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Generalizations of Hermite-Hadamard type inequalities for MT-convex functions

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Abstract

In this paper, we discover two novel integral identities for twice differentiable functions. Under the utility of these identities, we establish some generalized inequalities for classical integrals and Riemann-Liouville fractional integrals of the Hermite-Hadamard type via functions whose derivatives absolute values are MT-convex. At the end, we present applications for special means and several error approximations for the trapezoidal formula.

Keywords: MT-convex function, Hermite-Hadamard inequality, fractional integrals, trapezoidal formula, error approximation.
2010 MSC: 26D15, 26A51, 26D20.

1. Introduction

A real-valued function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on the interval I if the inequality

$$\phi(tz + (1-t)w) \leq t\phi(z) + (1-t)\phi(w) \tag{1.1}$$

holds for all $z, w \in I$ and $t \in [0,1]$. ϕ is said to be concave on I if the inequality given in (1.1) holds in the reverse direction.

A number of important inequalities have been obtained for the class of convex functions, when the idea of convexity was introduced more than a hundred years ago. But among those one of the most prominent inequality is the Hermite-Hadamard inequality. It can be stated more appropriately like (see [11]):

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If the function $\phi : I \to \mathbb{R}$ is convex on I, then the double inequality

$$\phi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \phi(z)dz \leq \frac{\phi(c)+\phi(d)}{2} \tag{1.2}$$

holds for all $c, d \in I$ with $c < d$. If the function ϕ is concave on I, then both the inequalities in (1.2) hold in the reverse order. It gives an estimate from both sides of the mean, that is, from above and below of the mean value of a convex function and ensures the integrability of any convex function too. It is also a matter of great interest and one has to note that some of the classical important inequalities for means can be obtained from the Hermite-Hadamard inequality under the utility of peculiar convex functions ϕ. These inequalities for convex functions play a crucial role in analysis as well as in other areas of pure and applied mathematics.

The following important result is due to Dragomir and Agarwal related to the right hand side of (1.2).

Theorem 1.1 ([4])**.** *Let $I \subseteq \mathbb{R}$ be an interval and $\phi : I^\circ \to \mathbb{R}$ be a differentiable function on I°. Then*

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(z)dz\right| \leq \frac{(d-c)(|\phi'(c)|+|\phi'(d)|)}{8},$$

if $c, d \in I^\circ$ with $c < d$ and $|\phi'|$ is convex on $[c, d]$, where and in what follows I° denotes the interior of I.

Before writing the Hermite-Hadamard inequality for fractional integrals, we first recall the definition of fractional integrals [10, 21].

Let $\eta > 0$, $d > c \geq 0$ and $\phi \in L[c, d]$. Then the left-sided and right-sided Riemann-Liouville fractional integrals $J^\eta_{c^+}\phi$ and $J^\eta_{d^-}\phi$ of order η are defined by

$$J^\eta_{c^+}\phi(z) = \frac{1}{\Gamma(\eta)}\int_c^z (z-s)^{\eta-1}\phi(s)ds \quad (z > c)$$

and

$$J^\eta_{d^-}\phi(z) = \frac{1}{\Gamma(\eta)}\int_z^d (s-z)^{\eta-1}\phi(s)ds \quad (z < d),$$

respectively, where $\Gamma(\eta)$ is the Gamma function given by

$$\Gamma(\eta) = \int_0^\infty e^{-u}u^{\eta-1}du.$$

It is also important to note that $J^0_{c^+}\phi(z) = J^0_{d^-}\phi(z) = \phi(z)$, and the fractional integral shrinks to the classical integral in the case of $\eta = 1$.

Sarikaya et al. [22] established the Hermite-Hadamard's inequalities for fractional integrals as follows.

Theorem 1.2 ([22])**.** *Let $d > c \geq 0$ and $\phi : [c, d] \to \mathbb{R}^+$ be a convex function and $\phi \in L[c, d]$. Then*

$$\phi\left(\frac{c+d}{2}\right) \leq \frac{\Gamma(\eta+1)}{2(d-c)^\eta}\left[J^\eta_{c^+}\phi(d) + J^\eta_{d^-}\phi(c)\right] \leq \frac{\phi(c)+\phi(d)}{2}.$$

In [29] (see also [26–28]), Tunç and Yidirim defined the so-called MT-convex function as follows.

A nonnegative function $\phi : I \to \mathbb{R}$ is said to be MT-convex on the interval I if the inequality

$$\phi(sz + (1-s)w) \leq \frac{\sqrt{s}}{2\sqrt{1-s}}\phi(z) + \frac{\sqrt{1-s}}{2\sqrt{s}}\phi(w)$$

holds for all $z, w \in I$ and $s \in (0, 1)$.

The following important Theorem 1.3 for the class of MT-convex functions also can be found in the literature [29].

Theorem 1.3 ([29]). *Let ϕ be an MT-convex function on the interval I and $c, d \in I$ with $c < d$. Then*

$$\phi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \phi(z)dz, \quad \frac{2}{d-c}\int_c^d \tau(z)\phi(z)dz \leq \frac{\phi(c)+\phi(d)}{2},$$

if $\phi \in L[c,d]$, where $\tau(z) = \sqrt{(d-z)(z-c)}/(d-c)$.

Liu et. al. [18] presented Theorems 1.4 and 1.5 for the class of MT-convex functions as follows.

Theorem 1.4 ([18]). *Let $c, d \in I$ with $c < d$ and $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I such that $\phi' \in L[c,d]$. If $|\phi'|^q$ is MT-convex on $[c,d]$ with $q > 1$, $p^{-1} = 1 - q^{-1}$ and $|\phi'(z)| \leq K$ for $z \in [c,d]$, then the inequality*

$$\left|\frac{\phi(d)(d-z)+\phi(c)(z-c)}{(d-c)} - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq \frac{K}{(1+p)^{1/p}}\left(\frac{\pi}{2}\right)^{1/q}\frac{\left[(z-c)^2+(d-z)^2\right]}{(d-c)}$$

holds for each $z \in [c,d]$.

Theorem 1.5 ([18]). *Let $c, d \in I$ with $c < d$ and $\phi : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on I such that $\phi' \in L[c,d]$. If $|\phi'|^q$ is MT-convex on $[c,d]$ with $q > 1$, and $|\phi'(z)| \leq K$ for $z \in [c,d]$, then*

$$\begin{aligned}\left|\frac{\phi(d)(d-z)+\phi(c)(z-c)}{(d-c)} - \frac{1}{d-c}\int_c^d \phi(u)du\right| &\leq K\left(\frac{1}{2}\right)^{1+1/q}\pi^{1/q}\frac{(z-c)^2+(d-z)^2}{(d-c)} \\ &< \frac{K\pi\left[(z-c)^2+(d-z)^2\right]}{4(d-c)}\end{aligned}$$

for each $z \in [c,d]$.

For more recent results, extensions, generalizations and refinements concerning to Hermite-Hadamard inequality can be found in the literatures [1–3, 5–9, 12–17, 19, 20, 23–25, 30].

In this paper, we discover two novel integral identities for twice differentiable functions. We use these identities to establish some general inequalities for functions whose second derivatives absolute values are MT-convex. These general inequalities give us some new estimates for the right-hand side of classical integrals and Riemann-Liouville fractional integrals inequalities of Hermite-Hadamard type. At the end, we give applications for some means and error estimates for the trapezoidal formula.

2. Hermite-Hadamard type inequalities for MT-Convex functions via classical integrals

Before starting our main results we write the following hypotheses which will be utilized throughout the paper.

- $\mathbf{H_1}$: Suppose that $I \subseteq \mathbb{R}$ is an interval, $\phi : I \to \mathbb{R}$ is twice differentiable on I°, $c, d \in I^\circ$ with $c < d$ and $\phi'' \in L[c,d]$.
- $\mathbf{H_2}$: Suppose that $\mathbf{H_1}$ holds and $|\phi''|$ is MT-convex on $[c,d]$ such that $|\phi''(z)| \leq K$ for $z \in [c,d]$.
- $\mathbf{H_3}$: Suppose that $\mathbf{H_1}$ holds, $q > 1$ and $|\phi''|^q$ is MT-convex on $[c,d]$ with $|\phi''(z)| \leq K$ for $z \in [c,d]$.
- $\mathbf{H_4}$: Suppose that $\mathbf{H_1}$ holds for $I \subset [0,\infty)$ and $\eta > 0$.
- $\mathbf{H_5}$: Suppose that $\mathbf{H_4}$ holds and $|\phi''|$ is MT-convex on $[c,d]$ such that $|\phi''(z)| \leq K$ for $z \in [c,d]$.
- $\mathbf{H_6}$: Suppose that $\mathbf{H_4}$ holds, $q > 1$ and $|\phi''|^q$ is MT-convex on $[c,d]$ with $|\phi''(z)| \leq K$ for $z \in [c,d]$.

For simplicity we take the function Δ as

$$\Delta(z) = \frac{\phi'(z)\left((z-c)^2-(d-z)^2\right)+2\phi(d)(d-z)+2\phi(c)(z-c)}{2(d-c)}, \quad (z \in [c,d]).$$

To give the main results of this section, we need to prove the following Lemma 2.1.

Lemma 2.1. *The identity*

$$\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du = \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sc+(1-s)z)ds + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sd+(1-s)z)ds$$

holds under the hypothesis $\mathbf{H_1}$.

Proof. It follows from the integration by parts and changes of variables that

$$\begin{aligned}
&\frac{(z-c)^3}{2(d-c)}\int_0^1 (1-s^2)\phi''(sc+(1-s)z)ds + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\phi''(sd+(1-s)z)ds \\
&= \frac{(z-c)^3}{2(d-c)}\left[-\frac{\phi'(z)}{c-z} + 2\left(\frac{\phi(c)}{(c-z)^2} - \frac{1}{(c-z)^3}\int_z^c \phi(u)du\right)\right] \\
&\quad + \frac{(d-z)^3}{2(d-c)}\left[-\frac{\phi'(z)}{d-z} + 2\left(\frac{\phi(d)}{(d-z)^2} - \frac{1}{(d-z)^3}\int_z^d \phi(u)du\right)\right] \\
&= \Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du.
\end{aligned}$$

□

Now we start to prove our main results.

Theorem 2.2. *Under the hypothesis* $\mathbf{H_2}$, *the inequality given below*

$$\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \le \frac{5K\pi\left[(z-c)^3+(d-z)^3\right]}{32(d-c)}$$

is valid for all $z \in [c,d]$.

Proof. It follows from the MT-convexity of $|\phi''|$ and Lemma 2.1 together with the elementary properties of Euler Beta function that

$$\begin{aligned}
&\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \\
&\le \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sc+(1-s)z)|ds + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sd+(1-s)z)|ds \\
&\le \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)| + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|\right]ds \\
&\quad + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)\left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(d)| + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|\right]ds \\
&\le \frac{K[(z-c)^3+(d-z)^3]}{4(d-c)}\int_0^1 \left(s^{1/2}(1-s)^{-1/2} - s^{5/2}(1-s)^{-1/2} + s^{-1/2}(1-s)^{1/2} - s^{3/2}(1-s)^{1/2}\right)ds \\
&= \frac{5K\pi[(z-c)^3+(d-z)^3]}{32(d-c)}.
\end{aligned}$$

□

Remark 2.3. Let $z = (c+d)/2$. Then Theorem 2.2 leads to

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(u)du\right| \le \frac{5K\pi(d-c)^2}{128}.$$

Theorem 2.4. *Suppose that the hypothesis* $\mathbf{H_3}$ *holds. If* $p, q > 1$ *such that* $p^{-1} = 1 - q^{-1}$*, then the inequality*

$$\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}\frac{[(z-c)^3 + (d-z)^3]}{2(d-c)}$$

takes place for each $z \in [c, d]$.

Proof. Making use of Lemma 2.1 and the well-known Hölder inequality, we get

$$\begin{aligned}\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq &\frac{(z-c)^3}{2(d-c)}\left(\int_0^1 \left(1-s^2\right)^p ds\right)^{1/p}\left(\int_0^1 \mid \phi''(sc + (1-s)z) \mid^q ds\right)^{1/q}\\ &+ \frac{(d-z)^3}{2(d-c)}\left(\int_0^1 \left(1-s^2\right)^p ds\right)^{1/p}\left(\int_0^1 \mid \phi''(sd + (1-s)z) \mid^q ds\right)^{1/q}.\end{aligned}$$

It follows from the MT-convexity of $|\phi''|^q$ and boundedness of $|\phi''(z)|$ that

$$\int_0^1 |\phi''(sc + (1-s)z)|^q dt \leq \int_0^1 \left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)|^q + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds \leq \frac{K^q\pi}{2},$$

$$\int_0^1 |\phi''(sd + (1-s)z)|^q ds \leq \int_0^1 \left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(d)|^q + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds \leq \frac{K^q\pi}{2}.$$

It is not difficult to verify that

$$\left(\int_0^1 \left(1-s^2\right)^p ds\right)^{1/p} = \left(\frac{1}{2}\int_0^1 (1-z)^p z^{-1/2} dz\right)^{1/p} = \left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}.$$

Therefore, the required conclusion follows easily from the above inequalities and identity. □

Remark 2.5. Let $z = (c+d)/2$. Then, Theorem 2.4 leads to

$$\left|\frac{\phi(c) + \phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq K\left(\frac{\pi}{2}\right)^{1/q}(d-c)^2\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}.$$

Theorem 2.6. *Suppose that the hypothesis* $\mathbf{H_3}$ *is true, then the inequality*

$$\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq K\left(\frac{5\pi}{16}\right)^{1/q}\left(\frac{2}{3}\right)^{1-1/q}\left[\frac{(z-c)^3 + (d-z)^3}{2(d-c)}\right]$$

is valid for each $z \in [c, d]$.

Proof. It follows from Lemma 2.1 and the well-known Hölder inequality that

$$\begin{aligned}&\left|\Delta(z) - \frac{1}{d-c}\int_c^d \phi(u)du\right|\\ &\leq \frac{(z-c)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sc + (1-s)z)|ds + \frac{(d-z)^3}{2(d-c)}\int_0^1 \left(1-s^2\right)|\phi''(sd + (1-s)z)|ds\\ &\leq \frac{(z-c)^3}{2(d-c)}\left(\int_0^1 (1-s^2)ds\right)^{1-1/q}\left(\int_0^1 \left(1-s^2\right)|\phi''(sc + (1-s)z)|^q ds\right)^{1/q}\\ &\quad + \frac{(d-z)^3}{2(d-c)}\left(\int_0^1 (1-s^2)ds\right)^{1-1/q}\left(\int_0^1 \left(1-s^2\right)|\phi''(sd + (1-s)z)|^q ds\right)^{1/q}.\end{aligned}$$

From the MT-convexity of $|\phi''|^q$ and $|\phi''(z)| \leq K$ on $[c,d]$ we clearly see that

$$\int_0^1 (1-s^2)|\phi''(sc+(1-s)z)|^q ds \leq \int_0^1 \left[\frac{(1-s^2)\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)|^q + \frac{(1-s^2)\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds \leq \frac{5K^q\pi}{16},$$

$$\int_0^1 (1-s^2)|\phi''(sd+(1-s)z)|^q ds \leq \int_0^1 \left[\frac{(1-s^2)\sqrt{s}}{2\sqrt{1-s}}|\phi''(d)|^q + \frac{(1-s^2)\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds \leq \frac{5K^q\pi}{16}.$$

Note that

$$\left(\int_0^1 (1-s^2)ds\right)^{1-1/q} = \left(\frac{2}{3}\right)^{1-1/q}.$$

Therefore, Theorem 2.6 follows from the above inequalities and identity. □

Remark 2.7. By choosing $z=(c+d)/2$ in Theorem 2.6, we get

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(u)du\right| \leq K\left(\frac{5\pi}{16}\right)^{1/q}\left(\frac{2}{3}\right)^{1-1/q}\frac{(d-c)^2}{8}.$$

3. Hermite-Hadamard type inequalities for MT-convex functions via fractional integrals

In this section we present new Hermite-Hadamard type inequalities for MT-convex functions. For this purpose we first establish fractional integral identity in the following lemma.

For simplicity we denote the function $\tilde{\Delta}$ by

$$\tilde{\Delta}(z) = \frac{\phi'(z)\left((z-c)^{\eta+1} - (d-z)^{\eta+1}\right) + (\eta+1)\phi(c)(z-c)^\eta + (\eta+1)\phi(d)(d-z)^\eta}{(\eta+1)(d-c)}.$$

Lemma 3.1. *Under the hypothesis* $\mathbf{H_4}$*, the following identity is valid*

$$\begin{aligned}&\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^\eta_{c^+}\phi(z) + J^\eta_{d^-}\phi(z)\right]\\ &= \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sc+(1-s)z)ds + \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sd+(1-s)z)ds.\end{aligned}$$

Proof. By integration by parts and then by changing of variables, we obtain

$$\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sc+(1-s)z)ds = \frac{\phi'(z)}{z-c} + \frac{(\eta+1)\phi(c)}{(c-z)^2} - \frac{\Gamma(\eta+2)}{(z-c)^{\eta+2}}J^\eta_{c^+}\phi(z). \tag{3.1}$$

Similarly,

$$\int_0^1 \left(1-s^{\eta+1}\right)\phi''(sd+(1-s)z)ds = -\frac{\phi'(z)}{d-z} + \frac{(\eta+1)\phi(d)}{(d-z)^2} - \frac{\Gamma(\eta+2)}{(d-z)^{\eta+2}}J^\eta_{d^-}\phi(z). \tag{3.2}$$

Now multiplying (3.1) by $(z-c)^{\eta+2}[(\eta+1)(d-c)]$ and (3.2) by $(d-z)^{\eta+2}/[(\eta+1)(d-c)]$ and then adding we get the desired result. □

Remark 3.2. Lemma 3.1 shrinks to Lemma 2.1 by setting $\eta=1$.

Theorem 3.3. *If the hypothesis* $\mathbf{H_5}$ *is true, then inequality*

$$\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^\eta_{c^+}\phi(z) + J^\eta_{d^-}\phi(z)\right]\right| \leq \frac{K[(z-c)^{\eta+2} + (d-z)^{\eta+2}]}{2(\eta+1)(d-c)}\left[\pi - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{\Gamma(\eta+2)}\right]$$

holds for all $z \in [c,d]$.

Proof. It follows from Lemma 3.1 and the MT-convexity of $|\phi''|$ that

$$\begin{aligned}
&\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right| \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\left[\int_0^1 |1-s^{\eta+1}||\phi''(sc+(1-s)z)|ds\right] \\
&\quad + \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\left[\int_0^1 |1-s^{\eta+1}||\phi''(sd+(1-s)z)|ds\right] \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)| + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|\right]ds \\
&\quad + \frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)\left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(d)| + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|\right]ds \\
&\leq \frac{K(z-c)^{\eta+2}}{2(\eta+1)(d-c)}\int_0^1 (1-s^{\eta+1})\left(s^{1/2}(1-s)^{-1/2} + s^{-1/2}(1-s)^{1/2}\right)ds \\
&\quad + \frac{K(d-z)^{\eta+2}}{2(\eta+1)(d-c)}\int_0^1 (1-s^{\eta+1})\left(s^{1/2}(1-s)^{-1/2} + s^{-1/2}(1-s)^{1/2}\right)ds \\
&= \frac{K[(z-c)^{\eta+2} + (d-z)^{\eta+2}]}{2(\eta+1)(d-c)}\int_0^1 (1-s^{\eta+1})\left(s^{1/2}(1-s)^{-1/2} + s^{-1/2}(1-s)^{1/2}\right)ds \\
&= \frac{K[(z-c)^{\eta+2} + (d-z)^{\eta+2}]}{2(\eta+1)(d-c)}\left[\pi - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{\Gamma(\eta+2)}\right].
\end{aligned}$$

□

Remark 3.4. Let $z = (c+d)/2$. Then Theorem 3.3 leads to

$$\begin{aligned}
&\left|\left(\frac{d-c}{2}\right)^{\eta-1}\frac{\phi(c)+\phi(d)}{2} - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi\left(\frac{c+d}{2}\right) + J^{\eta}_{d^-}\phi\left(\frac{c+d}{2}\right)\right]\right| \\
&\quad\leq \frac{K}{2(\eta+1)}\left(\frac{d-c}{2}\right)^{\eta+1}\left[\pi - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{\Gamma(\eta+2)}\right].
\end{aligned}$$

Remark 3.5. By putting $\eta = 1$ in Theorem 3.3, we obtain the inequality given in Theorem 2.2.

Theorem 3.6. *Suppose that the hypothesis* $\mathbf{H_6}$ *holds and* $p, q > 1$ *such that* $p^{-1} = 1 - q^{-1}$*, then*

$$\begin{aligned}
&\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right| \\
&\quad\leq \frac{K[(z-c)^{\eta+2} + (d-z)^{\eta+2}]}{(\eta+1)(d-c)}\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1+p)\Gamma(1/(\eta+1))}{(\eta+1)\Gamma(1+p+1/(\eta+1))}\right)^{1/p}
\end{aligned}$$

for all $z \in [c, d]$.

Proof. As in Theorem 3.3, making use of Lemma 3.1 and the Hölder inequality again, we have

$$\begin{aligned}
&\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right| \\
&\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 |1-s^{\eta+1}||\phi''(sc+(1-s)z)|ds \\
&\quad + \frac{(d-z)^{\eta+2}}{(\eta+2)(d-c)}\int_0^1 |1-s^{\eta+1}||\phi''(sd+(1-s)z)|ds
\end{aligned}$$

$$\leq \frac{(z-a)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 \left(1-s^{\eta+1}\right)^p ds\right)^{1/p}\left(\int_0^1 |\phi''(sc+(1-s)z)|^q ds\right)^{1/q}$$
$$+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 \left(1-s^{\eta+1}\right)^p ds\right)^{1/p}\left(\int_0^1 |\phi''(sd+(1-s)z)|^q ds\right)^{1/q}.$$

It follows from the MT-convexity of $|\phi''|^q$ and $|\phi''(z)| \leq K$ that

$$\int_0^1 |\phi''(sc+(1-s)z)|^q ds \leq \int_0^1 \left[\frac{\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)|^q + \frac{\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds = \frac{\pi}{4}\left[|\phi''(c)|^q + |\phi''(z)|^q\right] \leq \frac{\pi}{2}K^q.$$

Similarly,

$$\int_0^1 |\phi''(sd+(1-s)z)|^q ds \leq \frac{\pi}{2}K^q.$$

Note that

$$\int_0^1 \left(1-s^{\eta+1}\right)^p ds = \frac{\int_0^1 (1-s)^p s^{1/(\eta+1)-1} ds}{\eta+1} = \frac{\Gamma(1+p)\Gamma[1/(\eta+1)]}{(\eta+1)\Gamma[1+p+1/(\eta+1)]}.$$

Therefore, Theorem 3.6 follows easily from the above inequalities and identity. □

Remark 3.7. Let$z = (c+d)/2$ in Theorem 3.6. Then we have

$$\left|\left(\frac{d-c}{2}\right)^{\eta-1}\frac{\phi(c)+\phi(d)}{2} - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi\left(\frac{c+d}{2}\right) + J^{\eta}_{d^-}\phi\left(\frac{a+d}{2}\right)\right]\right|$$
$$\leq \frac{K}{\eta+1}\left(\frac{d-c}{2}\right)^{\eta+1}\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1+p)\Gamma(1/(\eta+1))}{(\eta+1)\Gamma(1+p+1/(\eta+1))}\right)^{1/p}.$$

Remark 3.8. Taking $\eta = 1$ in Theorem 3.6, then we get the inequality in Theorem 2.4.

Theorem 3.9. *If the hypothesis* $\mathbf{H_6}$ *is true, then the inequality*

$$\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{(d-c)}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right|$$
$$\leq K\left(\frac{\eta+1}{\eta+2}\right)^{1-1/q}\left(\frac{\pi}{2} - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{2\Gamma(\eta+2)}\right)^{1/q}\left[\frac{(z-c)^{\eta+2}+(d-z)^{\eta+2}}{(\eta+1)(d-c)}\right]$$

is valid for each $z \in [c,d]$.

Proof. Making use of the Hölder inequality and Lemma 3.1, we have

$$\left|\tilde{\Delta}(z) - \frac{\Gamma(\eta+1)}{d-c}\left[J^{\eta}_{c^+}\phi(z) + J^{\eta}_{d^-}\phi(z)\right]\right|$$
$$\leq \frac{(z-c)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sc+(1-s)z)|ds$$
$$+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sd+(1-s)z)|ds$$
$$\leq \frac{(z-a)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 (1-s^{\eta+1})ds\right)^{1-1/q}\left(\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sc+(1-s)z)|^q ds\right)^{1/q}$$
$$+\frac{(d-z)^{\eta+2}}{(\eta+1)(d-c)}\left(\int_0^1 (1-s^{\eta+1})ds\right)^{1-1/q}\left(\int_0^1 \left(1-s^{\eta+1}\right)|\phi''(sd+(1-s)z)|^q ds\right)^{1/q}.$$

It follows from the MT-convexity of $|\phi''|^q$ and $|\phi''(z)| \le K$ on $z \in [c,d]$ that

$$\begin{aligned}\int_0^1 (1-s^{\eta+1})|\phi''(sc+(1-s)z)|^q ds &\le \int_0^1 \left[\frac{(1-s^{\eta+1})\sqrt{s}}{2\sqrt{1-s}}|\phi''(c)|^q + \frac{(1-s^{\eta+1})\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds\\ &\le \frac{K^q}{2}\int_0^1 (1-s^{\eta+1})\left(s^{1/2}(1-s)^{-1/2}+s^{-1/2}(1-s)^{1/2}\right) ds\\ &= \frac{K^q}{2}\left[\pi - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{\Gamma(\eta+2)}\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^1 (1-s^{\eta+1})|\phi''(sd+(1-s)z)|^q ds &\le \int_0^1 \left[\frac{(1-s^{\eta+1})\sqrt{s}}{2\sqrt{1-s}}|\phi''(d)|^q + \frac{(1-s^{\eta+1})\sqrt{1-s}}{2\sqrt{s}}|\phi''(z)|^q\right] ds\\ &\le \frac{K^q}{2}\int_0^1 (1-s^{\eta+1})\left(s^{1/2}(1-s)^{-1/2}+s^{-1/2}(1-s)^{1/2}\right) ds\\ &= \frac{K^q}{2}\left[\pi - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{\Gamma(\eta+2)}\right].\end{aligned}$$

Note that

$$\left(\int_0^1 (1-s^{\eta+1})ds\right)^{1-1/q} = \left(\frac{\eta+1}{\eta+2}\right)^{1-1/q}.$$

Therefore, Theorem 3.9 follows easily from the above inequalities and identity. □

Remark 3.10. Let $z=(c+d)/2$. Then Theorem 3.9 leads to

$$\left|\frac{\phi(c)+\phi(d)}{2} - \frac{1}{d-c}\int_c^d \phi(u)du\right| \le \frac{K}{\eta+1}\left(\frac{\pi}{2} - \frac{\Gamma(\eta+3/2)\Gamma(1/2)}{2\Gamma(\eta+2)}\right)^{1/q}\left(\frac{\eta+1}{\eta+2}\right)^{1-1/q}\left(\frac{d-c}{2}\right)^{\eta+1}.$$

Remark 3.11. In Theorem 3.9, if we put $\eta = 1$, then we get Theorem 2.6.

4. Applications to special means

In this section we present applications of our main results obtained in Section 2 to the following special means.

(1) The arithmetic mean

$$A = A(c,d) = \frac{c+d}{2} \quad (c,d>0);$$

(2) The logarithmic mean

$$L(c,d) = \frac{d-c}{\log d - \log c} \quad (c \ne d, c,d>0);$$

(3) The generalized logarithmic mean

$$L_n(c,d) = \left[\frac{d^{n+1}-c^{n+1}}{(d-c)(n+1)}\right]^{1/n} \quad (n\in\mathbb{Z}\setminus\{-1,0\},\ c,d>0, c\ne d).$$

The following two propositions are valid in the light of the above results.

Proposition 4.1. *Let $c,d\in\mathbb{R}^+$ with $c<d$ and $n\ge 3$ be a positive integer. Then the inequalities*

$$|A(c^n,d^n) - L_n^n(c,d)| \le \frac{5K\pi(d-c)^2}{128},$$

$$|A(c^n,d^n)-L_n^n(c,d)|\leq K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(p+1)\Gamma(1/2)}{2\Gamma(p+3/2)}\right)^{1/p}(d-c)^2$$

and

$$|A(c^n,d^n)-L_n^n(c,d)|\leq K\left(\frac{5\pi}{16}\right)^{1/q}\left(\frac{2}{3}\right)^{1-1/q}\frac{(d-c)^2}{8}$$

hold for all $q\geq 1$

Proof. Let $z>0$ and $\phi(z)=z^n$. Then the desired results follow directly from Remarks 2.3, 2.5, and 2.7. □

Proposition 4.2. *Let* $c,d\in\mathbb{R}^+$ *with* $c<d$*. Then the inequalities*

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)|\leq\frac{5K\pi(d-c)^2}{128},$$

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)|\leq K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}(d-c)^2$$

and

$$|A(c^{-1},d^{-1})-L^{-1}(c,d)|\leq K\left(\frac{5\pi}{16}\right)^{1/q}\left(\frac{2}{3}\right)^{1-1/q}\frac{(d-c)^2}{8}$$

are valid for all $q\geq 1$*.*

Proof. Let $z>0$ and $\phi(z)=1/z$. Then the desired results follow directly from Remarks 2.3, 2.5, and 2.7. □

Remark 4.3. In the above propositions we have used the fact that every positive convex function is a MT-convex function [28].

5. Applications to error estimates for trapezoidal formula

Let $p=\{z_1,z_2,..,z_n\}$, $z_i\in[c,d]$, $i=\overline{1,n}$ with $c=z_0$, $z_n=d$ and $z_i<z_{i+1}$ for $i=\overline{1,n}$. Then the well known quadrature formula for the partition p is

$$\int_c^d\phi(z)dz=\tau(\phi,p)+e(\phi,p),$$

where

$$\tau(\phi,p)=\sum_{i=0}^{n-1}\frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)$$

denotes the trapezoidal formula and $e(\phi,p)$ represents the error approximation associated to it.

Proposition 5.1. *Let the hypothesis* $\mathbf{H_3}$ *holds and* $p,q>1$ *such that* $p^{-1}=1-q^{-1}$*, then we have*

$$|e(\phi,p)|\leq K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^3.$$

Proof. We consider the subintervals $[z_i,z_{i+1}]$ $(i=\overline{0,n-1})$ of the partition p and make use of Remark 2.5, we have

$$\left|\frac{\phi(z_i)+\phi(z_{i+1})}{2}-\frac{\int_{z_i}^{z_{i+1}}\phi(z)dz}{z_{i+1}-z_i}\right|\leq K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}(z_{i+1}-z_i)^2,$$

$$\begin{aligned}\left|\int_c^d \phi(z)dz-\tau(\phi,p)\right| &= \left|\sum_{i=0}^{n-1}\left\{\int_{z_i}^{z_{i+1}}\phi(z)dz-\frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)\right\}\right| \\ &\le \sum_{i=0}^{n-1}\left|\left\{\int_{z_i}^{z_{i+1}}\phi(z)dz-\frac{\phi(z_i)+\phi(z_{i+1})}{2}(z_{i+1}-z_i)\right\}\right| \\ &\le K\left(\frac{\pi}{2}\right)^{1/q}\left(\frac{\Gamma(1/2)\Gamma(p+1)}{2\Gamma(p+3/2)}\right)^{1/p}\sum_{i=0}^{n-1}(z_{i+1}-z_i)^3.\end{aligned}$$

□

Proposition 5.2. *Under the hypothesis* $\mathbf{H_3}$*, the following inequality holds*

$$|e(\phi,p)|\le K\left(\frac{5\pi}{16}\right)^{1/q}\left(\frac{2}{3}\right)^{1-1/q}\sum_{i=0}^{n-1}\frac{(z_{i+1}-z_i)^3}{8}.$$

Proof. Proceeding on the same lines like in the Proposition 5.1, we can prove the inequality in Proposition 5.2 by making use of Remark 2.7. □

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