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GENERALIZATIONS OF EULER NUMBERS AND POLYNOMIALS

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The concepts of Euler numbers and Euler polynomials are generalized and some basic properties are investigated.

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1. Introduction. It is well known that the Euler numbers and polynomials can be defined by the following definitions.

Definition 1.1 (see [1]). The Euler numbers E_k are defined by the following expansion:

$$\operatorname{sech} t = \frac{2e^t}{e^{2t}+1} = \sum_{k=0}^{\infty} \frac{E_k}{k!} t^k, \quad |t| \le \pi. \tag{1.1}$$

In [6, page 5], the Euler numbers are defined by

$$\frac{2e^{t/2}}{e^t+1} = \operatorname{sech}\frac{t}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n E_n}{(2n)!} \left(\frac{t}{2}\right)^{2n}, \quad |t| \le \pi. \tag{1.2}$$

Definition 1.2 (see [1, 6]). The Euler polynomials $E_k(x)$ for $x \in \mathbb{R}$ are defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty} \frac{E_k(x)}{k!} t^k, \quad |t| < \pi. \tag{1.3}$$

Let $\mathbb{N}$ denote the set of all positive integers. It can also be shown that the polynomials $E_i(t)$, $i \in \mathbb{N}$, are uniquely determined by the following two properties:

$$\begin{gathered} E_i'(t) = iE_{i-1}(t), \quad E_0(t) = 1, \\ E_i(t+1) + E_i(t) = 2t^i. \end{gathered} \tag{1.4}$$

Euler polynomials are related to the Bernoulli numbers. For information about Bernoulli numbers and polynomials, we refer to [1, 2, 3, 5, 6].

In this note, we give some generalizations of the concepts of Euler numbers and Euler polynomials and research their basic properties. In fact, motivations and ideas of this note and other articles, see, for example, [2, 3, 4], originate essentially from [5].

2. Generalizations of Euler numbers and polynomials.

In this section, we give two definitions, the generalized Euler number and the generalized Euler polynomial, which generalize the concepts of Euler number and Euler polynomial.

Definition 2.1. For positive numbers a, b, and c, the generalized Euler numbers $E_k(a,b,c)$ are defined by

$$\frac{2c^t}{b^{2t}+a^{2t}} = \sum_{k=0}^{\infty} \frac{E_k(a,b,c)}{k!} t^k. \tag{2.1}$$

Definition 2.2. For any given positive numbers a, b, and c and $x \in \mathbb{R}$, the generalized Euler polynomials $E_k(x;a,b,c)$ are defined by

$$\frac{2c^{xt}}{b^t+a^t} = \sum_{k=0}^{\infty} \frac{E_k(x;a,b,c)}{k!} t^k. \tag{2.2}$$

Taking $a = 1$ and $b = c = e$, then Definitions 1.1 and 1.2 can be deduced from Definitions 2.1 and 2.2, respectively. Thus, Definitions 2.1 and 2.2 generalize the concepts of Euler numbers and polynomials.

3. Some properties of the generalized Euler numbers.

In this section, we study some basic properties of the generalized Euler numbers defined in Definition 2.1.

Theorem 3.1. *For positive numbers a, b, and c and real number $x \in \mathbb{R}$,*

$$E_0(a,b,c) = 1, \qquad E_k(1,e,e) = E_k, \qquad E_k(1,e^{1/2},e^x) = E_k(x), \tag{3.1}$$

$$E_k(a,b,c) = 2^k(\ln b - \ln a)^k E_k\left(\frac{\ln c - 2\ln a}{2(\ln b - \ln a)}\right), \tag{3.2}$$

$$E_k(a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln b - \ln a)^j (\ln c - \ln a - \ln b)^{k-j} E_j. \tag{3.3}$$

Proof. The formulas in (3.1) follow from Definitions 1.1, 1.2, and 2.1 easily. By Definitions 1.2 and 2.1 and direct computation, we have

$$\begin{aligned} \frac{2c^t}{b^{2t}+a^{2t}} &= \frac{2\exp\left((\ln c - 2\ln a)/2(\ln b - \ln a)\cdot 2t(\ln b - \ln a)\right)}{\exp\left(2t(\ln b - \ln a)\right)+1} \\ &= \sum_{k=0}^{\infty} 2^k(\ln b - \ln a)^k E_k\left(\frac{\ln c - 2\ln a}{2(\ln b - \ln a)}\right)\frac{t^k}{k!}. \end{aligned} \tag{3.4}$$

Then, formula (3.2) follows.

Substituting $E_k(x) = \sum_{j=0}^{k} 2^{-j}\binom{k}{j}(x-1/2)^{k-j}E_j$ into the formula (3.2) yields formula (3.3). The proof of the classical result for $E_k(x)$ follows from the more general proof that will be given for (4.1). □

Theorem 3.2. *For $k \in \mathbb{N}$,*

$$E_k(a,b,c) = -\frac{1}{2}\sum_{j=0}^{k-1}\binom{k}{j}\left[(2\ln b-\ln c)^{k-j}+(2\ln a-\ln c)^{k-j}\right]E_j(a,b,c), \tag{3.5}$$

$$E_k(a,b,c) = E_k(b,a,c), \tag{3.6}$$

$$E_k(a^\alpha,b^\alpha,c^\alpha) = \alpha^k E_k(a,b,c). \tag{3.7}$$

Proof. By Definition 2.1, direct calculation yields

$$\begin{aligned}
1 &= \frac{1}{2}\left[\left(\frac{b^2}{c}\right)^t+\left(\frac{a^2}{c}\right)^t\right]\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\left(\ln\frac{b^2}{c}\right)^k+\left(\ln\frac{a^2}{c}\right)^k\right]\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}\left[\left(\ln\frac{b^2}{c}\right)^{k-j}+\left(\ln\frac{a^2}{c}\right)^{k-j}\right]E_j(a,b,c)\right)\frac{t^k}{k!}.
\end{aligned} \tag{3.8}$$

Equating coefficients of t^k in (3.8) gives us

$$\sum_{j=0}^{k}\binom{k}{j}\left[\left(\ln\frac{b^2}{c}\right)^{k-j}+\left(\ln\frac{a^2}{c}\right)^{k-j}\right]E_j(a,b,c) = 0. \tag{3.9}$$

Formula (3.5) follows.

The other formulas follow from Definition 2.1 and formula (3.2). □

Remark 3.3. For positive numbers a, b, and c, we have

$$\begin{aligned}
&E_0(a,b,c) = 1,\\
&E_1(a,b,c) = \ln c-\ln a-\ln b,\\
&E_2(a,b,c) = (\ln c-2\ln a)(\ln c-2\ln b),\\
&E_3(a,b,c) = \left[(\ln c-\ln a-\ln b)^2-3(\ln b-\ln a)^2\right](\ln c-\ln a-\ln b).
\end{aligned} \tag{3.10}$$

Since it is well known and easily established that the E_k are integers, $E_j = 0$ if j is odd, and $E_j(0) = 0$ if j is positive and even, it follows from (3.3) and (3.2) that $E_k(a,b,c)$ is an integer polynomial in $\ln a$, $\ln b$, and $\ln c$ which is homogeneous of degree k and which is divisible by $\ln c-\ln a-\ln b$ if k is odd, and divisible by $(\ln c-2\ln a)(\ln c-2\ln b)$ if k is even and positive.

4. Some properties of the generalized Euler polynomials. In this section, we investigate properties of the generalized Euler polynomials defined by Definition 2.2.

Theorem 4.1. *For any given positive numbers a, b, and c and $x \in \mathbb{R}$,*

$$E_k(x;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} \frac{(\ln c)^{k-j}}{2^j} \left(x - \frac{1}{2}\right)^{k-j} E_j(a,b,c), \tag{4.1}$$

$$E_k(x;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} \left(\ln \frac{b}{a}\right)^{j} \left(x - \frac{1}{2}\right)^{k-j} E_j\left(\frac{\ln c - 2\ln a}{2(\ln b - \ln a)}\right), \tag{4.2}$$

$$E_k(x;a,b,c) = \sum_{j=0}^{k} \sum_{\ell=0}^{j} \binom{k}{j} \binom{j}{\ell} \frac{(\ln c)^{k-j}}{2^j} \left[\ln \frac{b}{a}\right]^{\ell} \left[\ln \frac{c}{ab}\right]^{j-\ell} \left[x - \frac{1}{2}\right]^{k-j} E_\ell, \tag{4.3}$$

$$E_k(a,b,c) = 2^k E_k\left(\frac{1}{2};a,b,c\right), \tag{4.4}$$

$$E_k(x) = E_k(x;1,e,e). \tag{4.5}$$

Proof. By Definitions 2.1 and 2.2, we have

$$\begin{aligned}
\frac{2c^{2xt}}{b^{2t}+a^{2t}} &= \sum_{k=0}^{\infty} 2^k E_k(x;a,b,c) \frac{t^k}{k!}, \\
\frac{2c^{2xt}}{b^{2t}+a^{2t}} &= \frac{2c^{t}}{b^{2t}+a^{2t}} \cdot c^{(2x-1)t} \\
&= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(a,b,c)\right) \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} (2x-1)^k (\ln c)^k\right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} (2x-1)^{k-j} E_j(a,b,c)\right) \frac{t^k}{k!}.
\end{aligned} \tag{4.6}$$

Equating the coefficients of $t^k/k!$ in (4.6) yields

$$2^k E_k(x;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} (2x-1)^{k-j} E_j(a,b,c). \tag{4.7}$$

Formula (4.1) follows.

The other formulas follow directly from substituting formulas (3.2) and (3.3) into (4.1) and taking $x = 1/2$ in (4.1), respectively. □

Theorem 4.2. *For positive integer* $1 \le p \le k$*,*

$$\frac{\partial^p}{\partial x^p} E_k(x;a,b,c) = \frac{k!}{(k-p)!}(\ln c)^p E_{k-p}(x;a,b,c), \tag{4.8}$$

$$\int_\beta^x E_k(t;a,b,c)\,dt = \frac{1}{(k+1)\ln c}\left[E_{k+1}(x;a,b,c) - E_{k+1}(\beta;a,b,c)\right]. \tag{4.9}$$

Proof. Differentiating equation (2.2) with respect to x yields

$$\frac{\partial}{\partial x} E_k(x;a,b,c) = k(\ln c) E_{k-1}(x;a,b,c). \tag{4.10}$$

Using formula (4.10) and by mathematical induction, formula (4.8) follows. Rearranging formula (4.10) produces

$$E_k(x;a,b,c) = \frac{1}{(k+1)\ln c}\frac{\partial}{\partial x} E_{k+1}(x;a,b,c). \tag{4.11}$$

Formula (4.9) follows from integration on both sides of formula (4.11). □

Theorem 4.3. *For positive numbers* a*,* b*, and* c *and* $x \in \mathbb{R}$*,*

$$E_k(x+1;a,b,c) = \sum_{j=0}^{k} \binom{k}{j} (\ln c)^{k-j} E_j(x;a,b,c), \tag{4.12}$$

$$E_k(x+1;a,b,c) = 2x^k(\ln c)^k + \sum_{j=0}^{k}\binom{k}{j}\left[(\ln c)^{k-j} - (\ln b)^{k-j} - (\ln a)^{k-j}\right] E_j(x;a,b,c), \tag{4.13}$$

$$E_k(x+1;a,b,c) = E_k\left(x;\frac{a}{c},\frac{b}{c},c\right). \tag{4.14}$$

Proof. From Definition 2.2 and straightforward calculation, we have

$$\begin{aligned}\frac{2c^{xt}}{b^t+a^t}\cdot c^t &= \left[\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}(\ln c)^k\right]\\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}E_j(x;a,b,c)\right]\frac{t^k}{k!},\end{aligned} \tag{4.15}$$

$$\frac{2c^{xt}}{b^t+a^t}\cdot c^t = \frac{2c^{(x+1)t}}{b^t+a^t} = \sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c).$$

Therefore, from equating the coefficients of $t^k/k!$ in (4.15), formula (4.12) follows.

Similarly, we obtain

$$\begin{aligned}\frac{2c^{(x+1)t}}{b^t+a^t} &= \sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c) = 2c^{xt}+\frac{2c^{xt}}{b^t+a^t}\left(c^t-b^t-a^t\right)\\ &= 2\sum_{k=0}^{\infty}\frac{t^k}{k!}x^k(\ln c)^k\\ &\quad+\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty}\left((\ln c)^k-(\ln b)^k-(\ln a)^k\right)\frac{t^k}{k!}\right]\\ &= \sum_{k=0}^{\infty}\left[2x^k(\ln c)^k\right.\\ &\qquad\left.+\sum_{j=0}^{k}\binom{k}{j}\left[(\ln c)^{k-j}-(\ln b)^{k-j}-(\ln a)^{k-j}\right]E_j(x;a,b,c)\right]\frac{t^k}{k!}.\end{aligned} \tag{4.16}$$

By equating coefficients of $t^k/k!$, we obtain formula (4.13).

Since

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+1;a,b,c) = \frac{2c^{(x+1)t}}{b^t+a^t} &= \frac{2c^{xt}}{(b/c)^t+(a/c)^t}\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}E_k\left(x;\frac{a}{c},\frac{b}{c},c\right),\end{aligned} \tag{4.17}$$

by equating coefficients, we obtain formula (4.14). The proof is complete. □

Corollary 4.4. *The following formulas are valid for positive numbers a, b, and c and real number x:*

$$E_k(x+1)+E_k(x) = 2x^k, \tag{4.18}$$

$$E_k(x+1) = \sum_{j=0}^{k}\binom{k}{j}E_j(x), \tag{4.19}$$

$$E_k(x+1;1,b,b)+E_k(x;1,b,b) = 2x^k(\ln b)^k, \tag{4.20}$$

$$E_k(x+1;1,b,b) = \sum_{j=0}^{k}\binom{k}{j}E_j(x;1,b,b)(\ln b)^{k-j}, \tag{4.21}$$

$$\sum_{j=0}^{k-1}\binom{k}{j}E_j(x;1,b,b)(\ln b)^{k-j}+2E_k(x;1,b,b) = 2x^k(\ln b)^k, \tag{4.22}$$

$$\int_x^{x+1}E_k(t;a,b,c)\,dt = \frac{1}{(k+1)\ln c}\sum_{j=0}^{k}\binom{k+1}{j}(\ln c)^{k-j}E_j(x;a,b,c). \tag{4.23}$$

Theorem 4.5. *For positive numbers* $a,b,c>0$, $x\in\mathbb{R}$, *and nonnegative integer* k,

$$E_k(1-x;a,b,c)=(-1)^kE_k\left(x;\frac{c}{a},\frac{c}{b},c\right), \tag{4.24}$$

$$E_k(1-x;a,b,c)=E_k\left(-x;\frac{a}{c},\frac{b}{c},c\right). \tag{4.25}$$

Proof. From Definition 2.2 and easy computation, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(1-x;a,b,c)&=\frac{2c^{(1-x)t}}{b^t+a^t}=\frac{2c^t\cdot c^{-xt}}{b^t+a^t}=\frac{2c^{-xt}}{(c/b)^{-t}+(c/a)^{-t}}\\&=\sum_{k=0}^{\infty}\frac{t^k}{k!}(-1)^kE_k\left(x;\frac{c}{a},\frac{c}{b},c\right).\end{aligned} \tag{4.26}$$

Equating coefficients of t^k above leads to formula (4.24).

By the same procedure, we can establish formula (4.25). □

Theorem 4.6. *For positive numbers* $a,b,c>0$, *nonnegative natural number* k, *and* $x,y\in\mathbb{R}$,

$$\begin{aligned}E_k(x+y;a,b,c)&=\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}y^{k-j}E_j(x;a,b,c),\\E_k(x+y;a,b,c)&=\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}x^{k-j}E_j(y;a,b,c).\end{aligned} \tag{4.27}$$

Proof. These two formulas can be deduced from the following calculation and considering symmetry of x and y:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x+y;a,b,c)&=\frac{2c^{(x+y)t}}{b^t+a^t}=\frac{2c^{xt}\cdot c^{yt}}{b^t+a^t}\\&=\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)\right]\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}(\ln c)^ky^k\right]\\&=\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}(\ln c)^{k-j}y^{k-j}E_j(x;a,b,c)\right]\frac{t^k}{k!}.\end{aligned} \tag{4.28}$$

The proof is complete. □

Theorem 4.7. *For natural numbers* k *and* m *and positive number* b,

$$\sum_{\ell=1}^{m}(-1)^{\ell}\ell^k=\frac{1}{2(\ln b)^k}\left[(-1)^mE_k(m+1;1,b,b)-E_k(1;1,b,b)\right]. \tag{4.29}$$

Proof. Rearranging formula (4.20) gives us

$$x^k = \frac{1}{2(\ln b)^k}\left[E_k(x+1;1,b,b)+E_k(x;1,b,b)\right]. \tag{4.30}$$

Replacing x by $\ell \in \mathbb{N}$ and summing up ℓ from 1 to m yields

$$\begin{aligned}\sum_{\ell=1}^{m}(-1)^{\ell}\ell^k &= \frac{1}{2(\ln b)^k}\sum_{\ell=1}^{m}(-1)^{\ell}\left[E_k(\ell+1;1,b,b)+E_k(\ell;1,b,b)\right]\\ &= \frac{1}{2(\ln b)^k}\left[(-1)^m E_k(m+1;1,b,b)-E_k(1;1,b,b)\right].\end{aligned} \tag{4.31}$$

The proof is complete. □

Remark 4.8. Finally, we give several concrete formulas as follows:

$$\begin{aligned}E_0(x;a,b,c) &= 1,\\ E_1(x;a,b,c) &= \left(x-\frac{1}{2}\right)\ln c+\frac{1}{2}(\ln c-\ln a-\ln b),\\ E_2(x;a,b,c) &= \left(x-\frac{1}{2}\right)^2(\ln c)^2+\left(x-\frac{1}{2}\right)(\ln c-\ln b-\ln a)\ln c\\ &\quad+\frac{1}{4}(\ln c-2\ln a)(\ln c-2\ln b).\end{aligned} \tag{4.32}$$

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