GENERALIZATIONS OF A COTANGENT SUM ASSOCIATED TO THE ESTERMANN ZETA FUNCTION

HELMUT MAIER AND MICHAEL TH. RASSIAS

ABSTRACT. Cotangent sums are associated to the zeros of the Estermann zeta function. They have also proven to be of importance in the Nyman-Beurling criterion for the Riemann Hypothesis.
The main result of the paper is the proof of the existence of a unique positive measure μ on $\mathbb{R}$, with respect to which certain normalized cotangent sums are equidistributed.
Improvements as well as further generalizations of asymptotic formulas regarding the relevant cotangent sums are obtained. We also prove an asymptotic formula for a more general cotangent sum as well as asymptotic results for the moments of the cotangent sums under consideration. We also give an estimate for the rate of growth of the moments of order $2k$, as a function of k.

Key words: Cotangent sums, equidistribution, moments, asymptotics, Estermann zeta function, Riemann Hypothesis, fractional part.
2000 Mathematics Subject Classification: 11L03, 11M06.

1. Introduction

Cotangent sums are associated to the zeros of the Estermann zeta function. R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown [3], used properties of the Estermann zeta function to prove asymptotic formulas for mean-values of the product consisting of the Riemann zeta function and a Dirichlet polynomial. Period functions and families of cotangent sums appear in recent work of S. Bettin and J. B. Conrey (cf. [6]). They generalize the Dedekind sum and share with it the property of satisfying a reciprocity formula. They prove a reciprocity formula for the V. I. Vasyunin's sum [31], which appears in the Nyman-Beurling criterion for the Riemann Hypothesis.
In the present paper, improvements as well as further generalizations of asymptotic formulas regarding the relevant cotangent sums are obtained. We also prove an asymptotic formula for a more general cotangent sum as well as asymptotic results and upper bounds for the moments of the cotangent sums under consideration. Furthermore, we obtain detailed information about the distribution of the values of these cotangent sums. We also give an estimate for the rate of growth of the moments of order $2k$, as a function of k.

1.1. **The cotangent sum and its applications.**

The present paper is focused in the study of the following cotangent sum:

Date: September 19, 2014.

Definition 1.1.

$$c_0\left(\frac{r}{b}\right) := -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right) ,$$

where $r,\ b \in \mathbb{N}$, $b \geq 2$, $1 \leq r \leq b$ *and* $(r,b) = 1$.

The function $c_0(r/b)$ is odd and periodic of period 1 and its value is an algebraic number. Its properties of being odd and periodic are depicted in the following graphs:

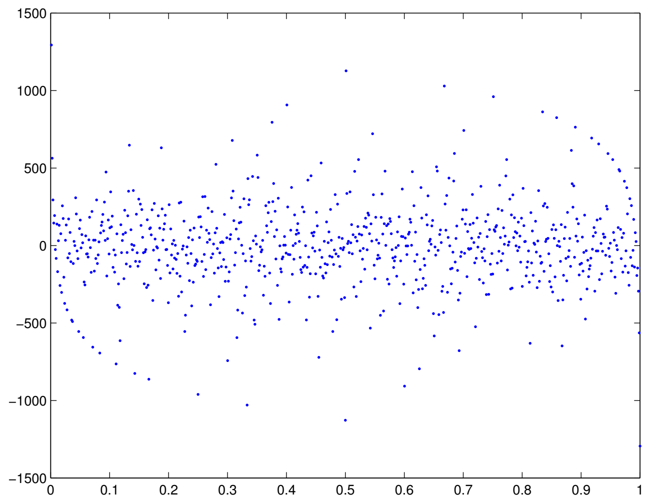

Figure 1: Graph of $c_0(r/b)$, for $1 \leq r \leq b,\ b = 757$, with $(r,b) = 1$.

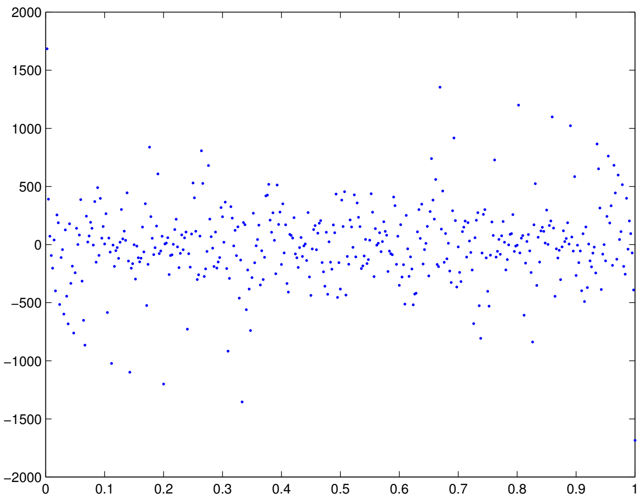

Figure 2: Graph of $c_0(r/b)$, for $1 \leq r \leq b,\ b = 946$, with $(r, b) = 1$.

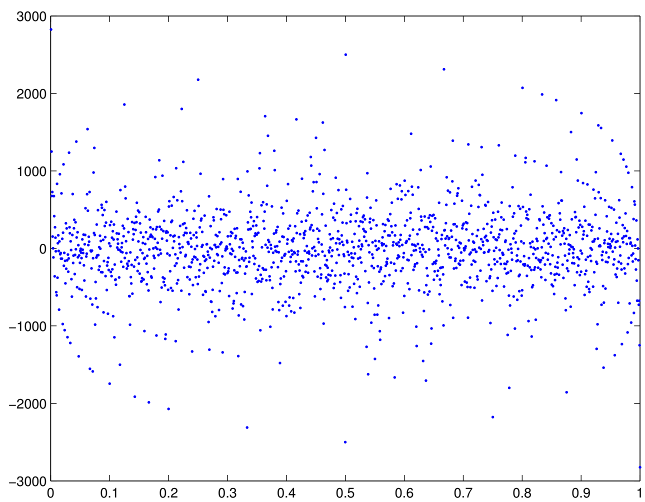

Figure 3: Graph of $c_0(r/b)$, for $1 \leq r \leq b,\ b = 1471$, with $(r, b) = 1$.

It is interesting to mention that for hundreds of integer values of k for which we have examined the graph of $c_0(r/b)$ by the use of MATLAB, the resulting figure always has a shape similar to an ellipse.

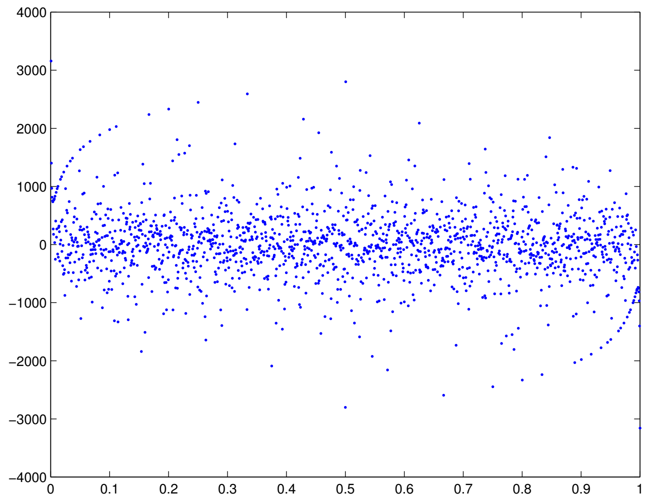

Figure 4: Graph of $c_0(r/b)$, for $1 \le r \le b,\ b = 1619$, with $(r, b) = 1$.

Part of our goal is to understand this phenomenon, and we will do it to some extent. The main result in this respect is contained in Theorem 1.5, which provides information about equidistribution and moments of these sums.

Before presenting the main results of the paper regarding this cotangent sum, we shall demonstrate its significance by exhibiting its relation to other important functions in number theory, such as the Estermann and the Riemann zeta functions, and its connections to major open problems in Mathematics, such as the Riemann Hypothesis.

Definition 1.2. *The Estermann zeta function $E\left(s, \frac{r}{b}, \alpha\right)$ is defined by the Dirichlet series*

$$E\left(s, \frac{r}{b}, \alpha\right) = \sum_{n \ge 1} \frac{\sigma_\alpha(n) \exp\left(2\pi i n r / b\right)}{n^s} \,,$$

where $Re\, s > Re\, \alpha + 1,\ b \ge 1,\ (r, b) = 1$ and

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha \,.$$

It is worth mentioning that T. Estermann (see [16]) introduced and studied the above function in the special case when $\alpha = 0$. Much later, it was studied by I. Kiuchi (see [21]) for $\alpha \in (-1, 0]$.
The Estermann zeta function can be continued analytically to a meromorphic function, on the whole complex plane up to two simple poles $s = 1$ and $s = 1 + \alpha$ if $\alpha \ne 0$ or a double pole at $s = 1$ if $\alpha = 0$ (see [16], [18], [29]).

Moreover, it satisfies the functional equation:

$$E\left(s, \frac{r}{b}, \alpha\right) = \frac{1}{\pi}\left(\frac{b}{2\pi}\right)^{1+\alpha-2s} \Gamma(1-s)\Gamma(1+\alpha-s)$$
$$\times \left(\cos\left(\frac{\pi\alpha}{2}\right) E\left(1+\alpha-s, \frac{\bar{r}}{b}, \alpha\right) - \cos\left(\pi s - \frac{\pi\alpha}{2}\right) E\left(1+\alpha-s, -\frac{\bar{r}}{b}, \alpha\right)\right),$$

where $\bar{r}$ is such that $\bar{r}r \equiv 1 \pmod{b}$ and $\Gamma(s)$ stands for the Gamma function.
For more details regarding the functional equation of the Estermann zeta function, the reader is referred to the Appendix.
R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown [3], used properties of $E\left(0, \frac{r}{b}, 0\right)$ to prove an asymptotic formula for

$$I = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|A\left(\frac{1}{2}+it\right)\right|^2 dt\,,$$

where $A(s)$ is a Dirichlet polynomial.
Asymptotics for functions of the form of I are useful for theorems which provide a lower bound for the portion of zeros of the Riemann zeta-function $\zeta(s)$ on the critical line (see [19]).

M. Ishibashi (see [17]) presented a nice result concerning the value of $E\left(s, \frac{r}{b}, \alpha\right)$ at $s = 0$.

Theorem 1.3. *(Ishibashi) Let $b \geq 2$, $1 \leq r \leq b$, $(r, b) = 1$, $\alpha \in \mathbb{N} \cup \{0\}$. Then*
(1) For even α, it holds

$$E\left(0, \frac{r}{b}, \alpha\right) = \left(-\frac{i}{2}\right)^{\alpha+1} \sum_{m=1}^{b-1} \frac{m}{b} \cot^{(\alpha)}\left(\frac{\pi m r}{b}\right) + \frac{1}{4}\delta_{\alpha,0}\,,$$

where $\delta_{\alpha,0}$ is the Kronecker delta function.
(2) For odd α, it holds

$$E\left(0, \frac{r}{b}, \alpha\right) = \frac{B_{\alpha+1}}{2(\alpha+1)}\,.$$

In the special case when $r = b = 1$, we have

$$E(0, 1, \alpha) = \frac{(-1)^{\alpha+1} B_{\alpha+1}}{2(\alpha+1)}\,,$$

where by B_m we denote the m-th Bernoulli number, where $B_{2m+1} = 0$,

$$B_{2m} = 2\frac{(2m)!}{(2\pi)^{2m}} \sum_{\nu \geq 1} \nu^{-2m}.$$

Hence for $b \geq 2$, $1 \leq r \leq b$, $(r, b) = 1$, it follows that

$$E\left(0, \frac{r}{b}, 0\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{r}{b}\right)\,,$$

where $c_0(r/b)$ is the cotangent sum (see Definition 1.1).
This result gives a connection between the cotangent sum $c_0(r/b)$ and the Estermann zeta function.
Period functions and families of cotangent sums appear in recent work of S. Bettin

and J. B. Conrey [6], generalizing the Dedekind sums and sharing with it the property of satisfying a reciprocity formula. Bettin and Conrey proved the following reciprocity formula for $c_0(r/b)$:

$$c_0\left(\frac{r}{b}\right) + \frac{b}{r} c_0\left(\frac{b}{r}\right) - \frac{1}{\pi r} = \frac{i}{2}\psi_0\left(\frac{r}{b}\right) ,$$

where

$$\psi_0(z) = -2\,\frac{\log 2\pi z - \gamma}{\pi i z} - \frac{2}{\pi}\int_{\left(\frac{1}{2}\right)} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s}\, z^{-s}\, ds ,$$

and γ stands for the Euler-Mascheroni constant.
This reciprocity formula demonstrates that $c_0(r/b)$ can be interpreted as an "imperfect" quantum modular form of weight 1, in the sense of D. Zagier (see [5], [32]).

The cotangent sum $c_0(r/b)$ can be associated to the study of the Riemann Hypothesis, also through its relation with the so-called Vasyunin sum. The Vasyunin sum is defined as follows:

$$V\left(\frac{r}{b}\right) := \sum_{m=1}^{b-1} \left\{\frac{mr}{b}\right\} \cot\left(\frac{\pi m r}{b}\right) ,$$

where $\{u\} = u - \lfloor u \rfloor$, $u \in \mathbb{R}$.
It can be shown (see [5], [6]) that

$$V\left(\frac{r}{b}\right) = -c_0\left(\frac{\bar{r}}{b}\right),$$

where, as mentioned previously, $\bar{r}$ is such that $\bar{r}r \equiv 1 \pmod{b}$.
The Vasyunin sum is itself associated to the study of the Riemann hypothesis through the following identity (see [5], [6]):

(1)

$$\begin{aligned}\frac{1}{2\pi(rb)^{1/2}}\int_{-\infty}^{+\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{r}{b}\right)^{it} \frac{dt}{\frac{1}{4}+t^2} &= \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{r}+\frac{1}{b}\right)\\ &+ \frac{b-r}{2rb}\log\frac{r}{b} - \frac{\pi}{2rb}\left(V\left(\frac{r}{b}\right) + V\left(\frac{b}{r}\right)\right).\end{aligned}$$

Note that the only non-explicit function in the right hand side of (1) is the Vasyunin sum.
The above formula is related to the Nyman-Beurling-Baéz-Duarte-Vasyunin approach to the Riemann Hypothesis (see [2], [5]). According to this approach, the Riemann Hypothesis is true if and only if

$$\lim_{N \to +\infty} d_N = 0,$$

where

$$d_N^2 = \inf_{D_N} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|1 - \zeta\left(\frac{1}{2}+it\right) D_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\frac{1}{4}+t^2}$$

and the infimum is taken over all Dirichlet polynomials

$$D_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

Hence, from the above arguments it follows that from the behavior of $c_0(r/b)$, we understand the behavior of $V(r/b)$ and thus from (1) we may hope to obtain crucial information related to the Nyman-Beurling-Baéz-Duarte-Vasyunin approach to the Riemann Hypothesis.

Therefore, to sum up, one can see from all the above that the cotangent sum $c_0(r/b)$ is strongly related to important functions of Number Theory and its properties can be applied in the study of significant open problems, such as Riemann's Hypothesis.

1.2. **Main result.** We now come to the main result of the paper, which states the equidistribution of certain normalized cotangent sums with respect to a positive measure, which is also constructed in the following theorem.

Definition 1.4. *For $z \in \mathbb{R}$, let*

$$F(z) = meas\{\alpha \in [0,1] \, : \, g(\alpha) \leq z\},$$

where "meas" denotes the Lebesgue measure,

$$g(\alpha) = \sum_{l=1}^{+\infty} \frac{1 - 2\{l\alpha\}}{l}$$

and

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \, : \, \forall \, \epsilon > 0, \, \exists \, a \text{ compact set } \mathcal{K}, \text{ such that } |f(x)| < \epsilon, \forall \, x \notin \mathcal{K}\}.$$

Remark. The convergence of this series has been investigated by R. de la Bretèche and G. Tenenbaum (see [8]). It depends on the partial fraction expansion of the number α.

Theorem 1.5. *i) F is a continuous function of z.*
ii) Let A_0, A_1 be fixed constants, such that $1/2 < A_0 < A_1 < 1$. Let also

$$H_k = \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx,$$

H_k is a positive constant depending only on k, $k \in \mathbb{N}$.
There is a unique positive measure μ on $\mathbb{R}$ with the following properties:
(a) For $\alpha < \beta \in \mathbb{R}$ we have

$$\mu([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha)).$$

(b)

$$\int x^k d\mu = \begin{cases} (A_1 - A_0) H_{k/2} \, , & \text{for even } k \\ 0 \, , & \text{otherwise} \, . \end{cases}$$

(c) For all $f \in C_0(\mathbb{R})$, we have

$$\lim_{b \to +\infty} \frac{1}{\phi(b)} \sum_{\substack{r \, : \, (r,b)=1 \\ A_0 b \leq r \leq A_1 b}} f\left(\frac{1}{b} c_0\left(\frac{r}{b}\right)\right) = \int f \, d\mu.$$

Remark. R. W. Bruggeman (see [9], [10]) and I. Vardi (see [30]) have investigated the equidistribution of Dedekind sums. In contrast with the work in this paper, they consider an additional averaging over the denominator.

1.3. **Outline of the proof and further results.** In [25], M. Th. Rassias proved the following asymptotic formula:

Theorem 1.6. *For $b \geq 2$, $b \in \mathbb{N}$, we have*

$$c_0\left(\frac{1}{b}\right) = \frac{1}{\pi}\, b \log b - \frac{b}{\pi}(\log 2\pi - \gamma) + O(1)\,.$$

In that paper, a method which applies properties of fractional parts in order to approach the cotangent sum in question is described. This method is generalized in the present paper, where some stronger results are being proved.
We initially provide a proof of an improvement of Theorem 1.6 as an asymptotic expansion. Namely, we prove the following:

Theorem 1.7. *Let $b, n \in \mathbb{N}$, $b \geq 6N$, with $N = \lfloor n/2 \rfloor + 1$. There exist absolute real constants $A_1, A_2 \geq 1$ and absolute real constants E_l, $l \in \mathbb{N}$ with $|E_l| \leq (A_1 l)^{2l}$, such that for each $n \in \mathbb{N}$ we have*

$$c_0\left(\frac{1}{b}\right) = \frac{1}{\pi} b \log b - \frac{b}{\pi}(\log 2\pi - \gamma) - \frac{1}{\pi} + \sum_{l=1}^{n} E_l b^{-l} + R_n^*(b)$$

where

$$|R_n^*(b)| \leq (A_2 n)^{4n}\, b^{-(n+1)}.$$

Additionally, we investigate the cotangent sum $c_0\left(\frac{r}{b}\right)$ for a fixed arbitrary positive integer value of r and for large integer values of b and prove the following results.

Proposition 1.8. *For r, $b \in \mathbb{N}$ with $(r, b) = 1$, it holds*

$$c_0\left(\frac{r}{b}\right) = \frac{1}{r}\, c_0\left(\frac{1}{b}\right) - \frac{1}{r} Q\left(\frac{r}{b}\right)\,,$$

where

$$Q\left(\frac{r}{b}\right) = \sum_{m=1}^{b-1} \cot\left(\frac{\pi m r}{b}\right) \left\lfloor \frac{rm}{b} \right\rfloor\,.$$

Theorem 1.9. *Let $r, b_0 \in \mathbb{N}$ be fixed, with $(b_0, r) = 1$. Let b denote a positive integer with $b \equiv b_0 \pmod{r}$. Then, there exists a constant $C_1 = C_1(r, b_0)$, with $C_1(1, b_0) = 0$, such that*

$$c_0\left(\frac{r}{b}\right) = \frac{1}{\pi r} b \log b - \frac{b}{\pi r}(\log 2\pi - \gamma) + C_1\, b + O(1),$$

for large integer values of b.

Theorem 1.10. *Let $k \in \mathbb{N}$ be fixed. Let also A_0, A_1 be fixed constants such that $1/2 < A_0 < A_1 < 1$. Then there exist explicit constants $E_k > 0$ and $H_k > 0$, depending only on k, such that*

(a)

$$\sum_{\substack{r:(r,b)=1 \\ A_0 b \leq r \leq A_1 b}} Q\left(\frac{r}{b}\right)^{2k} = E_k \cdot (A_1^{2k+1} - A_0^{2k+1}) b^{4k} \phi(b)(1 + o(1)), \quad (b \to +\infty).$$

(b)

$$\sum_{\substack{r:(r,b)=1\\ A_0 b\le r\le A_1 b}} Q\left(\frac{r}{b}\right)^{2k-1} = o\left(b^{4k-2}\phi(b)\right), \quad (b\to+\infty).$$

(c)

$$\sum_{\substack{r:(r,b)=1\\ A_0 b\le r\le A_1 b}} c_0\left(\frac{r}{b}\right)^{2k} = H_k\cdot(A_1-A_0)b^{2k}\phi(b)(1+o(1)), \quad (b\to+\infty).$$

(d)

$$\sum_{\substack{r:(r,b)=1\\ A_0 b\le r\le A_1 b}} c_0\left(\frac{r}{b}\right)^{2k-1} = o\left(b^{2k-1}\phi(b)\right), \quad (b\to+\infty).$$

Using the method of moments, we deduce detailed information about the distribution of the values of $c_0(r/b)$, where $A_0b\le r\le A_1b$ and $b\to+\infty$. Namely, we prove Theorem 1.5.

Finally, we study the convergence of the series

$$\sum_{k\ge 0} H_k x^{2k}$$

and prove the following theorem:

Theorem 1.11. *The series*

$$\sum_{k\ge 0} H_k x^{2k},$$

converges only for $x=0$.

Another interesting question which we have investigated but have not reached a conclusion yet is whether the series

$$\sum_{k\ge 0} \frac{H_k}{(2k)!} x^{2k},$$

has a positive radius of convergence. This would lead to a simplification in the proof of our equidistribution result, since in this case we could apply results about distributions which are determined by their moments.

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Department of Mathematics, University of Ulm, Helmholtzstrasse 18, 8901 Ulm, Germany.

E-mail address: helmut.maier@uni-ulm.de

Department of Mathematics, ETH-Zürich, Rämistrasse 101, 8092 Zürich, Switzerland.

E-mail address: michail.rassias@math.ethz.ch, michailr@princeton.edu