Filomat 32:18 (2018), 6465–6474
https://doi.org/10.2298/FIL1818465S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalizations of the Aluthge Transform of Operators

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Abstract. Let A be an operator with the polar decomposition $A = U|A|$. The Aluthge transform of the operator A, denoted by $\tilde{A}$, is defined as $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. In this paper, first we generalize the definition of Aluthge transform for non-negative continuous functions f, g such that $f(x)g(x) = x$ $(x \geq 0)$. Then, by using this definition, we get some numerical radius inequalities. Among other inequalities, it is shown that if A is bounded linear operator on a complex Hilbert space $\mathbb{H}$, then

$$h(w(A)) \leq \frac{1}{4}\left\|h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right)\right\| + \frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right),$$

where f, g are non-negative continuous functions such that $f(x)g(x) = x$ $(x \geq 0)$, h is a non-negative and non-decreasing convex function on $[0, \infty)$ and $\tilde{A}_{f,g} = f(|A|)Ug(|A|)$.

1. Introduction and preliminaries

2. Introduction

Let $\mathbb{B}(\mathbb{H})$ denotes the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. In the case when $\dim\mathbb{H} = n$, we identify $\mathbb{B}(\mathbb{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. For an operator $A \in \mathbb{B}(\mathbb{H})$, let $A = U|A|$ (U is a partial isometry with $\ker U = \text{range}|A|^{\perp}$) be the polar decomposition of A. The Aluthge transform of the operator A, denoted by $\tilde{A}$, is defined as $\tilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. In [7, 21], a more general notion called t-Aluthge transform has been introduced which has later been studied. This is defined for any $0 < t \leq 1$ by $\tilde{A}_t = |A|^t U|A|^{1-t}$. Clearly, for $t = \frac{1}{2}$ we obtain the usual Aluthge transform. For the case $t = 1$, the operator $\tilde{A}_1 = |A|U$ is called the Duggal transform of $A \in \mathbb{B}(\mathbb{H})$. For $A \in \mathbb{B}(\mathbb{H})$, we generalize the Aluthge transform of the operator A to the form

$$\tilde{A}_{f,g} = f(|A|)Ug(|A|),$$

in which f, g are non-negative continuous functions such that $f(x)g(x) = x$ $(x \geq 0)$. The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

2010 *Mathematics Subject Classification*. Primary 47A12, Secondary 47A63, 47A30
Keywords. Aluthge transform, Numerical radius, Operator matrices, Polar decomposition.
Received: 09 July 2017; Accepted: 05 October 2017
Communicated by M.S. Moslehian
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It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathbb{H})$, $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$; see [8]. Let $r(\cdot)$ denote the spectral radius. It is well known that for every operator $A \in \mathbb{B}(\mathbb{H})$, we have $r(A) \leq w(A)$. An important inequality for $\omega(A)$ is the power inequality stating that $\omega(A^n) \leq \omega(A)^n$ $(n = 1, 2, \cdots)$. For further information about the numerical radius we refer the reader to [10–12] and references therein. The quantity $w(A)$ is useful in studying perturbation, convergence and approximation problems as well as integrative methods, etc. For more information see [3, 6, 9, 13–15, 17].

Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. The operator matrices $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ are called the diagonal and off-diagonal parts of the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, respectively.

In [16], it has been shown that if A is an operator in $\mathbb{B}(\mathbb{H})$, then

$$w(A) \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right). \tag{1}$$

Several refinements and generalizations of inequality (1) have been given; see [1, 4, 5, 21–24]. Yamazaki [22] showed that for $A \in \mathbb{B}(\mathbb{H})$ and $t \in [0, 1]$ we have

$$w(A) \leq \frac{1}{2}\left(\|A\| + w(\tilde{A}_t)\right). \tag{2}$$

Davidson and Power [7] proved that if A and B are positive operators in $\mathbb{B}(\mathbb{H})$, then

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}. \tag{3}$$

Inequality (3) has been generalized in [2, 20] and improved in [18, 19]. In [20], the author extended this inequality to the form

$$\|A + B^*\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2}\left(\left\||A|^t|B^*|^{1-t}\right\| + \left\||A^*|^{1-t}|B|^t\right\|\right), \tag{4}$$

in which $A, B \in \mathbb{B}(\mathbb{H})$ and $t \in [0, 1]$.

In this paper, by applying the generalized Aluthge transform of operators, we establish some inequalities involving the numerical radius. In particular, we extend inequalities (2) and (4) for two non-negative continuous functions. We also show some upper bounds for the numerical radius of 2×2 operator matrices.

3. main results

To prove our numerical radius inequalities, we need several known lemmas.

Lemma 3.1. *[1, Theorem 2.2] Let $X, Y, S, T \in \mathbb{B}(\mathbb{H})$. Then*

$$r(XY + ST) \leq \frac{1}{2}\left(w(YX) + w(TS)\right) + \frac{1}{2}\sqrt{\left(w(YX) - w(TS)\right)^2 + 4\|YS\|\|TX\|}.$$

Lemma 3.2. *[16, 22] Let $A \in \mathbb{B}(\mathbb{H})$. Then*

(a) $w(A) = \max_{\theta \in \mathbb{R}} \left\|Re\left(e^{i\theta}A\right)\right\|$.

(b) $w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = \frac{1}{2}\|A\|$.

Polarization identity: For all $x, y \in \mathbb{H}$, we have

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3}\left\|x + i^k y\right\|^2 i^k.$$

Now, we are ready to present our first result. The following theorem shows a generalization of inequality (2).

Theorem 3.3. *Let $A \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$. Then, for all non-negative and non-decreasing convex function h on $[0, \infty)$, we have*

$$h\left(w(A)\right) \leq \frac{1}{4}\left\|h\left(g^2\left(|A|\right)\right) + h\left(f^2\left(|A|\right)\right)\right\| + \frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right).$$

Proof. Let x be any unit vector. Then

$$\begin{aligned}
\mathrm{Re}\left\langle e^{i\theta}Ax, x\right\rangle &= \mathrm{Re}\left\langle e^{i\theta}U\left|A\right|x, x\right\rangle \\
&= \mathrm{Re}\left\langle e^{i\theta}Ug\left(|A|\right)f\left(|A|\right)x, x\right\rangle \\
&= \mathrm{Re}\left\langle e^{i\theta}f\left(|A|\right)x, g\left(|A|\right)U^*x\right\rangle \\
&= \frac{1}{4}\left\|\left(e^{i\theta}f\left(|A|\right) + g\left(|A|\right)U^*\right)x\right\|^2 - \frac{1}{4}\left\|\left(e^{i\theta}f\left(|A|\right) - g\left(|A|\right)U^*\right)x\right\|^2 \\
&\qquad\qquad \text{(by polarization identity)} \\
&\leq \frac{1}{4}\left\|\left(e^{i\theta}f\left(|A|\right) + g\left(|A|\right)U^*\right)x\right\|^2 \\
&\leq \frac{1}{4}\left\|\left(e^{i\theta}f\left(|A|\right) + g\left(|A|\right)U^*\right)\right\|^2 \\
&= \frac{1}{4}\left\|\left(e^{i\theta}f\left(|A|\right) + g\left(|A|\right)U^*\right)\left(e^{-i\theta}f\left(|A|\right) + Ug\left(|A|\right)\right)\right\| \\
&= \frac{1}{4}\left\|g^2\left(|A|\right) + f^2\left(|A|\right) + e^{i\theta}\tilde{A}_{f,g} + e^{-i\theta}\left(\tilde{A}_{f,g}\right)^*\right\| \\
&\leq \frac{1}{4}\left\|g^2\left(|A|\right) + f^2\left(|A|\right)\right\| + \frac{1}{4}\left\|e^{i\theta}\tilde{A}_{f,g} + e^{-i\theta}\left(\tilde{A}_{f,g}\right)^*\right\| \\
&= \frac{1}{4}\left\|g^2\left(|A|\right) + f^2\left(|A|\right)\right\| + \frac{1}{2}\left\|\mathrm{Re}\left(e^{i\theta}\tilde{A}_{f,g}\right)\right\| \\
&\leq \frac{1}{4}\left\|g^2\left(|A|\right) + f^2\left(|A|\right)\right\| + \frac{1}{2}w\left(\tilde{A}_{f,g}\right).
\end{aligned}$$

Now, taking the supremum over all unit vectors $x \in \mathbb{H}$ and applying Lemma 3.2 in the above inequality produces

$$w\left(A\right) \leq \frac{1}{4}\left\|g^2\left(|A|\right) + f^2\left(|A|\right)\right\| + \frac{1}{2}w\left(\tilde{A}_{f,g}\right).$$

Therefore,

$$\begin{aligned} h(w(A)) &\leq h\left(\frac{1}{4}\left\|g^2(|A|)+f^2(|A|)\right\|+\frac{1}{2}w\left(\tilde{A}_{f,g}\right)\right) \\ &= h\left(\frac{1}{2}\left\|\frac{g^2(|A|)+f^2(|A|)}{2}\right\|+\frac{1}{2}w\left(\tilde{A}_{f,g}\right)\right) \\ &\leq \frac{1}{2}h\left(\left\|\frac{g^2(|A|)+f^2(|A|)}{2}\right\|\right)+\frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) \\ &\qquad\text{(by the convexity of } h) \\ &= \frac{1}{2}\left\|h\left(\frac{g^2(|A|)+f^2(|A|)}{2}\right)\right\|+\frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) \\ &\leq \frac{1}{4}\left\|h\left(g^2(|A|)\right)+h\left(f^2(|A|)\right)\right\|+\frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) \\ &\qquad\text{(by the convexity of } h). \end{aligned}$$

□

Theorem 3.3 includes some special cases as follows.

Corollary 3.4. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for all non-negative and non-decreasing convex function h on $[0,\infty)$ and all $t \in [0,1]$, we have*

$$h(w(A)) \leq \frac{1}{4}\left\|h\left(|A|^{2t}\right)+h\left(|A|^{2(1-t)}\right)\right\|+\frac{1}{2}h\left(w\left(\tilde{A}_t\right)\right). \tag{5}$$

Corollary 3.5. *Let $A \in \mathbb{B}(\mathbb{H})$. Then, for all $t \in [0,1]$ and $s \geq 1$, we have*

$$w^s(A) \leq \frac{1}{4}\| \, |A|^{2ts}+|A|^{2(1-t)s} \, \| + \frac{1}{2}w^s\left(\tilde{A}_t\right).$$

In particular,

$$w^s(A) \leq \frac{1}{2}\left(\|A\|^s + w^s\left(\tilde{A}\right)\right).$$

Proof. The first inequality follows from inequality (5) for the function $h(x) = x^s$ $(s \geq 1)$. For the particular case, it is enough to put $t = \frac{1}{2}$. □

Theorem 3.3 gives the next result for the off-diagonal operator matrix $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$.

Theorem 3.6. *Let $A, B \in \mathbb{B}(\mathbb{H})$, f, g be two non-negative continuous functions on $[0,\infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$ and $s \geq 1$. Then*

$$\begin{aligned} w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{4}\max\left(\left\|g^{2s}(|A|)+f^{2s}(|A|)\right\|, \left\|g^{2s}(|B|)+f^{2s}(|B|)\right\|\right) \\ &\quad + \frac{1}{4}\left(\|f(|B|)g(|A^*|)\|^s + \|f(|A|)g(|B^*|)\|^s\right). \end{aligned}$$

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of A and B, respectively, and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. It follows from the polar decomposition of $T = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} |B| & 0 \\ 0 & |A| \end{bmatrix}$ that

$$\begin{aligned}\tilde{T}_{f,g} &= f(|T|)\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}g(|T|) \\ &= \begin{bmatrix} f(|B|) & 0 \\ 0 & f(|A|) \end{bmatrix}\begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}\begin{bmatrix} g(|B|) & 0 \\ 0 & g(|A|) \end{bmatrix} \\ &= \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}.\end{aligned}$$

Using $|A^*|^2 = AA^* = U|A|^2U^*$ and $|B^*|^2 = BB^* = V|B|^2V^*$ we have $g(|A|) = U^*g(|A^*|)U$ and $g(|B|) = V^*g(|B^*|)V$ for every non-negative continuous function g on $[0, \infty)$. Therefore,

$$\begin{aligned}w\left(\tilde{T}_{f,g}\right) &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & 0 \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}\right) \\ &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(W^*\begin{bmatrix} 0 & f(|A|)Vg(|B|) \\ 0 & 0 \end{bmatrix}W\right) \\ &= w\left(\begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & f(|A|)Vg(|B|) \\ 0 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\|f(|B|)Ug(|A|)\| + \frac{1}{2}\|f(|A|)Vg(|B|)\| \\ &\qquad \text{(by Lemma 3.2(b))} \\ &= \frac{1}{2}\|f(|B|)UU^*g(|A^*|)U\| + \frac{1}{2}\|f(|A|)VV^*g(|B^*|)V\| \\ &\leq \frac{1}{2}\|f(|B|)g(|A^*|)\| + \frac{1}{2}\|f(|A|)g(|B^*|)\|, \qquad (6)\end{aligned}$$

where $W = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is unitary. Applying Theorem 3.3 and inequality (6), we have

$$\begin{aligned}w^s(T) &\leq \frac{1}{4}\left\|g^{2s}(|T|) + f^{2s}(|T|)\right\| + \frac{1}{2}\left(w^s\left(\tilde{T}_{f,g}\right)\right) \\ &\leq \frac{1}{4}\max\left(\left\|g^{2s}(|A|) + f^{2s}(|A|)\right\|, \left\|g^{2s}(|B|) + f^{2s}(|B|)\right\|\right) \\ &\quad + \frac{1}{2}\left[\frac{1}{2}\left(\|f(|B|)g(|A^*|)\| + \|f(|A|)g(|B^*|)\|\right)\right]^s \\ &\leq \frac{1}{4}\max\left(\left\|g^{2s}(|A|) + f^{2s}(|A|)\right\|, \left\|g^{2s}(|B|) + f^{2s}(|B|)\right\|\right) \\ &\quad + \frac{1}{4}\|f(|B|)g(|A^*|)\|^s + \frac{1}{4}\|f(|A|)g(|B^*|)\|^s \\ &\qquad \text{(by the convexity } h(x) = x^s\text{)}.\end{aligned}$$

□

Corollary 3.7. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then, for all $t \in [0,1]$ and $s \geq 1$, we have*

$$\begin{aligned} w^{\frac{s}{2}}(AB) &\leq \frac{1}{4}\max\left(\left\||A|^{2ts}+|A|^{2(1-t)s}\right\|, \left\||B|^{2ts}+|B|^{2(1-t)s}\right\|\right) \\ &+\frac{1}{4}\left(\left\||A|^{t}|B^*|^{1-t}\right\|^{s}+\left\||B|^{t}|A^*|^{1-t}\right\|^{s}\right). \end{aligned}$$

Proof. Applying the power inequality of the numerical radius ($w(A^n) \leq w^n(A)$ [19]), we have

$$\begin{aligned} w^{\frac{s}{2}}(AB) &\leq \max\left(w^{\frac{s}{2}}(AB), w^{\frac{s}{2}}(BA)\right) \\ &= w^{\frac{s}{2}}\left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}\right) \\ &= w^{\frac{s}{2}}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^2\right) \\ &\leq w^{s}\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{4}\max\left(\left\||A|^{2ts}+|A|^{2(1-t)s}\right\|, \left\||B|^{2ts}+|B|^{2(1-t)s}\right\|\right) \\ &+\frac{1}{4}\left(\left\||A|^{t}|B^*|^{1-t}\right\|^{s}+\left\||B|^{t}|A^*|^{1-t}\right\|^{s}\right) \\ &\qquad \text{(by Theorem 3.6)}. \end{aligned}$$

□

Corollary 3.8. *Let $A, B \in \mathbb{B}(\mathbb{H})$ be positive operators. Then, for all $t \in [0,1]$ and $s \geq 1$, we have*

$$\begin{aligned} \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^{s} &\leq \frac{1}{4}\max\left(\left\|A^{ts}+A^{(1-t)s}\right\|, \left\|B^{ts}+B^{2(1-t)s}\right\|\right) \\ &+\frac{1}{4}\left(\left\|A^{t}B^{1-t}\right\|^{s}+\left\|B^{t}A^{1-t}\right\|^{s}\right). \end{aligned}$$

Proof. Since the spectral radius of any operator is dominated by its numerical radius, then $r^{\frac{1}{2}}(AB) \leq w^{\frac{1}{2}}(AB)$. Applying a commutativity property of the spectral radius, we get

$$\begin{aligned} r^{\frac{s}{2}}(AB) &= r^{\frac{s}{2}}\left(A^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}\right) \\ &= r^{\frac{s}{2}}\left(A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}\right) \\ &= r^{\frac{s}{2}}\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\right)^{*}\right) \\ &= \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\right)^{*}\right\|^{\frac{s}{2}} \\ &= \left\|A^{\frac{1}{2}}B^{\frac{1}{2}}\right\|^{s}. \end{aligned} \tag{7}$$

Now, the result follows from Corollary 3.7. □

An important special case of Theorem 3.6, which generalizes inequality (4) can be stated as follows.

Corollary 3.9. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and $s \geq 1$. Then*

$$\begin{aligned} \|A+B\|^{s} &\leq \frac{1}{2^{2-s}}\max\left(\left\||A|^{2ts}+|A|^{2(1-t)s}\right\|, \left\||B^*|^{2ts}+|B^*|^{2(1-t)s}\right\|\right) \\ &+\frac{1}{2^{2-s}}\left(\left\||A|^{t}|B|^{1-t}\right\|^{s}+\left\||B^*|^{t}|A^*|^{1-t}\right\|^{s}\right). \end{aligned}$$

In particular, if A and B are normal, then

$$\|A+B\|^s \leq \frac{1}{2^{1-s}} \max\left(\|A\|^s, \|B\|^s\right) + \frac{1}{2^{1-s}} \|AB\|^{\frac{s}{2}}.$$

Proof. Applying Lemma 3.2 and Theorem 3.3, we have

$$\begin{aligned}
\|A+B^*\|^s &= \|T+T^*\|^s \\
&\leq 2^s \max_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}T\right)\right\|^s \\
&= 2^s w^s(T) \\
&\leq \frac{2^s}{4}\max\left(\left\||A|^{2ts}+|A|^{2(1-t)s}\right\|, \left\||B|^{2ts}+|B|^{2(1-t)s}\right\|\right) \\
&\quad + \frac{2^s}{4}\left(\left\||A|^t\,|B^*|^{1-t}\right\|^s + \left\||B|^t\,|A^*|^{1-t}\right\|^s\right) \\
&\qquad\qquad \text{(by Theorem 3.6)},
\end{aligned}$$

where $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$, $f(x) = x^t$, and $g(x) = x^{1-t}$. Now, the desired result follows by replacing B by B^*. For the particular case $t = \frac{1}{2}$. If A and B are normal, then $|B^*| = |B|$ and $|A^*| = |A|$. Applying equality (7) for the operators $|A|^{\frac{1}{2}}$ and $|B|^{\frac{1}{2}}$, we have

$$\begin{aligned}
\left\||A|^{\frac{1}{2}}\,|B|^{\frac{1}{2}}\right\|^s &= r^{\frac{s}{2}}\left(|A|\,|B|\right) \\
&\leq \||A|\,|B|\|^{\frac{s}{2}} \\
&= \|U^*AB^*V\|^{\frac{s}{2}} \\
&= \|AB^*\|^{\frac{s}{2}},
\end{aligned}$$

where $A = U|A|$ and $B = V|B|$ are the polar decompositions of the operators A and B. This completes the proof of the corollary. □

In the next result, we show another generalization of inequality (2).

Theorem 3.10. *Let $A \in \mathbb{B}(\mathbb{H})$ and f, g, h be non-negative and non-decreasing continuous functions on $[0,\infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$ and h is convex. Then*

$$h\left(w(A)\right) \leq \frac{1}{2}\Big(h\left(w\left(\tilde{A}_{f,g}\right)\right) + \|h(|A|)\|\Big).$$

Proof. Let $A = U|A|$ be the polar decomposition of A. Then for every $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
\|\mathrm{Re}\left(e^{i\theta}A\right)\| &= r\left(\mathrm{Re}\left(e^{i\theta}A\right)\right) \\
&= \frac{1}{2}r\left(e^{i\theta}A + e^{-i\theta}A^*\right) \\
&= \frac{1}{2}r\left(e^{i\theta}U|A| + e^{-i\theta}|A|U^*\right) \\
&= \frac{1}{2}r\left(e^{i\theta}Ug(|A|)f(|A|) + e^{-i\theta}f(|A|)g(|A|)U^*\right). \qquad (8)
\end{aligned}$$

Now, if we put $X = e^{i\theta}Ug(|A|)$, $Y = f(|A|)$, $S = e^{-i\theta}f(|A|)$ and $T = g(|A|)U^*$ in Lemma 3.1, then we get

$$\begin{aligned}
&r\Big(e^{i\theta}Ug(|A|)f(|A|) + e^{-i\theta}f(|A|)g(|A|)U^*\Big)\\
&\quad\le \frac{1}{2}\Big(w(f(|A|)Ug(|A|)) + w(g(|A|)U^*f(|A|))\Big)\\
&\qquad + \frac{1}{2}\sqrt{4\|e^{-i\theta}f(|A|)g(|A|)\|\|g(|A|)U^*e^{i\theta}Uf(|A|)\|}\\
&\qquad\qquad\text{(by Lemma 3.1)}\\
&\quad\le w(f(|A|)Ug(|A|)) + \sqrt{\|f(|A|)\|\|f(|A|)\|\|g(|A|)\|\|g(|A|)\|}\\
&\quad= w(f(|A|)Ug(|A|)) + \sqrt{f(\|A\|)g(\|A\|)g(\|A\|)f(\|A\|)}\\
&\quad= w(f(|A|)Ug(|A|)) + \sqrt{\|A\|\|A\|}\\
&\quad= w\left(\tilde{A}_{f,g}\right) + \|A\|. \qquad (9)
\end{aligned}$$

Note that, since $w(X) = w(X^*)$ $(X \in \mathbb{B}(\mathbb{H}))$, in the first inequality we have

$$w(YX) - w(TS) = w(f(|A|)Ug(|A|)) - w(g(|A|)U^*f(|A|)) = 0.$$

Using inequalities (8), (9) and Lemma 3.2 we get

$$\omega(A) = \max_{\theta\in\mathbb{R}} \left\|\mathrm{Re}\left(e^{i\theta}A\right)\right\| \le \frac{1}{2}\Big(w\left(\tilde{A}_{f,g}\right) + \|A\|\Big).$$

Hence

$$\begin{aligned}
h(w(A)) &\le h\left(\frac{1}{2}\left[w\left(\tilde{A}_{f,g}\right) + \|A\|\right]\right)\\
&\qquad\text{(by the monotonicity of } h)\\
&\le \frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{2}h(\|A\|)\\
&\qquad\text{(by the convexity of } h)\\
&= \frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{2}\|h(|A|)\|,
\end{aligned}$$

as required. □

Remark 3.11. *We can obtain Theorem 3.3 from Theorem 3.10, but we keep the proof for the readers. To see this, first note that by the hypotheses of Theorem 3.3 we have*

$$\begin{aligned}
h(|A|) &= h(g(|A|)f(|A|))\\
&\le h\left(\frac{g^2(|A|) + f^2(|A|)}{2}\right) \qquad \textit{(by the arithmetic-geometric inequality)}\\
&\le \frac{1}{2}\left(h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right)\right) \qquad \textit{(by the convexity of } h). \qquad (10)
\end{aligned}$$

Hence, using Theorem 3.10 and inequality (10) *we get*

$$\begin{aligned}
h(w(A)) &\le \frac{1}{2}\left[h\left(w\left(\tilde{A}_{f,g}\right)\right) + \|h(|A|)\|\right]\\
&\le \frac{1}{2}\left[h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{2}\left\|h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right)\right\|\right]\\
&= \frac{1}{2}h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{4}\left\|h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right)\right\|.
\end{aligned}$$

Remark 3.12. *For the special case $f(x) = x^t$ and $g(x) = x^{1-t}$ $(t \in [0,1])$, we obtain the inequality* (2)

$$w(A) \leq \frac{1}{2}\left(w\left(\tilde{A}_t\right) + \|A\|\right),$$

where $A \in \mathbb{B}(\mathbb{H})$.

Let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Using Theorem 3.10, we get the following result.

Corollary 3.13. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative and non-decreasing continuous functions such that $f(x)g(x) = x$ $(x \geq 0)$. Then*

$$2w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \max\{\|A\|^s, \|B\|^s\} + \frac{1}{2}\Big(\left\|f(|B|)g(|A^*|)\right\|^s + \left\|f(|A|)g(|B^*|)\right\|^s\Big),$$

where $s \geq 1$.

Proof. Using Theorem 3.10 and inequality (6), we have

$$\begin{aligned} 2w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \left\|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right\|^s + w^s\left(\tilde{T}_{f,g}\right) \\ &= \max\{\|A\|^s, \|B\|^s\} + \left(\frac{1}{2}\Big[\left\|f(|B|)g(|A^*|)\right\| + \|f(|A|)g(|B^*|)\|\Big]\right)^s \\ &\leq \max\{\|A\|^s, \|B\|^s\} + \frac{1}{2}\Big(\left\|f(|B|)g(|A^*|)\right\|^s + \|f(|A|)g(|B^*|)\|^s\Big) \end{aligned}$$

and the proof is complete. □

Using similar arguments to the proof of Corollary 3.9, we get the following result.

Corollary 3.14. *Let $A, B \in \mathbb{B}(\mathbb{H})$ and f, g be two non-negative and non-decreasing continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geq 0)$. Then*

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2}\Big(\left\|f(|B|)g(|A|)\right\| + \left\|f(|A^*|)g(|B^*|)\right\|\Big).$$

Acknowledgement. The second author would like to thank the Tusi Mathematical Research Group (TMRG).

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