GENERALIZATION OF THE DIVISOR PRODUCTS AND PROPER DIVISOR PRODUCTS SEQUENCES

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Abstract Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n, $q_d(n)$ denotes the product of all proper divisors of n. In this paper, we study the properties of the sequences of $\{p_d(n)\}$ and $\{q_d(n)\}$, and prove that the generalized results for the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$.

Keywords: Divisor and proper divisor product; Generalization ; Sequence.

§1. Introduction and results

Let n be a positive integer, $p_d(n)$ denotes the product of all positive divisors of n. That is, $p_d(n) = \prod_{d|n} d$. For example, $p_d(1) = 1$, $p_d(2) = 2$, $p_d(3) = 3$, $p_d(4) = 8, \cdots, p_d(p) = p, \cdots$. $q_d(n)$ denotes the product of all proper divisors of n. That is, $q_d(n) = \prod_{d|n,\ d<n} d$. For example, $q_d(1) = 1$, $q_d(2) = 1$, $q_d(3) = 1$, $q_d(4) = 2, \cdots$. In problem 25 and 26 of [1], Professor F. Smarandache asked us to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$. About this problem, Liu Hongyan and Zhang Wenpeng in [2] have studied it and proved the Makowsiki & Schinzel conjecture in [3] hold for $\{p_d(n)\}$ and $\{q_d(n)\}$. One of them is that for any positive integer n, we have the inequality:

$$\sigma(\phi(p_d(n))) \geq \frac{1}{2} p_d(n), \tag{1}$$

where $\sigma(n)$ is the divisor sum function, $\phi(n)$ is the Euler's function.

In this paper, as the generalization of [2], we will consider the properties of the sequences of $\{p_d(n)\}$ and $\{q_d(n)\}$ for k-th divisor sum function, and give two more general results. That is, we shall prove the following:

Theorem 1. Let $n = p^\alpha$, p be a prime and α be a positive integer. Then for any fixed positive integer k, we have the inequality

$$\sigma_k(\phi(p_d(n))) \geq \frac{1}{2^k} p_d^k(n),$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the k-th divisor sum function.

Theorem 2. Let $n = p^\alpha$, p be a prime and α be a positive integer. Then for any fixed positive integer k, we have the inequality

$$\sigma_k(\phi(q_d(n))) \geq \frac{1}{2^k} q_d^k(n).$$

§2. Proof of the theorems

In this section, we shall complete the proof of the theorem. First we need two Lemmas as following:

Lemma 1. For any positive integer n, then we have the identity $p_d(n) = n^{\frac{d(n)}{2}}$ and $q_d(n) = n^{\frac{d(n)}{2}-1}$,

where $d(n) = \sum_{d|n} 1$ is the divisor function.

Proof. (See Reference [2] Lemma 1).

Lemma 2. For any positive integer n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ with $\alpha_i \geq 2$ $(1 \leq i \leq s)$, $p_j (1 \leq j \leq s)$ are some different primes with $p_1 < p_2 \cdots p_s$. Then for any fixed positive integer k, we have the estimate

$$\sigma_k(\phi(n)) \geq \phi^k(n) \cdot \prod_{p|n} \left(1 + \frac{1}{p^k}\right).$$

Proof. From the properties of the Euler's function we have

$$\begin{aligned} \phi(n) &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_s^{\alpha_s}) \\ &= p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_s^{\alpha_s - 1}(p_1 - 1)(p_2 - 1) \cdots (p_s - 1). \end{aligned} \tag{2}$$

Here, let $(p_1 - 1)(p_2 - 1) \cdots (p_s - 1) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}$, where $\beta_i \geq 0$, $1 \leq i \leq s; r_j \geq 1, 1 \leq j \leq t$ and $q_1 < q_2 < \cdots < q_t$ are different primes. Note that $\sigma_k(p^\alpha) = 1^k + p^k + \cdots + p^{k\alpha} = \frac{p^{k(\alpha+1)}-1}{p^k-1}$, for any $k > 0$.

Then for (2), we deduce that

$$\begin{aligned} \sigma_k(\phi(n)) &= \sigma_k(p_1^{\alpha_1+\beta_1-1} p_2^{\alpha_2+\beta_2-1} \cdots p_s^{\alpha_s+\beta_s-1} q_1^{r_1} q_2^{r_2} \cdots q_t^{r_t}) \\ &= \prod_{i=1}^{s} \frac{p_i^{k(\alpha_i+\beta_i)} - 1}{p_i^k - 1} \prod_{j=1}^{t} \frac{q_j^{k(r_j+1)} - 1}{q_j^k - 1} \\ &= p_1^{k(\alpha_1+\beta_1)} p_2^{k(\alpha_2+\beta_2)} \cdots p_s^{k(\alpha_s+\beta_s)} q_1^{kr_1} q_2^{kr_2} \cdots q_t^{kr_t} \end{aligned} \tag{3}$$

$$\times \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{p_i^k-1} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}}$$

$$= \quad n^k \cdot \prod_{i=1}^{s}\left(1-\frac{1}{p_i}\right)^k \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{1-\frac{1}{p_i^x}} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}}.$$

Because

$$\phi(n) = n \cdot \prod_{p|n}\left(1-\frac{1}{p}\right), \tag{4}$$

then from (3) and (4) we get

$$\begin{aligned} \sigma_k(\phi(n)) &= n^k \cdot \frac{\phi^k(n)}{n^k} \cdot \prod_{i=1}^{s} \frac{1-\frac{1}{p_i^{k(\alpha_i+\beta_i)}}}{1-\frac{1}{p_i^x}} \prod_{j=1}^{t} \frac{1-\frac{1}{q_j^{k(r_j+1)}}}{1-\frac{1}{q_j^k}} \\ &= \phi^k(n) \cdot \prod_{i=1}^{s}(1+\frac{1}{p_i^k}+\cdots+\frac{1}{p_i^{k(\alpha_i+\beta_i)-1}}) \\ &\quad \times \prod_{j=1}^{t}(1+\frac{1}{q_j^k}+\cdots+\frac{1}{q_j^{k(r_j+1)-1}}) \\ &\geq \phi^k(n) \cdot \prod_{p|n}\left(1+\frac{1}{p^k}\right). \end{aligned}$$

This completes the proof of Lemma 2.

Now we use Lemma 1 and Lemma 2 to complete the proof of Theorem 1. Here we will debate this problem in two cases:

(i) If n is a prime, then $d(n) = 2$. So from Lemma 1 we have

$$P_d(n) = n^{\frac{d(n)}{2}} = n. \tag{5}$$

Noting that $\phi(n) = n - 1$, then from (5) we immediately get

$$\sigma_k(\phi(P_d(n))) = \sigma_k(n-1) = \sum_{d|(n-1)} d^k \geq (n-1)^k \geq \frac{1}{2^k} \cdot n^k = \frac{1}{2^k} P_d^k(n).$$

(ii) If $n = p^\alpha$, p be a prime and $\alpha > 1$ be any positive integer. Then $d(n) = \alpha + 1$. So that

$$P_d(n) = n^{\frac{d(n)}{2}} = p^{\frac{\alpha(\alpha+1)}{2}}. \tag{6}$$

Using Lemma 2 and (6), we can easily deduce that

$$\sigma_k(\phi(P_d(n))) \quad = \quad \sigma_k(\phi(p^{\frac{\alpha(\alpha+1)}{2}}))$$

$$\begin{aligned}
&\geq \phi^k(p^{\frac{\alpha(\alpha+1)}{2}}) \prod_{p_1 | p^{\frac{\alpha(\alpha+1)}{2}}} \left(1+\frac{1}{p_1^k}\right) \\
&= p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \left(1-\frac{1}{p}\right)^k \cdot \left(1+\frac{1}{p^k}\right) \\
&\geq p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \left(1-\frac{1}{p}\right)^k \\
&\geq p^{\frac{k\alpha(\alpha+1)}{2}} \cdot \frac{1}{2^k} = \frac{1}{2^k} P_d^k(n).
\end{aligned}$$

This completes the proof of Theorem 1.

Similarly, we can easily prove Theorem 2. That is,

(i) If n is a prime, then $d(n) = 2$. So from Lemma 1 we have

$$q_d(n) = n^{\frac{d(n)}{2}-1} = 1, \tag{7}$$

hence

$$\sigma_k(\phi(q_d(n))) = \sigma_k(1) = 1 \geq \frac{1}{2^k} q_d^k(n).$$

(ii) If $n = p^\alpha$, p be a prime and $\alpha > 1$ be any positive integer. Then $d(n) = \alpha + 1$, so that

$$q_d(n) = n^{\frac{d(n)}{2}-1} = p^{\frac{\alpha(\alpha-1)}{2}}. \tag{8}$$

Using Lemma 2 and (8), we have

$$\sigma_k(\phi(q_d(n))) = \sigma_k(\phi(p^{\frac{\alpha(\alpha-1)}{2}})) \geq \frac{1}{2^k} q_d^k(n).$$

This completes the proof of Theorem 2.

References

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