the q-index of the attractor. Moreover, if we insert the "evolution parameter" t, then the translation of a q-Gaussian to a density in $\mathcal{G}_q[\alpha]$ changes t to $t^{2/\alpha}$. Hence, applying these two facts to the general case, $0<\alpha\le 2$, and taking into account that the attractor index in our case is $q^*_{\alpha,k-1}$, we obtain the formula for the scaling rate,

$$\delta = \frac{2}{\alpha(2-q^*_{\alpha,k-1})}. \tag{66}$$

In accordance with Proposition V.2, $2-q^*_{\alpha,k-1}=1/q^*_{\alpha,k+1}$. Consequently,

$$\delta = \frac{2}{\alpha} q^*_{\alpha,k+1} = \frac{2}{\alpha}\frac{\alpha-(k-1)(q-1)}{\alpha-(k+1)(q-1)}. \tag{67}$$

Finally, in terms of $Q=2q-1$ the formula (67) takes the form

$$\delta = \frac{2}{\alpha}\frac{2\alpha-(k-1)(Q-1)}{2\alpha-(k+1)(Q-1)}. \tag{68}$$

In Ref. 7 it was noticed that the scaling rate in the nonlinear Fokker–Planck equation can be derived from the model corresponding to the case $k=1$. Taking this fact into account we can *conjecture that the scaling rate in the fractional generalization of the nonlinear Fokker–Planck equation is*

$$\delta = \frac{2}{\alpha+1-Q},$$

which can be derived from (68) setting $k=1$. In the case $\alpha=2$ we get the known result $\delta=2/(3-Q)$ obtained in Ref. 41.

VII. ON ADDITIVE AND MULTIPLICATIVE DUALITIES

In the nonextensive statistical mechanical literature, there are two transformations that appear quite frequently in various contexts. They are sometimes referred to as *dualities*. The *multiplicative duality* is defined through

$$\mu(q) = 1/q, \tag{69}$$

and the *additive duality* is defined through

$$\nu(q) = 2 - q. \tag{70}$$

They satisfy $\mu^2 = \nu^2 = \mathbf{1}$, where $\mathbf{1}$ represents the *identity*, i.e., $\mathbf{1}(q) = q, \forall q$. We also verify that

$$(\mu\nu)^m(\nu\mu)^m = (\nu\mu)^m(\mu\nu)^m = \mathbf{1} \quad (m = 0,1,2,\ldots).$$

Consistently, we define $(\mu\nu)^{-m} \equiv (\nu\mu)^m$ and $(\nu\mu)^{-m} \equiv (\mu\nu)^m$.

Also, for $m = 0, \pm 1, \pm 2, \ldots$, and $\forall q$,

$$(\mu\nu)^m(q) = \frac{m - (m-1)q}{m + 1 - mq} = \frac{q + m(1-q)}{1 + m(1-q)},$$

$$\nu(\mu\nu)^m(q) = \frac{m + 2 - (m+1)q}{m + 1 - mq} = \frac{2 - q + m(1-q)}{1 + m(1-q)}, \tag{71}$$

and

$$(\mu\nu)^m\mu(q) = \frac{-m + 1 + mq}{-m + (m+1)q} = \frac{1 - m(1-q)}{q - m(1-q)}.$$

We can easily verify, from Eqs. (54) and (71), that the sequences $q_{2,n}$ $(n = 0, \pm 2, \pm 4, \ldots)$ and $q_{1,n}$ $(n = 0, \pm 1, \pm 2, \ldots)$ coincide with the sequence $(\mu\nu)^m(q)$ $(m = 0, \pm 1, \pm, 2, \ldots)$.

VIII. CLASSIFICATION OF (q, α)-STABLE DISTRIBUTIONS AND SOME CONJECTURES

The q-CLT formulated in Ref. 7 states that the appropriately scaling limit of sums of q_k-independent random variables with a finite $(2q_k - 1)$-variance is a q_k^*-Gaussian, which is a q_k^*-Fourier preimage of a q_k-Gaussian. Here q_k and q_k^* are sequences defined as

$$q_k = \frac{2q - k(q-1)}{2 - k(q-1)}, \quad k = 0, \pm 1, \ldots,$$

and

$$q_k^* = q_{k-1}, \quad k = 0, \pm 1, \ldots .$$

Schematically q-CLT in Ref. 7 can be represented as

$$\{f : \sigma_{2q_k - 1}(f) < \infty\} \overset{F_{q_k}}{\longrightarrow} \mathcal{G}_{q_k}[2] \overset{F_{q_k^*}}{\longleftarrow} \mathcal{G}_{q_k^*}[2]. \tag{72}$$

We have also noticed that q-CLT can be described by the triplet $(P_{\mathrm{att}}, P_{\mathrm{cor}}, P_{\mathrm{scl}})$, where P_{att}, P_{cor}, and P_{scl} represent parameters of *the attractor*, *the correlation*, and *the scaling rate*, respectively. We found that (see details in Ref. 7) for q-CLT this triplet,

$$(P_{\mathrm{att}}, P_{\mathrm{cor}}, P_{\mathrm{scl}}) \equiv (q_{k-1}, q_k, q_{k+1}). \tag{73}$$

Schematically Theorem 3 of the current paper can be represented as

$$\mathcal{L}_q[\alpha] \overset{F_q}{\longrightarrow} \mathcal{G}_q[\alpha] \overset{F_q}{\longleftarrow} \mathcal{G}_{q^L}[2], \quad 0 < \alpha < 2, \tag{74}$$

where $\mathcal{L}_q[\alpha]$ is the set of (q, α)-stable distributions, $\mathcal{G}_{q^L}[2]$ is the set of q^L-Gaussians with index q^L defined as

$$q^L = q^L_\alpha(q) = \frac{3+(2q-1)\alpha}{1+\alpha}.$$

Recall that the case $\alpha=2$ was peculiar and we agree to refer to the scheme (72) in this case.

Theorem 4 generalizes the q-CLT (which corresponds to $\alpha=2$) to the whole range $0<\alpha\leq 2$. Schematically this theorem can be represented as

$$\mathcal{L}_{q_{\alpha,k}}[\alpha] \xrightarrow{F_{q_{\alpha,k}}} \mathcal{G}_{q_{\alpha,k}}[\alpha] \xleftarrow{F_{q^*_{\alpha,k}}} \mathcal{G}_{q^*_{\alpha,k}}[2], \quad 0<\alpha\leq 2, \tag{75}$$

generalizing the scheme (72). The sequences $q_{\alpha,k}$ and $q^*_{\alpha,k}$ in this case read

$$q_{\alpha,k} = \frac{\alpha q + k(1-q)}{\alpha + k(1-q)}, \quad k=0, \pm 1, \ldots ,$$

and

$$q^*_{\alpha,k} = 1 - \frac{2(1-q)}{\alpha + k(1-q)}, \quad k=0, \pm 1, \ldots .$$

Note that the triplet $(P_{\text{att}}, P_{\text{cor}}, P_{\text{scl}})$ mentioned above, in this case, takes the form

$$(P_{\text{att}}, P_{\text{cor}}, P_{\text{scl}}) \equiv (q^*_{\alpha,k-1}, q_{\alpha,k}, (2/\alpha) q^*_{\alpha,k+1}), \tag{76}$$

recovering the triplet (73) in the case $\alpha=2$.

In connection with the above discussion about triplets, we note that the existence of a q-triplet, namely, $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$, related, respectively, to sensitivity to the initial conditions, relaxation, and stationary state was conjectured in Ref. 42. Later it was observed in the solar wind at the distant heliosphere.[43,44] The triplet in (73) obtained theoretically might be useful hint for its understanding.

Finally, unifying the schemes (74) and (75) we obtain the general picture for the description of (q,α)-stable distributions,

$$\begin{array}{c} \mathcal{L}_{q_{\alpha,k}}[\alpha] \xrightarrow{F_{q_{\alpha,k}}} \mathcal{G}_{q_{\alpha,k}}[\alpha] \xleftarrow{F_{q^*_{\alpha,k}}} \mathcal{G}_{q^*_{\alpha,k}}[2], \\ \\ \updownarrow F_q, \\ \\ \mathcal{G}_{q^L_{\alpha,k}}[2], \end{array} \tag{77}$$

where

$$q^L_{\alpha,k} = q^L_\alpha(q_{\alpha,k}) = \frac{3+(2q_{\alpha,k}-1)\alpha}{1+\alpha}.$$

In Fig. 1 the dependence of q^L and q^* on parameters $(Q,\alpha)\in\mathcal{Q}$ in the case $k=0$ is represented. If $Q=1$ and $\alpha=2$ (the blue box in the figure), then the random variables are independent in the usual sense and have *finite* variance. The standard CLT applies, and the attractors are classic Gaussians.

If Q belongs to the interval (1,3) and $\alpha=2$ (the blue straight line on the top), the random variables are *not* independent. If the random variables have a *finite* Q-variance, then q-CLT (Ref. 7) applies, and the attractors belong to the family of q^*-Gaussians. Note that q^* runs in [1,5/3). Thus, in this case, attractors (q^*-Gaussians) have *finite* classic variance (i.e., 1-variance) in addition to *finite* q^*-variance.

If $Q=1$ and $0<\alpha<2$ (the vertical green line in the figure), we have the classic Lévy distributions, and random variables are independent, and have *infinite* variance. Their scaling limits-

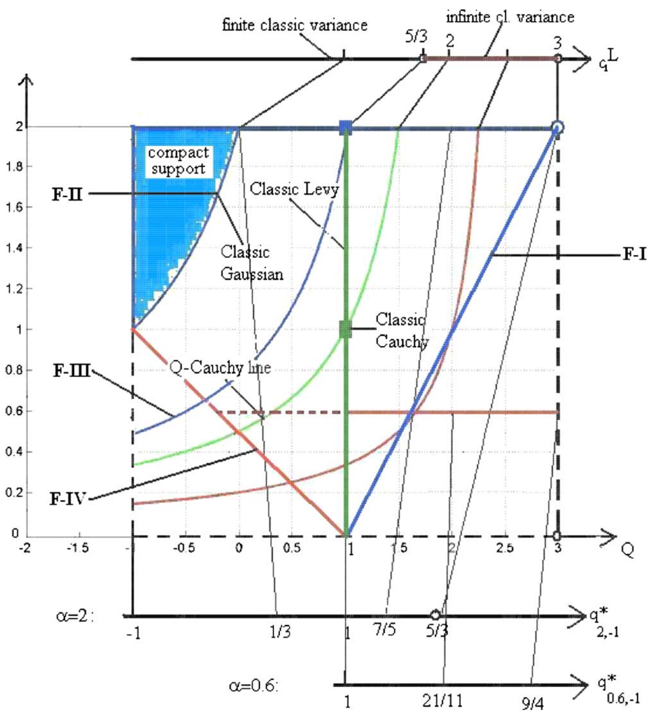

FIG. 1. (Color online) (Q,α)-regions.

attractors belong to the family of α-stable Lévy distributions. It follows from (44) that in terms of q-Gaussians classic symmetric α-stable distributions correspond to $\cup_{5/3<q<3}\mathcal{G}_q$.[8]

If $0<\alpha<2$ and Q belong to the interval (1,3) we observe the rich variety of possibilities of (q,α)-stable distributions. In this case random variables are *not* independent and have *infinite* variance and *infinite Q-variance*. The rectangle $\{1<Q<3;0<\alpha<2\}$, at the right of the classic Lévy line, is covered by nonintersecting curves,

$$C_{q^L} \equiv \left\{ (Q,\alpha): \frac{3+Q\alpha}{\alpha+1} = q^L \right\}, \quad 5/3 < q^L < 3.$$

This family of curves describes all (Q,α)-stable distributions based on the mapping (74) with q-Fourier transform. The constant q^L is the index of the q^L-Gaussian attractor corresponding to the points (Q,α) on the curve C_{q^L}. For example, the curve (green) corresponding to $q^L=2$, which passes through point $(1,1)$ (the green box in the figure), describes all Q-Cauchy distributions, recovering the classic Cauchy–Poisson distribution if $\alpha=1$. The figure also represents the curve (brown) describing all the distributions corresponding to $q^L=2.5$.

Every point (Q,α) lying on the brown curve corresponds to $q^L=2.5$.

The second classification of (Q,α)-stable distributions presented in the current paper and based on the mapping (75) with q^*-Fourier transform leads to a covering of $\mathcal{Q}$ by curves distinct from C_{q^L}. Namely, in this case we have the following family of straight lines:

$$L_{q^*} \equiv \left\{ (Q,\alpha): \frac{4\alpha}{Q+2\alpha-1} = 3-q^* \right\}, \quad 1 \le q^* < 3, \tag{78}$$

which are obtained from (56) replacing $n=-1$ and $2q-1=Q$. For instance, every (Q,α) on the line F-I (the blue diagonal of the rectangle in the figure) identifies q^*-Gaussians with $q^*=5/3$. This line is the frontier of points (Q,α) with finite and infinite classic variances. Namely, all (Q,α) above the line F-I identify attractors with *finite* variance, and points on this line and below identify attractors with *infinite* classic variance. Two bottom lines in Fig. 1 reflect the sets of q^* corresponding to lines $\{1 \le Q < 3; \alpha = 2)\}$ (the top boundary of the rectangle in the figure) and $\{1 \le Q < 3; \alpha = 0.6)\}$ (the brown horizontal line in the figure).

Some conjectures. Both classifications of (Q,α)-stable distributions are restricted to the region $Q = \{1 \le Q < 3, 0 < \alpha \le 2\}$. This limitation is caused by the tool used for these representations, namely, Q-Fourier transform is defined for $Q \ge 1$. However, at least two facts, the positivity of $\mu_{q,\alpha}$ in Proposition II.3 for $q > \max\{0, 1-1/\alpha\}$ (or, the same, $Q > \max\{-1, 1-2/\alpha\}$) and continuous extensions of curves in the family C_{q^L}, strongly indicate to following conjectures, regarding the region $\{Q < 1\}$ on the left to the vertical green line (the classic Lévy line) in Fig. 1. In this region we see three frontier lines, F-II, F-III, and F-IV.

Conjecture VIII.1. The line F-II splits the regions where the random variables have finite and infinite Q-variances. More precisely, the random variables corresponding to (Q,α) *on and above the line F-II have a finite Q -variance, and, consequently, q-CLT* (Ref. 7) *applies. Moreover, as seen in the figure, the* q^L *-attractors corresponding to the points on the line F-II are the classic Gaussians, because* $q^L = 1$ *for these* (Q,α)*. It follows from this fact that* q^L *-Gaussians corresponding to points above F-II have compact support (the blue region in the figure), and* q^L *-Gaussians corresponding to points on this line and below have infinite support.*

Conjecture VIII.2: The line F-III splits the points (Q,α) *whose* q^L *-attractors have finite or infinite classic variances. More precisely, the points* (Q,α) *above this line identify attractors (in terms of* q^L *-Gaussians) with finite classic variance, and the points on this line and below identify attractors with infinite classic variance.*

Conjecture VIII.3: The frontier line F-IV with the equation $Q + 2\alpha - 1 = 0$ *and joining the points (1,0) and* $(-1,1)$ *is related to attractors in terms of* q^*-Gaussians. It follows from *(78) that for* (Q,α) *lying on the line F-IV, the index* $q^* = -\infty$*. Thus the horizontal lines corresponding to* $\alpha < 1$ *can be continued only up to the line F-IV with* $q^* \in (-\infty, 3 - 4\alpha/(Q + 2\alpha - 1))$ *(see the dashed horizontal brown line in the figure). If* $\alpha \to 0$ *, the Q -interval becomes narrower, but* q^* *-interval becomes larger tending to* $(-\infty, 3)$.

Results confirming or refuting any of these conjectures would be an essential contribution to deeper understanding of the nature of (Q,α)-stable distributions, and nonextensive statistical mechanics, in particular.

Finally, we note that Fig. 1 corresponds to the case $k=0$ in the description (4). The cases $k \ne 0$ can be treated in the same way.

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[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[2] D. Prato and C. Tsallis, Phys. Rev. E **60**, 2398 (1999).
[3] M. Gell-Mann and C. Tsallis, *Nonextensive Entropy–Interdisciplinary Applications* (Oxford University Press, New York, 2004).
[4] J. P. Boon and C. Tsallis, Europhys. News **36**, 185 (2005).
[5] C. Tsallis, *Entropy*, Encyclopedia of Complexity and Systems Science (Springer, Berlin, 2009).
[6] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics–Approaching a Complex World* (Springer, New York, 2009).
[7] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008).
[8] S. Umarov and C. Tsallis, AIP Conf. Proc. **965**, 34 (2007).
[9] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, 1954).
[10] W. Feller, *An Introduction to Probability Theory and its Applications II* (Wiley, New York, 1966).
[11] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes* (Chapman and Hall, New York, 1994).
[12] V. V. Uchaykin and V. M. Zolotarev, *Chance and Stability. Stable Distributions and their Applications* (VSP, Utrecht, 1999).
[13] M. M. Meerschaert and H.-P. Scheffler, *Limit Distributions for Sums of Independent Random Vectors. Heavy Tails in Theory and Practice* (Wiley, New York, 2001).
[14] G. Zaslavsky, Phys. Rep. **371**, 461 (2002).

[15] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[16] B. Mandelbrot, *Fractals and Scaling in Finance* (Springer, New York, 1997).
[17] J. H. McCulloch, in *Handbook of Statistics*, edited by G. S. Maddala and C. R. Rao (North-Holland, New York, 1996), Vol. 14.
[18] S. Mittnik and S. Rachev, Econometric Rev. **12**, 261 (1993).
[19] E. W. Montroll and M. F. Shlesinger, Proc. Natl. Acad. Sci. U.S.A. **79**, 3380 (1982).
[20] E. W. Montroll and J. T. Bendler, J. Stat. Phys. **34**, 129 (1984).
[21] R. D. Pierce, *Proceedings of the National Radar Conference* (IEEE, New York, 1996), pp. 154–159.
[22] C. Beck and F. Schloegel, *Thermodynamics of Chaotic Systems: An Introduction* (Cambridge University Press, Cambridge, 1993).
[23] F. G. Schmitt and L. Seuront, Physica A **301**, 375 (2001).
[24] F. Caruso, A. Pluchino, V. Latora, S. Vinciguerra, and A. Rapisarda, Phys. Rev. E **75**, 055101 (2007).
[25] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).
[26] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).
[27] C. Tsallis and S. M. D. Queiros, AIP Conf. Proc. **965**, 8 (2007).
[28] S. M. D. Queiros and C. Tsallis, AIP Conf. Proc. **965**, 21 (2007).
[29] E. P. Borges, J. Phys. A **31**, 5281 (1998).
[30] A. Rodriguez, V. Schwammle, and C. Tsallis, J. Stat. Mech.: Theory Exp. (2008) P09006.
[31] R. Hanel, S. Thurner, and C. Tsallis, Eur. Phys. J. B **72**, 263 (2009).
[32] L. G. Moyano, C. Tsallis, and M. Gell-Mann, Europhys. Lett. **73**, 813 (2006).
[33] W. J. Thistleton, J. A. Marsh, K. P. Nelson, and C. Tsallis, Cent. Eur. J. Phys. **7**, 387 (2009).
[34] H. J. Hilhorst and G. Schehr, J. Stat. Mech.: Theory Exp. (2007) P06003.
[35] M. Hahn, X. Jiang, and S. Umarov, "On q-Gaussians and exchangeability," J. Phys. A: Math. Theor. (to be published).
[36] L. Nivanen, A. Le Mehaute, and Q. A. Wang, Rep. Math. Phys. **52**, 437 (2003).
[37] E. P. Borges, Physica A **340**, 95 (2004).
[38] H. J. Hilhorst, Braz. J. Phys. **39**, (2A), 371 (2009).
[39] S. Umarov and C. Tsallis, Phys. Lett. A **372**, 4874 (2008).
[40] R. S. Mendes and C. Tsallis, Phys. Lett. A **285**, 273 (2001).
[41] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).
[42] C. Tsallis, Physica A **340**, 1 (2004).
[43] L. F. Burlaga and A. F. Vinas, Physica A **356**, 375 (2005).
[44] L. F. Burlaga, N. F. Ness, and M. H. Acuna, Astrophys. J. **668**, 1246 (2007).