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Generalization of the topological algebra $(C_b(X), \beta)$

by

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Abstract. We study subalgebras of $C_b(X)$ equipped with topologies that generalize both the uniform and the strict topology. In particular, we study the Stone–Weierstrass property and describe the ideal structure of these algebras.

1. Introduction. Let X be a completely regular Hausdorff space. The algebra $C_b(X)$ of all continuous and bounded complex-valued functions on X is one of the most studied objects in modern analysis. Usually it is equipped with the supremum norm topology (denoted by σ) which makes it a Banach algebra. However, the structure of $(C_b(X), \sigma)$ is quite complicated. For example, its Gelfand space (the set of all regular maximal ideals equipped with the relative weak*-topology) is homeomorphic to the Stone–Čech compactification $\beta(X)$ of X. Another difficulty is that $(C_b(X), \sigma)$ does not have the Stone–Weierstrass property, i.e., a point-separating symmetric subalgebra which is bounded away from zero is not necessarily uniformly dense in $C_b(X)$. So the topology on $C_b(X)$ defined by the supremum norm is not the "best" topology from this point of view.

Another well-known and useful topology on $C_b(X)$ is the so-called strict topology (denoted by β) defined by the family of weighted supremum seminorms with weights running through all bounded (or equivalently, upper semicontinuous) non-negative functions on X which vanish at infinity. The topological algebra $(C_b(X), \beta)$ is in some sense easier to handle than $(C_b(X), \sigma)$. For example, its Gelfand space is homeomorphic to X and it has the Stone–Weierstrass property. On the other hand, even though the structure of $(C_b(X), \beta)$ has been extensively studied, it appears that its closed ideals have not been described yet. In [23] it was claimed without proof that in the case when X is locally compact, every closed ideal of $(C_b(X), \beta)$ consists of those functions on $C_b(X)$ which vanish on some closed subset E of X.

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The structures of $(C_b(X), \sigma)$ and $(C_b(X), \beta)$ have been generalized in [1–5, 15] so as to contain also unbounded functions. In these generalizations functions behave locally on certain subsets S of X as functions in $C_b(S)$. In this paper, we generalize the structures of $(C_b(X), \sigma)$ and $(C_b(X), \beta)$ in another direction. Namely, we study certain types of (possibly proper) subalgebras of $C_b(X)$ and provide them with topologies that generalize both the uniform and the strict topologies. We then study the Stone–Weierstrass property and the ideal structure of these algebras. In particular, a detailed description of the closed ideals and quotient algebras (modulo closed ideals) is given with respect to a topology that generalizes the strict topology. Therefore, the results we get for these subalgebras imply some old and new results for $(C_b(X), \beta)$.

The algebras we are interested in consist of functions which vanish at certain fixed points very quickly. A motivation for their study comes from connections to many classical results on Banach and function algebras. These include Gelfand representation of commutative Banach algebras and Stone–Weierstrass type approximation theorems. In particular, the structures we will study are required to get a satisfactory Gelfand representation also for those Banach algebras for which the Gelfand transform algebra is not complete with respect to the supremum norm (see [7]). Standard methods, like addition of a unit element, do not help in describing the structure of those algebras. Further, the use of uniform and strict topologies leads to various difficulties.

2. Function algebras. In this paper, X will denote a completely regular Hausdorff space. Let $F(X)$ be the set of all complex-valued functions on X. We denote by $B(X)$ the set of all bounded functions in $F(X)$, by $B_0(X)$ the set of all functions in $B(X)$ which vanish at infinity, and by $B_{00}(X)$ the set of all functions in $B(X)$ which have a compact support. By $C(X)$ we denote the set of all continuous functions in $F(X)$ and by $C_b(X)$ the set of all continuous functions in $B(X)$. If X is locally compact, then we set $C_0(X) = C(X) \cap B_0(X)$ and $C_{00}(X) = C(X) \cap B_{00}(X)$. It is easy to verify that with respect to pointwise algebraic operations, all these sets are algebras.

Let now A be a subset of $F(X)$. We denote by $Z(A) = \{t \in X : f(t) = 0$ for all $f \in A\}$ the zero set of A. We say that A is *bounded away from zero* if for each t in X there is some f in A for which $f(t) \neq 0$, i.e., the zero set of A is empty. Further, we say that A *separates the points* of X if for any s and t in X with $s \neq t$ there is some f in A for which $f(s) \neq f(t)$. If A separates the points of X and is bounded away from zero, then we say that A *strongly separates the points* of X. Finally, A is said to be *symmetric* if it is closed under complex conjugation.

Suppose now that A is a symmetric subalgebra of $C(X)$ which separates the points of $X \setminus Z(A)$. For a given subset B of A, we define $S(B) = \{f \in B : \bar{f} \in B\}$ (here $\bar{f}$ is the complex conjugate of f). Obviously, B is symmetric if and only if $B = S(B)$. The following lemma plays an important role in several theorems of this paper. For the proof, see [8, Lemma 2.2].

LEMMA 2.1. *Suppose that A is a symmetric subalgebra of $C(X)$ which separates the points of $X \setminus Z(A)$. Let I be an ideal of A, and M a maximal ideal of A. Then*

(i) *$S(I)$ is a symmetric ideal of A.*
(ii) *$Z(I) = Z(S(I))$.*
(iii) *I strongly separates the points of $X \setminus Z(I)$.*
(iv) *$S(I)$ strongly separates the points of $X \setminus Z(I)$.*
(v) *M separates the points of $X \setminus Z(A)$.*
(vi) *$S(M)$ separates the points of $X \setminus Z(A)$.*

Let $\| \|_\infty$ be the usual supremum norm on $C_b(X)$. Throughout the paper, σ will denote the *uniform topology* on $C_b(X)$ defined by $\| \|_\infty$, and β will denote the *strict topology* on $C_b(X)$ defined by the seminorms p_ϕ, $\phi \in B_0(X)$, where $p_\phi(f) = \|\phi f\|_\infty$ for every $f \in C_b(X)$. The compact-open topology on $C(X)$ will be denoted by κ. Our main tool in this paper will be the Stone–Weierstrass property. It is well-known that both $(C_b(X), \beta)$ and $(C(X), \kappa)$ have this property. That is, if B is a symmetric subalgebra of $C_b(X)$ (respectively, of $C(X)$) which strongly separates the points of X, then the β-closure of B is $C_b(X)$ (respectively, the κ-closure of B is $C(X)$). We will now study the Stone–Weierstrass property from a more general point of view. For this, we give the following definition.

DEFINITION 2.2. Let X be a completely regular Hausdorff space and let A be a subalgebra of $C(X)$ which is symmetric and separates the points of $X \setminus Z(A)$. Let T be a topology on A making it a topological algebra, i.e., (A, T) is a topological vector space and multiplication on A is separately continuous. We say that (A, T) is a *Stone–Weierstrass algebra* if for each symmetric subalgebra B of A which strongly separates the points of $X \setminus Z(A)$, we have $\mathrm{cl}_T(B) = A$ (here $\mathrm{cl}_T(B)$ stands for the closure of B with respect to the topology T).

So $(C_b(X), \beta)$ and $(C(X), \kappa)$ are Stone–Weierstrass algebras. Further, if X is locally compact, also $(C_0(X), \sigma)$ is a Stone–Weierstrass algebra. Note that a Stone–Weierstrass algebra (A, T) is not necessarily complete. For example, $(C_b(X), \beta)$ and $(C(X), \kappa)$ are complete if and only if X is a $k_{\mathbb{R}}$-space (see [16, p. 72]). On the other hand, $(C_b(X), \sigma)$ is complete but not a Stone–Weierstrass algebra unless X is compact. The concept of Stone–Weierstrass algebra was introduced by the first author in his talk at ICTAA 2005 Athens (see [6]). This was motivated by the application of the Stone–Weierstrass property in describing the ideal structure of topological algebras (see for example [8, Theorem 4.3], [17, Theorem 3.2] and [19, Theorem 8.3.2]). It is interesting to note that topological algebras with the Stone–Weierstrass property appear to have a very similar structure of closed ideals. In [8, Theorem 2.3], it was shown that if M is a closed maximal ideal of a Stone–Weierstrass algebra (A, T), then it is the kernel of some point evaluation, more precisely we have the following:

Theorem 2.3. *Let M be a closed maximal ideal of a Stone–Weierstrass algebra (A, T). Then there exists a unique $t_0 \in X \setminus Z(A)$ such that $M = \{f \in A : f(t_0) = 0\}$.*

Let now I be an arbitrary closed ideal of the Stone–Weierstrass algebra $(C(X), \kappa)$. It is well-known that I is of the form $\{f \in C(X) : f(t) = 0$ for all $t \in E\}$ with E some closed subset of X. The same holds true in $(C_0(X), \sigma)$ with X locally compact.We will show later that for $(C_b(X), \beta)$ this property is also valid. In fact, it appears to hold for every concrete Stone–Weierstrass algebra (A, T) that comes to mind. It is an interesting question whether it is a general property of Stone–Weierstrass algebras.

3. Basic properties of $C_b^v(X)$. In this paper, we are interested in proper subalgebras of $C_b(X)$. We now introduce a method of generating such algebras.

Let v be an upper semicontinuous real-valued function on X for which $\inf_{t \in X} v(t) > 0$. We define

$$C_b^v(X) = \{f \in C(X) : vf \in B(X)\}.$$

If X is locally compact, then we set

$$C_0^v(X) = \{f \in C(X) : vf \in B_0(X)\}.$$

Obviously, both $C_b^v(X)$ and $C_0^v(X)$ are algebras with respect to pointwise operations. If v is unbounded, then $C_b^v(X)$ is clearly a proper subalgebra of $C_b(X)$, and $C_0^v(X)$ is a proper subalgebra of $C_0(X)$. On the other hand, if v is bounded, then obviously $C_b^v(X) = C_b(X)$ and $C_0^v(X) = C_0(X)$. Thus, $C_b(X)$ and $C_0(X)$ are special cases of $C_b^v(X)$ and $C_0^v(X)$.

Remark 3.1. The assumption $\inf_{t \in X} v(t) > 0$ is essential if we want to study $C_b^v(X)$ and $C_0^v(X)$ as algebras. For example, if $X = [0, \infty)$ and v is defined on X by $v(t) = e^{-t}$, then $C_b^v(X)$ and $C_0^v(X)$ are not algebras. To see this, take $f(t) = e^t$. Then $f \in C_b^v(X)$, but as $f^2(t) = e^{2t}$, we have $f^2 \notin C_b^v(X)$.

The following two results are needed later in this paper.

Lemma 3.2. *Let K be a compact subset of X. Then $C_b^v(X)_{|K} = C(K)$.*

Proof. It clearly suffices to show that for any given $f \in C(K)$, there exists $g \in C_b^v(X)$ such that $g_{|K} = f$. Since v is upper semicontinuous on X, there exists a constant M such that $v(t) < M$ for all $t \in K$. Let now $U = \{t \in X : v(t) < M\}$. Then the upper semicontinuity of v implies that U is an open subset of X. By [11, Lemma 2.1.1], there exists $g \in C_b(X)$ for which $g(X \setminus U) \subset \{0\}$ and $g_{|K} = f$. As vg is obviously bounded on X, the lemma follows. ■

COROLLARY 3.3. *Let K be a compact subset of X and let E be a closed subset of X disjoint from K. Then there exists $f \in C_b^v(X)$ such that $f(K) = \{1\}$, $f(E) = \{0\}$ and $0 \le f(t) \le 1$ for all $t \in X$.*

Note that the algebras $C_b^v(X)$ and $C_0^v(X)$ are special cases of the so-called Nachbin algebras. Different types of Nachbin algebras have been studied for example in [9, 10, 20, 21, 22]. Their importance is based on their connections to many classical results on function and topological algebras. These include Stone–Weierstrass type theorems and Gelfand representation theory (see for example [7]). In [8], the Banach algebra structure of $C_0^v(X)$ is studied. In this paper, we will study the structure of $C_b^v(X)$ as a Banach algebra and as a locally convex algebra.

Since $C_b^v(X)$ is a subset of $C_b(X)$, we can equip it with both the uniform and strict topologies. However, it is easy to see that if v is unbounded, then $(C_b^v(X), \sigma)$ and $(C_b^v(X), \beta)$ are not complete. Therefore, σ is not the natural norm topology and β is not the natural locally convex topology on $C_b^v(X)$. We will now introduce more suitable linear topologies for $C_b^v(X)$.

For $f \in C_b^v(X)$, define

$$\|f\|_v = \sup_{t \in X} v(t)|f(t)|.$$

Further, for $\phi \in B_0(X)$, set

$$p_{v,\phi}(f) = \sup_{t \in X} v(t)|\phi(t)||f(t)|.$$

Obviously, $\| \, \|_v$ is a norm and $\{p_{v,\phi} : \phi \in B_0(X)\}$ is a directed system of seminorms on $C_b^v(X)$.

DEFINITION 3.4. The *weighted uniform topology* σ_v on $C_b^v(X)$ is the norm topology defined by the weighted supremum norm $\| \, \|_v$. The *v-strict topology* β_v on $C_b^v(X)$ is the locally convex topology defined by the weighted seminorms $p_{v,\phi}$, where ϕ ranges over $B_0(X)$.

Note that if v is bounded, then σ_v is equivalent to σ, and β_v is equivalent to β. Thus, $(C_b(X), \sigma)$ is a special case of $(C_b^v(X), \sigma_v)$, and $(C_b(X), \beta)$ is a special case of $(C_b^v(X), \beta_v)$. On the other hand, if v is unbounded, then σ is strictly weaker than σ_v, and β is strictly weaker than β_v.

We now state some basic properties of the weighted uniform and v-strict topologies. These properties are similar to those of $C_b(X)$ with respect to the usual uniform and strict topologies. In the following, we denote by $S_{0,+}(X)$ the set of all upper semicontinuous non-negative functions on X which vanish at infinity.

THEOREM 3.5.

(i) *$\sigma \le \sigma_v$ and $\kappa \le \beta \le \beta_v \le \sigma_v$.*
(ii) *β_v and σ_v have the same bounded subsets on $C_b^v(X)$.*
(iii) *β_v and κ coincide on every β_v-bounded subset of $C_b^v(X)$.*
(iv) *$\beta_v = \sigma_v$ if and only if X is compact.*
(v) *β_v coincides with the topology defined by the seminorms $p_{v,\phi}$, where ϕ ranges over $S_{0,+}(X)$.*

Proof. (i): Set $m = \inf_{t\in X} v(t)$. Since $m > 0$, the result follows easily from the fact that for given $f \in C_b^v(X)$, $\phi \in B_0(X)$ and $t \in X$, we have

$$|\phi(t)|\,|f(t)| \le \frac{1}{m}\, v(t)|\phi(t)|\,|f(t)|.$$

(ii): By (i), it suffices to show that every β_v-bounded subset B of $C_b^v(X)$ is σ_v-bounded. Suppose that B is not σ_v-bounded. Then there exist sequences $(f_n)_{n=1}^{\infty}$ in B and $(t_n)_{n=1}^{\infty}$ in X such that $v(t_n)|f_n(t_n)| > n^2$ for every n. Let ϕ be a function on X defined by $\phi(t_n) = 1/n$ for every n, and $\phi(t) = 0$ elsewhere. Obviously, $\phi \in B_0(X)$ and $p_{v,\phi}(f_n) > n$ for every n. Thus, B is not β_v-bounded, a contradiction.

(iii): The proof can be carried out as for $(C_b(X), \beta)$. See [16, p. 47].

(iv): If X is compact, then obviously $C_b^v(X) = C_b(X) = C(X)$ and $\kappa = \beta = \beta_v = \sigma_v$. Conversely, suppose that $\beta_v = \sigma_v$. Then by (iii), κ and σ_v coincide on the unit ball $B^v = \{f \in C_b^v(X) : \|f\|_v \le 1\}$ of $(C_b^v(X), \sigma_v)$. Thus, there exists a compact set K in X and a positive constant M such that $\|f\|_v \le Mp_K(f)$ for all $f \in B^v$. If $K \ne X$, then by Corollary 3.3, there exists $g \in C_b^v(X)$ such that $g(t) \ne 0$ for some $t \in X \setminus K$ and $g(K) = \{0\}$. Since clearly it can be assumed that $\|g\|_v \le 1$, we have $\|g\|_v \le Mp_K(g) = 0$, a contradiction. Thus, $K = X$.

(v): Since $S_{0,+}(X) \subset B_0(X)$, the topology on $C_b^v(X)$ defined by the directed system $\{p_{v,\psi} : \psi \in S_{0,+}(X)\}$ of seminorms is weaker than β_v. On the other hand, by [18, Theorem 3.7], for each $\phi \in B_0(X)$, there exists $\psi \in S_{0,+}(X)$ such that $|\phi(t)| \le \psi(t)$ for all $t \in X$. This clearly implies the result. ∎

Note that if v is unbounded, then the σ-bounded subsets of $C_b^v(X)$ do not coincide with the σ_v-bounded subsets. To see this, note that the unit ball $B = \{f \in C_b^v(X) : \|f\|_\infty \le 1\}$ of $(C_b^v(X), \sigma)$ is bounded with respect to σ_v if and only if there exists a constant M such that $\|f\|_v \le M\|f\|_\infty$ for all $f \in C_b^v(X)$. However, the latter is clearly valid only if v is bounded. In a similar way, it can be shown that if v is unbounded, then the β-bounded subsets of $C_b^v(X)$ do not coincide with the β_v-bounded subsets.

We now show that σ_v and β_v are the natural linear topologies on $C_b^v(X)$.

THEOREM 3.6.

(i) $(C_b^v(X), \sigma_v)$ *is complete.*
(ii) $(C_b^v(X), \beta_v)$ *is complete if* X *is a* $k_{\mathbb{R}}$*-space.*

Proof. (i): Let (f_n) be a Cauchy sequence in $(C_b^v(X), \sigma_v)$. Then (f_n) is also a Cauchy sequence in $(C_b(X), \sigma)$. Since $(C_b(X), \sigma)$ is complete, there exists $f \in C_b(X)$ such that $\|f_n - f\|_\infty \to 0$. Then also $\|f_n - f\|_v \to 0$. To see this, let $\varepsilon > 0$ be given and choose a positive integer n_ε such that $n, k \geq n_\varepsilon$ implies $\|f_n - f_k\|_v < \varepsilon$. Then for any $n \geq n_\varepsilon$ and $t \in X$, we have $v(t)|f_n(t) - f(t)| = \lim_k v(t)|f_n(t) - f_k(t)| \leq \varepsilon$, since $v(t)|f_n(t) - f_k(t)| < \varepsilon$ as soon as $k \geq n_\varepsilon$. Hence, $\|f_n - f\|_v \to 0$. To complete the proof, we have to show that $f \in C_b^v(X)$. For this, choose a positive integer n_0 such that $\|f_{n_0} - f\|_v < 1$. Then for all $t \in X$, we have $v(t)|f(t)| \leq v(t)|f(t) - f_{n_0}(t)| + v(t)|f_{n_0}(t)| < 1 + \|f_{n_0}\|_v$. Thus, vf is bounded, and so $f \in C_b^v(X)$.

(ii): Let (f_α) be a Cauchy net in $(C_b^v(X), \beta_v)$. Then (f_α) is also a Cauchy net in $(C_b(X), \beta)$. Since $(C_b(X), \beta)$ is complete, (f_α) has a β-limit, say f, in $C_b(X)$. This implies that $p_{v,\phi}(f_\alpha - f) \to 0$ for all $\phi \in B_0(X)$. To see this, fix $\varepsilon > 0$, $\phi \in B_0(X)$, and choose α_ε such that $p_{v,\phi}(f_\alpha - f_\gamma) < \varepsilon$ for $\alpha, \gamma \geq \alpha_\varepsilon$. For any $\alpha \geq \alpha_\varepsilon$ and $t \in X$, we now have $v(t)|\phi(t)|\,|f_\alpha(t) - f(t)| = \lim_\gamma v(t)|\phi(t)|\,|f_\alpha(t) - f_\gamma(t)| \leq \varepsilon$, since $v(t)|\phi(t)||f_\alpha(t) - f_\gamma(t)| < \varepsilon$ as soon as $\gamma \geq \alpha_\varepsilon$. Hence, $p_{v,\phi}(f_\alpha - f) \to 0$. To complete the proof, we have to show that $vf \in B(X)$. Suppose that vf is unbounded. Then there exists a sequence $(t_n)_{n=1}^\infty$ in X such that $v(t_n)|f(t_n)| > 3^n$ for every n. Let ψ be a function on X defined by $\psi(t_n) = 1/2^n$ for every n, and $\psi(t) = 0$ elsewhere. Obviously, $\psi \in B_0(X)$ and $v\psi f$ is unbounded. But this clearly contradicts the condition $p_{v,\psi}(f_\alpha - f) \to 0$, and so vf is bounded. ∎

4. The ideal structure of $C_b^v(X)$. We now study approximation properties of $C_b^v(X)$ with respect to the weighted uniform and v-strict topologies. In particular, we are interested in the Stone–Weierstrass property and the structure of the closed ideals of $(C_b^v(X), \sigma_v)$ and $(C_b^v(X), \beta_v)$. We start by considering the former.

It is easy to see that $(C_b^v(X), \sigma_v)$ is not in general a Stone–Weierstrass algebra. For example, if X is locally compact, then the Stone–Weierstrass property of $(C_b^v(X), \sigma_v)$ would imply that $C_0^v(X) = C_b^v(X)$, which is clearly not the case. On the other hand, it is an interesting question whether there exists a stronger condition that would imply a symmetric subalgebra to be dense in $(C_b^v(X), \sigma_v)$. As is well-known, for $(C_b(X), \sigma)$ such a condition exists: a symmetric subalgebra B is dense in $(C_b(X), \sigma)$ if and only if B *separates the zero sets of* X (for any disjoint zero sets Z_1 and Z_2 of X there exists $f \in B$ such that $f(Z_1)$ and $f(Z_2)$ have disjoint closures) and there exists $f \in B$ for which $\inf_{t\in X} f(t) > 0$. Note that these conditions are not in general valid for $(C_b^v(X), \sigma_v)$. In fact, it may happen that $C_b^v(X)$ does not even separate the zero sets of X. For example, if X is locally compact and v is such that $v(t_\alpha) \to \infty$ for all nets (t_α) in X for which $t_\alpha \to t_\infty$ (here t_∞ denotes the point at infinity of X), then $C_b^v(X)$ is a subalgebra of $C_0(X)$, and so it cannot separate the zero sets of X. On the other hand, finding similar approximation properties for $(C_b^v(X), \sigma_v)$ seems difficult. This is due to the complicated ideal structure of $(C_b^v(X), \sigma_v)$. We consider this in greater detail below.

Topological algebras without the Stone–Weierstrass property often have a complicated structure of closed ideals. For example, the maximal ideal space of $(C_b(X), \sigma)$, which can be identified with the Stone–Čech compactification $\beta(X)$ of X, is known to be extremely complicated. On the other hand, if the weight function v is unbounded, then the structure of the closed maximal ideals of $(C_b^v(X), \sigma_v)$ turns out to be even more involved than $\beta(X)$. Indeed, as we now show, in some cases describing the structure of the closed maximal ideals of $(C_b^v(X), \sigma_v)$ would even require describing closed maximal subspaces of $(C_b^v(X), \sigma_v)$.

Example 4.1. Consider $(C_b^v(X), \sigma_v)$ with X locally compact and v such that $C_b^v(X) \subset C_0(X)$. Let L be an arbitrary subspace of $C_b^v(X)$ such that $C_0^v(X) \subset L$. Then, surprisingly, L is automatically an ideal of $C_b^v(X)$. For if $f, g \in C_b^v(X)$, then the inclusion $C_b^v(X) \subset C_0(X)$ implies that $fg \in C_0^v(X)$, and so $C_b^v(X)L = \{fg : f \in C_b^v(X) \text{ and } g \in L\} \subset C_0^v(X) \subset L$. By using the Hahn–Banach Theorem, it is now easy to generate σ_v-closed maximal ideals in $C_b^v(X)$. For example, if f is a function on $C_b^v(X)$ such that $f \notin C_0^v(X)$, then there exists a closed maximal subspace M of $C_b^v(X)$ such that $C_0^v(X) \subset M$ and $f \notin M$. By the above, M is a closed maximal ideal of $(C_b^v(X), \sigma_v)$.

Note that in Example 4.1, every ideal I of $C_b^v(X)$ with $C_0^v(X) \subset I$ is non-regular. In fact, even though the ideal structure of $(C_b^v(X), \sigma_v)$ is in general very complicated, in some cases it is possible to describe the regular maximal ideals of $C_b^v(X)$. For $t \in X$, set $M_t^v = \{f \in C_b^v(X) : f(t) = 0\}$. Obviously, M_t^v is a regular maximal ideal of $C_b^v(X)$. We now have the following:

Theorem 4.2. *Suppose that X is locally compact and v is such that $C_b^v(X) \subset C_0(X)$. Then every regular maximal ideal M of $C_b^v(X)$ is of the form $M = M_t^v$ for some unique $t \in X$.*

Proof. By Lemma 2.1, M separates the points of X. Hence, the zero set of M is either empty or a single point. If $Z(M) = \emptyset$, then $S(M)$ is, by Lemma 2.1, a symmetric ideal of $C_b^v(X)$ strongly separating the points of X. Therefore, $C_0^v(X) \cap S(M)$ is a symmetric ideal of $C_0^v(X)$ which strongly separates the points of X. Since $(C_0^v(X), \sigma_v)$ is a Stone–Weierstrass algebra (see [8, Theorem 4.1]), we have $\mathrm{cl}_{\sigma_v}(C_0^v(X) \cap S(M)) = C_0^v(X) \subset \mathrm{cl}_{\sigma_v}(M) = M$. On the other hand, as M is a regular ideal of $C_b^v(X)$, there exists $g \in C_b^v(X)$ such that $fg - f \in M$ for all $f \in C_b^v(X)$. However, since $fg \in C_0^v(X)$ for all $f \in C_b^v(X)$, we have $f \in C_0^v(X) + M \subset M$ for all $f \in C_b^v(X)$, a contradiction. Thus, there exists a unique $t \in X$ such that $M \subset M_t^v$. But since M_t^v is a proper ideal of $C_b^v(X)$, we must have $M = M_t^v$. ∎

Let (A, T) be a (commutative) topological algebra. It is said to have an *approximate identity* if there exists a net $(e_\alpha)_{\alpha \in \Omega}$ in A for which $e_\alpha x \to x$ for all $x \in A$. An interesting difference between the structures of $(C_b(X), \sigma)$ and $(C_b^v(X), \sigma_v)$ with v unbounded is that unlike $(C_b(X), \sigma)$, which always contains an identity element, $(C_b^v(X), \sigma_v)$ does not in general contain even an approximate identity. For if $(C_b^v(X), \sigma_v)$ of Example 4.1 has an approximate identity, then $\mathrm{cl}_{\sigma_v}(C_b^v(X) C_b^v(X)) = C_b^v(X)$. However, clearly $\mathrm{cl}_{\sigma_v}(C_b^v(X) C_b^v(X)) = C_0^v(X)$.

REMARK 4.3. Although $(C_b^v(X), \sigma_v)$ does not in general contain an approximate identity, the subalgebra $(C_0^v(X), \sigma_v)$ always does. However, by [8, Corollary 3.6], the approximate identity of $(C_0^v(X), \sigma_v)$ is unbounded whenever v is.

We next consider the structure of $C_b^v(X)$ with respect to the v-strict topology. For technical reasons, we restrict to the seminorms $p_{v,\phi}$ with $\phi \in S_{0,+}(X)$. By Theorem 3.5, the topology on $C_b^v(X)$ generated by those seminorms is equivalent to β_v.

As mentioned earlier, $(C_b(X), \beta)$ is a Stone–Weierstrass algebra. This was first established by Giles [13] (for X locally compact it had already been done by Todd [23]). We now generalize this result by showing that also $(C_b^v(X), \beta_v)$ is a Stone–Weierstrass algebra.

THEOREM 4.4. *Let B be a symmetric subalgebra of $C_b^v(X)$ which strongly separates the points of X. Then B is dense in $(C_b^v(X), \beta_v)$.*

Proof. Let $f \in C_b^v(X)$, $\phi \in S_{0,+}(X)$ and $\varepsilon > 0$. Since $(C_b(X), \beta)$ is a Stone–Weierstrass algebra, B is β-dense in $C_b(X)$. Thus, as the identity function is in $C_b(X)$, there exists $g \in B$ such that $\phi(t)|1 - g(t)| < \varepsilon/2\|f\|_v$ for all $t \in X$. Similarly, as $f \in C_b(X)$, there is $h \in B$ such that $\phi(t)|f(t) - h(t)| < \varepsilon/2\|g\|_v$ for all $t \in X$. Now, for any $t \in X$, we have

$$\begin{aligned} v(t)\phi(t)|f(t) - g(t)h(t)| \\ &\le v(t)\phi(t)|f(t) - f(t)g(t)| + v(t)\phi(t)|f(t)g(t) - g(t)h(t)| \\ &= v(t)\phi(t)|f(t)|\,|1 - g(t)| + v(t)\phi(t)|g(t)|\,|f(t) - h(t)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

and so $p_{v,\phi}(f - gh) \le \varepsilon$. Since $gh \in B$, the theorem follows. ■

We next study the ideal structure of $(C_b^v(X), \beta_v)$. Since it is a Stone–Weierstrass algebra, we have the following:

THEOREM 4.5. *Let M be a closed maximal ideal of $(C_b^v(X), \beta_v)$. Then $M = M_t^v$ for some unique $t \in X$.*

Let now E be a closed subset of X. Set

$$\begin{aligned} I_{C(X)}(E) &= \{f \in C(X) : f(t) = 0 \text{ for all } t \in E\}, \\ I(E) &= \{f \in C_b(X) : f(t) = 0 \text{ for all } t \in E\}, \\ I^v(E) &= \{f \in C_b^v(X) : f(t) = 0 \text{ for all } t \in E\}. \end{aligned}$$

Obviously, $I^v(E)$ is a β_v-closed ideal of $C_b^v(X)$. We now show that the assumption we made earlier on the structure of closed ideals of Stone–Weierstrass algebras is valid for $(C_b^v(X), \beta_v)$. First we need the following lemma.

LEMMA 4.6. *Let E be a closed subset of X. Then $I_{C(X)}(E)$ is a Stone–Weierstrass algebra with respect to the relative compact-open topology inherited from $C(X)$.*

Proof. Obviously, $I_{C(X)}(E)$ is a symmetric subalgebra of $C(X)$ which separates the points of $X \setminus Z(I_{C(X)}(E)) = X \setminus E$. Let now B be a symmetric subalgebra of $I_{C(X)}(E)$ which strongly separates the points of $X \setminus E$. Further, let $f \in I_{C(X)}(E)$ and $\varepsilon > 0$. For an arbitrary compact subset K of X, define

$$B_K = \{h_{|K} : h \in B\}.$$

It is easy to see that B_K is a symmetric subalgebra of $C(K)$ which strongly separates the points of $K \setminus (K \cap E)$. Further, B_K is clearly contained in

$$I_{C(K)}(K \cap E) = \{f \in C(K) : f(t) = 0 \text{ for all } t \in K \cap E\}.$$

By [19, Theorem 8.3.2], $I_{C(K)}(K \cap E)$ and $C_0(K \setminus (K \cap E))$ can be identified by an isometric isomorphism (both equipped with the uniform topology), and so $I_{C(K)}(K \cap E)$ is a Stone–Weierstrass algebra with respect to the uniform topology. Thus, since $f_{|K} \in I_{C(K)}(K \cap E)$, there exists $g \in B$ such that $p_K(f - g) = \sup_{t \in K} |f(t) - g(t)| < \varepsilon$. Hence, $\mathrm{cl}_\kappa(B) = I_{C(X)}(E)$. ■

THEOREM 4.7. *Let I be a closed ideal of $(C_b^v(X), \beta_v)$. Then $I = I^v(E)$ for some closed subset E of X.*

Proof. Let E be the zero set of I. Then E is a closed subset of X and $I \subset I^v(E)$. To show that $I = I^v(E)$, let $f \in I^v(E)$, $\phi \in S_{0,+}(X)$ and $\varepsilon > 0$. Then there exists a compact set K in X such that $\phi(t) < \varepsilon/2\|f\|_v$ for all $t \in X \setminus K$. Further, since v is upper semicontinuous, there exists a constant $M > 1$ such that $v(t) < M$ for all $t \in K$. By Lemma 2.1, the set $S(I)$ is a symmetric subalgebra of $I_{C(X)}(E)$ which strongly separates the points of $X \setminus E$. Hence, by Lemma 4.6, $S(I)$ is dense in $I_{C(X)}(E)$ with respect to the compact-open topology, and so there exists $g \in I$ such that $|f(t) - g(t)| < \varepsilon/2M\|\phi\|_\infty$ for all $t \in K$. Define

$$U = \{t \in X : v(t) < M\},$$
$$V = \left\{t \in X : |f(t) - g(t)| < \frac{\varepsilon}{2M\|\phi\|_\infty}\right\},$$
$$W = U \cap V.$$

Obviously, W is an open subset of X which contains K. Thus, by Corollary 3.3, there exists $h \in C_b^v(X)$ such that $0 \le h(t) \le 1$ for all $t \in X$, $h(K) = \{1\}$ and $h(X \setminus W) \subset \{0\}$. Now $gh \in I$ and for all $t \in X$, we have

$$v(t)\phi(t)|f(t) - g(t)h(t)| \le v(t)\phi(t)|f(t)|\,|1 - h(t)| + v(t)\phi(t)|h(t)|\,|f(t) - g(t)|.$$

By considering the cases $t \in K$, $t \in W \setminus K$ and $t \in X \setminus W$, it is easy to verify that in every case $v(t)\phi(t)|f(t) - g(t)h(t)| < \varepsilon$. Thus, $p_{v,\phi}(f - gh) \le \varepsilon$, and so $I = \mathrm{cl}_{\beta_v}(I) = I^v(E)$. ■

Corollary 4.8. *Let I be a closed ideal of $(C_b(X), \beta)$. Then $I = I(E)$ for some closed subset E of X.*

It is well-known that the sum of two closed ideals of a C^*-algebra is always closed. Further, in [8, Theorem 4.7], it was shown that this is also valid for $(C_0^v(X), \sigma_v)$. However, it is not a general property of topological, or even of Banach, algebras (see for example [12]). Corollary 4.8 implies that $(C_b(X), \beta)$ has this property at least when X is normal.

Theorem 4.9. *Suppose that X is normal. Let I_1 and I_2 be closed ideals of $(C_b(X), \beta)$. Then $I_1 + I_2$ is a closed ideal of $(C_b(X), \beta)$.*

Proof. Set $E_i = Z(I_i)$, $i = 1, 2$. Then E_1 and E_2 are closed subsets of X and $I_1 + I_2$ is an ideal of $C_b(X)$ for which $Z(I_1 + I_2) = E_1 \cap E_2$. By Corollary 4.8, we have $I_i = I(E_i)$, $i = 1, 2$, and $\mathrm{cl}_\beta(I_1 + I_2) = I(E_1 \cap E_2)$. Let now $f \in I(E_1 \cap E_2)$ and define a function g on $E_1 \cup E_2$ by $g(t) = f(t)$ for all $t \in E_1$, and $g(t) = 0$ for all $t \in E_2$. Obviously, $g \in C_b(E_1 \cup E_2)$. Further, since $C_b(X)_{|E_1 \cup E_2} = C_b(E_1 \cup E_2)$, there exists $h \in C_b(X)$ for which $g = h_{|E_1 \cup E_2}$. Now $h \in I(E_2) = I_2$ and $f - h \in I(E_1) = I_1$. Thus, $f = (f - h) + h \in I_1 + I_2$, and so $I_1 + I_2 = \mathrm{cl}_\beta(I_1 + I_2) = I(E_1 \cap E_2)$. ■

Let I be a closed ideal of $(C_b^v(X), \beta_v)$. Since we now know the exact form of I, we can study the structure of the quotient algebra $C_b^v(X)/I$. Denote by $\widetilde{\beta}_v$ the usual quotient topology of $C_b^v(X)/I$ and by $\widetilde{p}_{v,\phi}$ the corresponding quotient seminorms. So for every $f + I \in C_b^v(X)/I$, we have

$$\widetilde{p}_{v,\phi}(f+I) = \inf_{g\in I} p_{v,\phi}(f+g).$$

Denote by E the zero set of I and set $w = v_{|E}$. Obviously, w is upper semicontinuous on E. Further, $f_{|E} \in C_b^w(E)$ for each $f \in C_b^v(X)$. Let now φ be a mapping from $(C_b^v(X)/I, \widetilde{\beta}_v)$ into $(C_b^w(E), \beta_w)$ defined by

$$\varphi(f+I) = f_{|E}. \tag{4.1}$$

Here β_w denotes of course the w-strict topology on $C_b^w(E)$. From now on, φ will always denote the mapping defined by (4.1). By Theorem 4.7, it is well-defined. We next show that φ is in fact a topological isomorphism from $(C_b^v(X)/I, \widetilde{\beta}_v)$ onto $(\varphi(C_b^v(X)/I), \beta_w)$. For this, we first need the following two lemmas.

LEMMA 4.10. *Let I be a closed ideal of $(C_b^v(X), \beta_v)$ and set $E = Z(I)$. Then $\widetilde{p}_{v,\phi}(f+I) = \sup_{t\in E} v(t)\phi(t)|f(t)|$ for all $f \in C_b^v(X)$ and $\phi \in S_{0,+}(X)$.*

Proof. Let $f \in C_b^v(X)$ and $\phi \in S_{0,+}(X)$. As $I = I^v(E)$, it is easy to see that $\sup_{t\in E} v(t)\phi(t)|f(t)| \le p_{v,\phi}(f+g)$ for all $g \in I$. Thus, $\sup_{t\in E} v(t)\phi(t)|f(t)| \le \widetilde{p}_{v,\phi}(f+I)$. To prove the converse inequality, let $\varepsilon > 0$ and set $k = \sup_{t\in E} v(t)\phi(t)|f(t)|$ and $F = \{t \in X : v(t)\phi(t)|f(t)| \ge k+\varepsilon\}$. Since $v\phi f \in B_0(X)$, there exists a compact subset Q of X such that $v(t)\phi(t)|f(t)| < k + \varepsilon/4$ for all $t \in X \setminus Q$. Clearly $F \subset Q$ and $F \subset X \setminus E$.

We next show that also $\mathrm{cl}(F) \subset Q \cap (X\setminus E)$. Suppose on the contrary that there exists $t_0 \in \mathrm{cl}(F)$ such that $t_0 \notin Q\cap(X\setminus E)$. Then $f(t_0) \ne 0$ (otherwise t_0 would clearly have a neighbourhood V in X such that $v(t)\phi(t)|f(t)| < \varepsilon$ for all $t \in V$, a contradiction with $F \cap V \ne \emptyset$). Further, as $t_0 \in (X \setminus Q) \cup E$, we have $v(t_0)\phi(t_0)|f(t_0)| < k + \varepsilon/4$. Let now

$$U_1 = \left\{ t \in X : |f(t)| < |f(t_0)| + \frac{|f(t_0)|\varepsilon}{4k+\varepsilon} \right\},$$
$$U_2 = \left\{ t \in X : \phi(t) < \frac{k}{v(t_0)|f(t_0)|} + \frac{\varepsilon}{4v(t_0)|f(t_0)|} \right\},$$
$$U_3 = \left\{ t \in X : v(t) < v(t_0) + \frac{v(t_0)\varepsilon}{2k+\varepsilon} \right\}.$$

As these are neighbourhoods of t_0 in X, so is $U = U_1 \cap U_2 \cap U_3$. Hence, $U \cap F \ne \emptyset$. On the other hand, it is easy to calculate that $v(t)\phi(t)|f(t)| < k+\varepsilon$ for every $t \in U$, a contradiction. Thus, $\mathrm{cl}(F) \subset Q \cap (X \setminus E)$.

Set now $K = \mathrm{cl}(F)$. Since $K \subset Q \cap (X \setminus E)$, it is a compact subset of X disjoint from E. Hence, by Corollary 3.3, there exists $h \in C_b^v(X)$ for which

$h(K) = \{1\}$, $h(E) = \{0\}$ and $0 \le h(t) \le 1$ for all $t \in X$. Let now $g = -fh$. Then $g \in I^v(E) = I$. Further, as $f(t) - f(t)h(t) = 0$ for all $t \in K$ and $v(t)\phi(t)|f(t)| < k + \varepsilon$ for all $t \in X \setminus K$, we have

$$\begin{aligned}\widetilde{p}_{v,\phi}(f + I) &\le p_{v,\phi}(f + g) = \sup_{t\in X} v(t)\phi(t)|f(t) + g(t)| \\ &= \sup_{t\in X} v(t)\phi(t)|f(t) - f(t)h(t)| = \sup_{t\in X\setminus K} v(t)\phi(t)|f(t) - f(t)h(t)| \\ &= \sup_{t\in X\setminus K} v(t)\phi(t)|f(t)|\,|1 - h(t)| \le k + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we have $\widetilde{p}_{v,\phi}(f + I) \le k$. Therefore, $\widetilde{p}_{v,\phi}(f + I) = \sup_{t\in E} v(t)\phi(t)|f(t)|$. ∎

LEMMA 4.11. *Let E be a closed subset of X. Then $S_{0,+}(X)_{|E} = S_{0,+}(E)$.*

Proof. It is easy to see that $S_{0,+}(X)_{|E} \subset S_{0,+}(E)$. To prove the converse inclusion, let $\phi \in S_{0,+}(E)$ and define a function ψ on X by $\psi(t) = \phi(t)$ for all $t \in E$, and $\psi(t) = 0$ for all $t \in X \setminus E$. Obviously, ψ vanishes at infinity. To see that ψ is upper semicontinuous, fix $m > 0$. Then $V = \{t \in E : \phi(t) < m\}$ is an open subset of E, and so there exists an open set W in X such that $V = W \cap E$. Since

$$\begin{aligned}\{t \in X : \psi(t) < m\} &= \{t \in E : \phi(t) < m\} \cup \{t \in X \setminus E : \psi(t) < m\} \\ &= (W \cap E) \cup (X \setminus E) = W \cup (X \setminus E)\end{aligned}$$

and $W \cup (X \setminus E)$ is open in X, the upper semicontinuity of ψ follows. Thus, $S_{0,+}(E) \subset S_{0,+}(X)_{|E}$. ∎

THEOREM 4.12. *Let I be a closed ideal of $(C_b^v(X), \beta_v)$ and set $E = Z(I)$ and $w = v_{|E}$. Then the mapping φ (see* (4.1)*) is a topological isomorphism from $(C_b^v(X)/I, \widetilde{\beta}_v)$ onto $(\varphi(C_b^v(X)/I), \beta_w)$.*

Proof. Obviously, φ is an algebra homomorphism from $C_b^v(X)/I$ into $C_b^w(E)$. Further, by Theorem 4.7, φ is injective. We next prove its continuity. Let V be a neighbourhood of zero in $(C_b^w(E), \beta_w)$. Then there exist $\phi \in S_{0,+}(E)$ and $\varepsilon > 0$ such that

$$\{f \in C_b^w(E) : p_{w,\phi}(f) < \varepsilon\} \subset V.$$

By Lemma 4.11, there exists $\psi \in S_{0,+}(X)$ such that $\psi_{|E} = \phi$. Let now

$$U = \{g + I \in C_b^v(X)/I : \widetilde{p}_{v,\psi}(g + I) < \varepsilon\}.$$

Then U is a neighbourhood of zero in $(C_b^v(X)/I, \widetilde{\beta}_v)$. By Lemma 4.10, we have $\widetilde{p}_{v,\psi}(g + I) = \sup_{t\in E} v(t)\psi(t)|g(t)| = p_{w,\phi}(\varphi(g + I))$ for all $g + I \in C_b^v(X)/I$. Thus, $\varphi(U) \subset V$, and so φ is continuous. In a similar way, it can be shown that also φ^{-1} from $(\varphi(C_b^v(X)/I), \beta_w)$ onto $(C_b^v(X)/I, \widetilde{\beta}_v)$ is continuous. This proves the theorem. ∎

Corollary 4.13. *Let I be a closed ideal of $(C_b^v(X), \beta_v)$ and set $E = Z(I)$ and $w = v_{|E}$. Then $(C_b^v(X)/I, \widetilde{\beta}_v)$ and $(C_b^v(X)_{|E}, \beta_w)$ are topologically isomorphic.*

It is well-known that a quotient space of a complete topological linear space is not necessarily complete. We now prove an interesting result on how the completeness of $(C_b^v(X)/I, \widetilde{\beta}_v)$ is connected with the topological structure of the zero set of I. First we need the following definition.

Definition 4.14. Let E be a closed subset of X and set $w = v_{|E}$. We say that E has the *v-extension property* if for every $f \in C_b^w(E)$, there exists $\widetilde{f} \in C_b^v(X)$ such that $\widetilde{f}_{|E} = f$.

Note that for v bounded, this notion coincides with the usual extension property: a closed subset E of X is said to have the extension property if for every $f \in C_b(E)$, there exists $\widetilde{f} \in C_b(X)$ such that $\widetilde{f}_{|E} = f$ (see [14]).

Theorem 4.15. *Let I be a closed ideal of $(C_b^v(X), \beta_v)$ and set $E = Z(I)$ and $w = v_{|E}$. If E is a $k_{\mathbb{R}}$-space, then the following conditions are equivalent*:

(i) *$(C_b^v(X)/I, \widetilde{\beta}_v)$ is complete.*
(ii) *φ is a topological isomorphism from $(C_b^v(X)/I, \widetilde{\beta}_v)$ onto $(C_b^w(E), \beta_w)$.*
(iii) *E has the v-extension property.*

Proof. By using Corollary 4.13, it is easy to verify that (ii)⇔(iii). Thus, it suffices to show that (i)⇔(iii). Suppose first that (iii) is valid. By Theorem 3.6, $C_b^w(E)$ is complete with respect to β_w. The completeness of $(C_b^v(X)/I, \widetilde{\beta}_v)$ is now a direct consequence of Corollary 4.13 and the v-extension property of E. Conversely, suppose that (i) is valid. Then Corollary 4.13 implies that $(C_b^v(X)_{|E}, \beta_w)$ is complete. Thus, as β_w is a Hausdorff topology, $C_b^v(X)_{|E}$ is closed in $C_b^w(E)$ with respect to β_w. On the other hand, since $C_b^v(X)_{|E}$ is obviously a symmetric subalgebra of $C_b^w(E)$ which strongly separates the points of E, the Stone–Weierstrass property of $(C_b^w(E), \beta_w)$ implies that $C_b^v(X)_{|E} = \mathrm{cl}_{\beta_w}(C_b^v(X)_{|E}) = C_b^w(E)$. Hence, E has the v-extension property. ∎

Corollary 4.16. *Let I be a closed ideal of $(C_b(X), \beta)$ and set $E = Z(I)$. If E is a $k_{\mathbb{R}}$-space, then the following conditions are equivalent*:

(i) *$(C_b(X)/I, \widetilde{\beta})$ is complete.*
(ii) *φ is a topological isomorphism from $(C_b(X)/I, \widetilde{\beta})$ onto $(C_b(E), \beta_E)$.*
(iii) *E has the extension property.*

Corollary 4.17. *Suppose that X is normal. Let I be a closed ideal of $(C_b(X), \beta)$ and set $E = Z(I)$. If E is a $k_{\mathbb{R}}$-space, then the following conditions are valid*:

(i) $(C_b(X)/I, \widetilde{\beta})$ *is complete.*
(ii) $(C_b(X)/I, \widetilde{\beta})$ *and* $(C_b(E), \beta_E)$ *are topologically isomorphic.*

In Corollaries 4.16 and 4.17, β_E denotes of course the usual strict topology on $C_b(E)$.

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