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GENERALIZATION OF TWO ASYMPTOTICALLY STATISTICAL EQUIVALENT THEOREMS

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Abstract. The goal of this paper is to present two theorems that characterize asymptotically statistical equivalent of multiple L and the regularity of asymptotically statistical convergence by using a sequence of infinite matrices.

1. Introduction and Background

In 1998 Kolk presented the notion of B-statistical convergence by considering a sequence of infinite matrices. In addition, the definition of asymptotically statistical equivalent sequences was presented in [6]. By combining the notions of B-statistical convergence and asymptotically statistical equivalent sequences we shall present answers to the following questions: Which type of summability matrices preserve asymptotically statistical equivalent of multiple L for a given sequence? What are the necessary and sufficient conditions that will ensure the regularity of asymptotically statistical for a given sequence? Let $l^1 = \{x = (x_k) : \sum_{k=1}^{\infty} |x_k| < \infty.\}$ and $d_A = \{x = (x_k) : \lim_n \sum_{k=1}^{\infty} a_{n,k}x_k = \text{ exists}\}$.

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Definition 1.1 (Fridy, [1])**.** *For each $x = (x_k)$ in l^1 the " remainder sequence" $[Rx]$ is the sequence whose n-th term is*

$$R_n x := \sum_{k \geq n} |x_k| .$$

Definition 1.2 (Marouf, [4])**.** *Two nonnegative sequences $x = (x_k)$, and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 1.3 (Fridy, [2])**.** *The sequence $x = (x_k)$ has statistical limit L provided that for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n}\{\textit{the number of } k \leq n : |x_k - L| \geq \epsilon\} = 0.$$

The next definition is natural combination of definition (1.2) and (1.3).

Definition 1.4. *Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L, briefly $x \overset{S_L}{\sim} y$, if for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| = 0,$$

where $|K|$ denotes the cardinality of K.

Let $P_\delta = \{x = (x_k) : x_k \geq \delta > 0 \text{ for all } k\}$ and let P_0 be the set of all nonnegative sequences which have at most a finite number of zero entries.

For $x = (x_k) \in l^1$ let $Rx = (R_n x) = (\sum_{k \geq n} |x_k|)$. For a sequence x and an infinite matrix $A = (a_{n,k})$ let $Ax = (\sum_k a_{n,k} x_k)$ provided that all series $\sum_k a_{n,k} x_k$ converge.

Definition 1.5. *If $\boldsymbol{B} = (\boldsymbol{B}_i)$ is a sequence of infinite matrices $\boldsymbol{B}_i = (b_{n,k}(i))$, then a sequence $x \in l_\infty$ is said to be $\boldsymbol{B}$-summable to the value x_0 if $\lim_n (\boldsymbol{B}_i x)_n = \lim_n \sum_k b_{n,k}(i) x_k = x_0$, uniformly in i.*

Taking in the theorems of Patterson [6] a summability method **B** instead of summability matrix A the authors give two analogous theorems.

Definition 1.6. *A summability matrix $\boldsymbol{B} = (\boldsymbol{B}_i)$ is asymptotically statistical regular provided that $\boldsymbol{B}_i x \overset{S_L}{\sim} \boldsymbol{B}_i y$ whenever $x \overset{S_L}{\sim} y$, $x \in P_0$, and $y \in P_\delta$ for some $\delta > 0$.*

2. Main Results

In this section we shall present two theorems concerning necessary and sufficient conditions of the matrix transformation that will preserve asymptotically statistical equivalents of multiple L of a given sequence and the regularity of asymptotically statistical convergence.

Theorem 2.1. *If $\boldsymbol{B} = (\boldsymbol{B}_i)$ is a sequence of infinite nonnegative matrices with $\boldsymbol{B}_i = (b_{n,k}(i))$ that maps bounded sequence $x = (x_k)$ into l^1 then the following statements are equivalent:*

(1) *If $x = (x_k)$ and $y = (y_k)$ are sequences such that $x \overset{S^L}{\sim} y$, $x \in P_0$ and $y \in P_\delta$ for some $\delta > 0$ then*

$$R_n(\boldsymbol{B}_i x) \overset{S_L}{\sim} R_n(\boldsymbol{B}_i y).$$

(2)

$$\lim_n \frac{1}{n} \max_i \left| \left\{ k \le n : \left| \frac{\sum_{p=k}^{\infty} b_{p,m}(i)}{\sum_{p=k}^{\infty} \sum_{j=1}^{\infty} b_{p,j}(i)} \right| \ge \epsilon \right\} \right| = 0 \text{ for each } m \text{ and } \epsilon > 0.$$

Proof. The definition of asymptotically statistically equivalent of multiple L can be interpreted as the following:

$$\left| \frac{x_s}{y_s} - L \right| \le \epsilon \text{ for almost all } s(\text{ denoted by a.a.s}).$$

This implies that

$$(L - \epsilon) y_s \le x_s \le (L + \epsilon) y_s \text{ a.a.s.} \tag{2.1}$$

Let us consider the for all i, $R_n(\mathbf{B}_i x) = \sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{pr(i)} x_r$; which implies the following

$$\begin{aligned} \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} &\le \frac{\sum_{r=1}^{R-1} \sum_{p=n}^{\infty} \max_{0 \le r \le R-1} \{b_{p,r}(i)\} x_r}{\sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i) y_r} \\ &+ \frac{\sum_{p=n}^{\infty} \sum_{r=R}^{\infty} b_{p,r}(i) x_r}{\sum_{p=n}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i) y_r}. \end{aligned}$$

Using (2.1) we obtain the following:

$$\frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \le \frac{\sum_{r=1}^{R-1} \sum_{p=n}^{\infty} \max_{0 \le r \le R-1} \{b_{p,r}(i)\}}{\delta \sum_{p=k}^{\infty} \sum_{r=1}^{\infty} b_{p,r}(i)} + (L + \epsilon) \text{ a.a.n}$$

Thus by Equation (2) for all i

$$\limsup_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \le (L + \epsilon) \text{ a.a.n }.$$

Inequality (2.1) can be used in a similar manner to obtain the following:

$$\liminf_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \ge (L - \epsilon) \text{ a.a.n }.$$

Thus

$$R_n(\mathbf{B}_i x) \overset{S_L}{\sim} R_n(\mathbf{B}_i y).$$

For the second part of this theorem let us consider the following two sequences:

$$x_s = \begin{cases} 0 & \text{if } s \leq K \\ 1 & \text{otherwise} \end{cases}$$

where K is a positive integer and $y_s = 1$ for all s . The two sequences imply the following: for all i,

$$\begin{aligned} R_n(\mathbf{B}_i x) = \sum_{k=n}^{\infty}(\mathbf{B}_i x) &= \sum_{k=n}^{\infty}\sum_{s=K+1}^{\infty} b_{k,s}(i) \\ &= \sum_{k=n}^{\infty}\sum_{s=1}^{\infty} b_{k,s}(i) - \sum_{k=n}^{\infty}\sum_{s=0}^{K} b_{k,s}(i). \end{aligned}$$

Therefore for all i

$$st - \liminf_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} \leq 1 - st - \limsup_n \frac{(K+1)\sum_{p=n}^{\infty} b_{p,k}(i)}{\sum_{p=n}^{\infty}\sum_{s=1}^{\infty} b_{p,s}(i)}$$

where $0 \leq k \leq K$. Since each nonconstant element of the last inequality has statistical limit zero we obtain the following for all i:

$$\lim_n \frac{R_n(\mathbf{B}_i x)}{R_n(\mathbf{B}_i y)} = 1 \text{ a.a.n .}$$

This completes the proof. □

In 1980 Pobyvanets presented definition for asymptotically equivalent sequences and asymptotic regular matrices. Using these definitions he also presented a Silverman Toeplitz type characterization for asymptotic equivalent sequences. In similar manner we have presented a definition for asymptotically statistical equivalent sequences via a sequence of infinite matrices and use it to present Silverman Toeplitz conditions similar to Poyvanents' results.

Theorem 2.2. *In order for a sequence of summability matrices* $\boldsymbol{B}$ *to be asymptotically statistical regular it is necessary and sufficient that for each fixed positive integer* k_0

(1) $\sum_{p=1}^{k_0} b_{n,p}(i)$ *is bounded for each* (n,i).

(2)

$$\lim_n \frac{1}{n}\max_i\left\{\text{the number of } k \leq n : \left|\frac{\sum_{p=1}^{k_0} b_{n,p}(i)}{\sum_{p=k1}^{\infty} b_{n,p}(i)}\right| \geq \epsilon \text{ for each } k_0 \text{ and } \epsilon > 0\right\} = 0.$$

Proof. The necessary part of this theorem is established in a manner similar to that of the necessary part of the last theorem. To establish the sufficient part of this theorem, let $\epsilon > 0$ and $x \overset{S^L}{\sim} y$, $x \in P_0$ and $y \in P_\delta$ for some $\delta > 0$ these conditions implies that

$$(L-\epsilon)y_{s+\alpha} \leq x_{s+\alpha} \leq (L+\epsilon)y_{s+\alpha} \text{ a.a.s for some } \alpha = 1, 2, 3, \ldots. \tag{2.2}$$

Let us consider the following:

$$\begin{aligned}\frac{(\mathbf{B}_i x)_n}{(\mathbf{B}_i y)_n} &= \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p + \sum_{p=1+\alpha}^{\infty} b_{n,p}(i)x_p}{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p + \sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} \\ &= \frac{\frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} + \frac{\sum_{p=\alpha+1}^{\infty} b_{n,p}(i)x_p}{\sum_{p=\alpha+1}^{\infty} b_{n,p}(i)y_p}}{\frac{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} + 1}.\end{aligned}$$

Inequality (2.2) implies that for all i,

$$\lim_n \frac{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = L, \text{ a.a.n.}$$

Since $x \in P_0$, $y \in P_\delta$, and condition (2) holds we obtain the following for all i

$$\lim_n \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)x_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = 0, \text{ a.a.n}$$

and

$$\lim_n \frac{\sum_{p=1}^{\alpha} b_{n,p}(i)y_p}{\sum_{p=1+\alpha}^{\infty} b_{n,p}(i)y_p} = 0, \text{ a.a.n.}$$

Thus for all i

$$\lim_n \frac{(\mathbf{B}_i x)_n}{(\mathbf{B}_i y)_n} = L, \text{ a.a.n.}$$

This implies that $Bx \overset{S^L}{\sim} By$ where $x \overset{S^L}{\sim} y$, $y \in P_0$, and $y \in P_\delta$ for some $\delta > 0$. This completes the proof. □

If we let $B_i = A$ for all i the above theorems reduces to Patterson's results in [6]

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