Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 32 (1982), No. 4, 516–528

Persistent URL: http://dml.cz/dmlcz/101831

Czechoslovak Mathematical Journal, 32 (107) 1982, Praha

GENERALIZATIONS OF THE CLASSICAL CHEBYSHEV POLYNOMIALS TO POLYNOMIALS IN TWO VARIABLES*)

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(Received November 12, 1979)

1. INTRODUCTION

Classical Chebyshev polynomials in one variable of the first and second kind have many well-known important analytic and numerical properties, see e.g., Szegö [20], Rivlin [18], but there are also arithmetic and algebraic properties of these polynomials, perhaps not as well-known as the analytic ones, see e.g., Fried [6], Lausch and Nöbauer [12; Chapter 4], Rivlin [18; Chapter 4], Schur [19; pp. 422–453], Bang [1] and Rankin [16].

Compared with orthogonal polynomials in one variable not very much is known about orthogonal polynomials in several variables. A short summary of results on the subject prior to 1953 is contained in [5], an excellent survey of more recent results is given by Koornwinder [11].

In this paper we consider several generalizations of the onevariable case of Chebyshev polynomials to polynomials in two variables and also give some of their arithmetical properties. We shall not consider any analytic properties, such as differential operators or orthogonality, for these polynomials. Section 2. gives a brief summary of some properties of the Chebyshev polynomials in one variable and also refers to a variation of these polynomials due to Dickson [3], see also Schur [19, p. 446].

Section 3 continues the investigation of a class of orthogonal polynomials in two variables introduced by Koornwinder [10]. Koornwindei defines polynomials $P^{\alpha}_{m,n}(z, \bar{z})$, with $z = x + iy$, $\bar{z} = x - iy$, as eigenfunctions of a second order differential operator. For particular values of the parameters and with a suitable transformation of variables this operator is the Laplace-Beltrami operator (see [11]) on a compact Riemannian symmetric space. These polynomials provide an important example of complete orthogonal systems of functions in 2 variables which cannot be factorized as products of functions in one variable. They are orthogonal on a region bounded by a closed three-cusped algebraic curve of fourth degree which is

*) Research supported by the Australian Research Grants Committee, Grant No. B 7815210 I.

known as Steiner's hypocycloid, given by $\mu(z, \bar{z}) = -z^2\bar{z}^2 + 4z^3 + 4\bar{z}^3 - 18z\bar{z} + 27$. The weight function is $\mu(z, \bar{z})^\alpha$. The choice of variables as complex conjugates and the method of orthogonalization in [10] are motivated by the special cases $\alpha = \pm\frac{1}{2}$, the "Chebyshev cases", since in these cases the functions $P_{m,n}^{-1/2}(z, \bar{z})$ and $\mu(z, \bar{z})^{1/2} P_{m-1,n-1}^{1/2}(z, \bar{z})$ can be expressed as explicit trigonometric polynomials. For details see §§ 2 and 3 in [10]. These orthogonal systems are natural generalizations of the two kinds of Chebyshev polynomials. We consider these polynomials from the formal algebraic viewpoint as polynomials in the variables x, y (not necessarily complex conjugates) over an arbitrary field K. The most important examples are of course the cases $K = \mathbb{C}$ or $\mathbb{R}$ for analytic properties and $K = GF(q)$, the finite field of prime power order, for algebraic properties. We denote Chebyshev polynomials of the first kind in two variables by $P_{m,n}^{-1/2}(x, y)$; $m + n$ is the total degree of this polynomial. In the case $n = 0$, these polynomials have been studied by Lidl [14], [15], Kaiser-Lidl [8], by Lidl and Wells [13] and Lidl [15] also in the k-dimensional case. The main results of section 3 are the proofs of a generating function, a recurrence relation and an explicit expression for the polynomial $P_{m,n}^{-1/2}(x, y)$.

Chebyshev polynomials of the second kind in two variables, denoted by $P_{m,n}^{1/2}(x, y)$, are studied in Section 4. These polynomials have been introduced by Koornwinder [10] in the case $K = \mathbb{C}$ for complex conjugate variables. The definition given in this paper is also similar to the definition of a general class of polynomials $Z_{n_1,n_2,n_3}(x,y,z)$ in Koornwinder [11, p. 483]. Analytic properties of the polynomials $P_{m,0}^{1/2}(x, y)$ have been studied by Lidl [15]. Here we construct generating function, recurrence relation and explicit expressions for $P_{m,n}^{1/2}(x, y)$.

A different class of Chebyshev polynomials, some of them are orthogonal on the unit disc, is given in Section 5. These polynomials $D_{m,n}^{-1/2}(x, y)$ have been mentioned to the second author in a private communication [2]. A relation between $D_{m,n}^{-1/2}(x, y)$ and $P_{m,0}^{-1/2}(x, y)$ enables us to find properties for $D_{m,n}^{-1/2}(x, y)$.

Section 5 also contains further generalizations of the types of Chebyshev polynomials mentioned above, the approach is similar to the one-dimensional case due to Schur. We also mention some other approaches to generalize the Chebyshev polynomials, namely Hays [7] and Ricci [17]. Some of the results in [17] are also contained in [15]. Koornwinder [11] considers generalized Jacobi polynomials and thus also generates the Chebyshev polynomials as special cases. He studies polynomials which are orthogonal on a region bounded by two straight lines $1 - x + y = 0$, $1 + x + y = 0$ and a parabola $x^2 - 4y = 0$ touching these lines. These polynomials result from orthogonalisation of the sequence $1, x, y, x^2, xy, y^2, \ldots$. Koornwinder in [11] includes altogether seven different classes of orthogonal polynomials in two variables, which contain "Chebyshev cases".

In a forthcoming paper we will consider non-trivial generalizations of Chebyshev polynomials in k-dimensions (see [21] and [23]).

2. THE POLYNOMIALS IN ONE VARIABLE

We summarize some of the properties of classical Chebyshev polynomials in one variable and present the polynomials in a form which is suitable for generalizations to two dimensions. The polynomial $T_n(x)$ over $\mathbb{C}$ of degree n defined by

$$T_n(x) = \cos n\theta\,, \quad x = \cos\theta\,, \tag{2.1}$$

where n is a nonnegative integer, is called the *Chebyshev polynomial of degree n of the first kind*. The polynomial $U_n(x)$ over $\mathbb{C}$ of degree n defined by

$$U_n(x) = (\sin(n+1)\theta)/\sin\theta\,, \quad x = \cos\theta\,, \tag{2.2}$$

is called the *Chebyshev polynomial of degree n of the second kind*. Now we define polynomials $P_n^{-1/2}(x)$ and $P_n^{1/2}(x)$ over a field K:

$$P_n^{-1/2}(x) = u^n + u^{-n}\,, \quad x = u + u^{-1}\,, \quad n \in \mathbb{Z}\,, \tag{2.3}$$

$$P_n^{1/2}(x) = (u - u^{-1})^{-1}\,(u^{n+1} - u^{-(n+1)})\,, \quad x = u + u^{-1}\,, \quad n \in \mathbb{Z}\,, \tag{2.4}$$

where u is an element of a suitable extension field of K. These polynomials have been studied in [4], [12, p. 209], [15], [19], they are closely related to the classical Chebyshev polynomials in the case $K = \mathbb{C}$, $n \geqq 0$, because

$$P_n^{-1/2}(2\cos\theta) = 2T_n(\cos\theta) \quad \text{and} \quad P_n^{1/2}(2\cos\theta) = U_n(\cos\theta)$$

by setting $u = e^{i\theta}$. Therefore we have

$$P_n^{-1/2}(2x) = 2T_n(x) \quad \text{and} \quad P_n^{1/2}(2x) = U_n(x)\,. \tag{2.5}$$

We list a few well-known properties for the Chebyshev polynomials in the form (2.3) and (2.4), see [15]. Generating functions for $P_n^{-1/2}(x)$, and $P_n^{1/2}(x)$, respectively are

$$\sum_{n=0}^{\infty} P_n^{-1/2}(x)\, z^n = \frac{2 - xz}{1 - xz + z^2} \tag{2.6}$$

$$\sum_{n=0}^{\infty} P_n^{1/2}(x)\, z^n = \frac{1}{1 - xz + z^2} \tag{2.7}$$

Recurrence relations for these polynomials are given by

$$\begin{aligned} &P_n^{-1/2}(x) = x\,P_{n-1}^{-1/2}(x) - P_{n-2}^{-1/2}(x) \quad \text{for} \quad n \in \mathbb{Z} - \{-1, 0, 1\}\,, \\ &P_{-1}^{-1/2}(x) = x,\ P_0^{-1/2}(x) = 2\,, \quad P_1^{-1/2}(x) = x\,, \end{aligned} \tag{2.8}$$

$$\begin{aligned} &P_n^{1/2}(x) = x\,P_{n-1}^{1/2}(x) - P_{n-2}^{1/2}(x) \quad \text{for} \quad n \in \mathbb{Z} - \{-1, 0, 1\}\,, \\ &P_{-1}^{1/2}(x) = x,\ P_0^{1/2}(x) = 1\,, \quad P_1^{1/2}(x) = x\,. \end{aligned} \tag{2.9}$$

Explicit expressions for these polynomials can be obtained from (2.3) and (2.4) by using Waring's formula, which gives the connection between the n-th power sums and the elementary symmetric polynomials, see [13, p. 109]. We have

$$(2.10)\qquad P_n^{-1/2}(x) = \sum_{i=0}^{[n/2]} \frac{n}{n-i}\binom{n-i}{i}(-1)^i x^{n-2i}$$

$$(2.11)\qquad P_n^{1/2}(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i}(-1)^i x^{n-2i}.$$

Generalizations of these polynomials to so called Dickson polynomials will be described in Section 5. Ordinary differential operators of order 2 for which these polynomials are eigenfunctions are given in [15].

3. THE POLYNOMIALS $P_{m,n}^{-1/2}(x, y)$

In this section we will define the polynomials $P_{m,n}^{-1/2}(x, y)$ over a field K as generalizations of the classical Chebyshev polynomials of the first kind. Let u, v, w be elements in a suitable extension field L of the field K. (If $K = \mathbb{C}$, then $L = \mathbb{C}$).

Definition 3.1.

$$P_{m,n}^{-1/2}(x, y) = (u^m + v^m + w^m)(u^{-n} + v^{-n} + w^{-n}) - (u^{m-n} + v^{m-n} + w^{m-n}),$$

where $x = u + v + w$, $y = uv + uw + vw$ and $uvw = 1 \in K$.

In the case $K = \mathbb{C}$ Koornwinder [10] defines these polynomials for $u = e^{i\sigma}$, $v = e^{-i\tau}$ and $w = e^{i(-\sigma+\tau)}$ as polynomials in the complex conjugates $z = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)}$ and $\bar{z} = e^{-i\sigma} + e^{i\tau} + e^{i(\sigma-\tau)}$. More generally, in [11] he gives the equivalent definition

(3.1)

$$(u_1u_2u_3)^{(1/3)(n_1+n_2+n_3)}\, P_{n_1-n_2,n_2-n_3}^{-1/2}\left(\frac{u_1+u_2+u_3}{(u_1u_2u_3)^{1/3}},\ \frac{u_1u_2+u_1u_3+u_2u_3}{(u_1u_2u_3)^{2/3}}\right) = \text{const.} \sum_{i_1,i_2,i_3} u_{i_1}^{n_1}u_{i_2}^{n_2}u_{i_3}^{n_3},$$

where the sum is taken over all permutations (i_1, i_2, i_3) of $(1, 2, 3)$ and the u_i are elements of $L = \mathbb{C}$. The special case $n = 0$ in Definition 3.1 gives $P_{m,0}^{-1/2}(x, y) = 2g_m(x, y)$, where the polynomials g_m are *Chebyshev polynomials* of the first kind introduced by Lidl and Wells [13] and studied in [14], [15]. It is shown in [10] that the polynomials $P_{m,n}^{-1/2}(x, y)$ over $\mathbb{C}$ are eigenfunctions of certain partial differential operators of orders 2 and 3, and that they are orthogonal on Steiner's hypocycloid.

The next theorem gives a generating function for the polynomials $P_{m,n}^{-1/2}(x, y)$. From Definition 3.1 follows immediately

Lemma 3.2. $P_{m,n}^{-1/2}(x, y) = \frac{1}{4}P_{m,0}^{-1/2}(x, y)\, P_{-n,0}^{-1/2}(x, y) - \frac{1}{2}P_{m-n,0}^{-1/2}(x, y)$.

Theorem 3.3 (Generating Function)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P_{m,n}^{-1/2}(x, y)\, s^m t^n = \frac{N}{(1 - xs + ys^2 - s^3)(1 - yt + xt^2 - t^3)}$$

where $N = 6 - 4xs + 2ys^2 - 4yt + 2xt^2 + (3xy - 3)\, st + (2y - 2x^2)\, st^2 + (2x - 2y^2)\, s^2 t + (xy - 3)\, s^2 t^2$.

Proof. We know from Eier and Lidl [4, p. 22] that $P_{m,0}^{-1/2}(x, y)$ (which is equal to $2g_m(x, y)$ in [4]) has the following generating function

$$\sum_{m=0}^{\infty} P_{m,0}^{-1/2}(x, y)\, s^m = \frac{6 - 4xs + 2ys^2}{1 - xs + ys^2 - s^3} \tag{3.2}$$

and

$$\sum_{m=0}^{\infty} P_{-m,0}^{-1/2}(x, y)\, t^m = \frac{6 - 4yt + 2xt^2}{1 - yt + xt^2 - t^3} \quad \text{for} \quad m \geqq 0\,. \tag{3.3}$$

Using Lemma 3.2 and (3.2) and (3.3) gives the result. □

If we multiply the equation in Theorem 3.3 by the denominator of the right hand side and compare coefficients of $s^m t^n$, we obtain the following

Theorem 3.4 (Recurrence Relations)

$$P_{m,n}^{-1/2}(x, y) = x\, P_{m-1,n}^{-1/2}(x, y) - y\, P_{m-2,n}^{-1/2}(x, y) + P_{m-3,n}^{-1/2}(x, y)\,,$$
$$P_{m,n}^{-1/2}(x, y) = y\, P_{m,n-1}^{-1/2}(x, y) - x\, P_{m,n-2}^{-1/2}(x, y) + P_{m,n-3}^{-1/2}(x, y)\,,$$

for $m > 2$ *and* $n > 2$ *respectively, where* $P_{00}^{-1/2} = 6$, $P_{10}^{-1/2} = 2x$, $P_{11}^{-1/2} = xy - 3$, $P_{20}^{-1/2} = 2x^2 - 4y$, $P_{21}^{-1/2} = x^2y - 2y^2 - x$, $P_{22}^{-1/2} = x^2y^2 - 2y^3 - 2x^3 + 4xy - 3$.

From Definition 3.1 the following Lemma is an immediate consequence.

Lemma 3.5 (i) $P_{m,n}^{-1/2}(x, y) = P_{n,m}^{-1/2}(y, x)$

(ii) $P_{m,n}^{-1/2}(x, y) = P_{-n,-m}^{-1/2}(x, y)$

If we correct a misprint in the exponent of -1 in the explicit formula for the polynomials $g_m(x, y)$ in Eier and Lidl [4] and notice that $P_{m,0}^{-1/2}(x, y) = 2g_m(x, y)$, then we obtain an explicit expression for $P_{m,0}^{-1/2}(x, y)$.

Theorem 3.6 (Explicit Expression)

$$P_{m,0}^{-1/2}(x, y) = \sum_{i=0}^{[m/2]} \sum_{j=0}^{[m/3]} \frac{2m(-1)^i}{m - i - 2j} \binom{m - i - 2j}{i + j} \binom{i + j}{i} x^{m-2i-3j} y^i$$

for $m > 0$.

By Lemma 3.5 (i) and (ii) we find $P_{-m,0}^{-1/2}(x, y) = P_{m,0}^{-1/2}(y, x)$, so we have an explicit expression for $P_{m,0}^{-1/2}(x, y)$ for arbitrary integer m. This also gives an eplicit expression for general polynomials $P_{m,n}^{-1/2}(x, y)$ by using Lemma 3.2. The more complicated k-dimensional case of these polynomials will be considered in a forthcoming paper.

4. THE POLYNOMIALS $P_{m,n}^{1/2}(x, y)$

We consider a generalization of the Chebyshev polynomials $P_n^{1/2}(x)$ of the second kind over a field K to two dimensions. Let L be a suitable extension field of the field K with elements u, v, w such that $uvw = 1 \in K$. Again, if $K = \mathbb{C}$ then $L = \mathbb{C}$. We introduce the following matrices. For $m, n \geqq 0$, $(m, n) \neq (0, 0)$ let

$$U_{m,n} = \begin{pmatrix} u^{m+2} & v^{m+2} & w^{m+2} \\ u & v & w \\ u^{-n} & v^{-n} & w^{-n} \end{pmatrix}, \quad U_{-m,-n} = (-1) \cdot \begin{pmatrix} u^{-(m+2)} & v^{-(m+2)} & w^{-(m+2)} \\ u^{-1} & v^{-1} & w^{-1} \\ u^{n} & v^{n} & w^{n} \end{pmatrix}.$$

Also define

$$U_{0,0}^{+} = \begin{pmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad U_{0,0}^{-} = (-1) \begin{pmatrix} u^{-2} & v^{-2} & w^{-2} \\ u^{-1} & v^{-1} & w^{-1} \\ 1 & 1 & 1 \end{pmatrix}.$$

Then clearly $\det U_{0,0}^{+} = \det U_{0,0}^{-}$, since $uvw = 1$. We denote this determinant by $\det U_{0,0}$.

We choose this particular arrangement for the rows of $U_{m,n}$ in view of later generalization to the k-dimensional case. The cases $m > 0$, $n < 0$ and $m < 0$, $n > 0$ are excluded.

Definition 4.1 $P_{m,n}^{1/2}(x, y) = (\det U_{m,n})(\det U_{0,0})^{-1}$, *where* $x = u + v + w$, $y = uv + uw + vw$ and $uvw = 1 \in K$.

This definition is equivalent to Koornwinder's definition of polynomials $P_{n_1-n_2,n_2-n_3}^{1/2}(x, y)$ in [11, p. 484]:

$$P_{n_1-n_2,n_2-n_3}^{1/2}(x, y) = (\det U)(\det U_{0,0})^{-1},$$

for integers $n_1 \geqq n_2 \geqq n_3 \geqq 0$, where the matrix U is given by

$$\begin{pmatrix} u^{n_1+2} & v^{n_1+2} & w^{n_1+2} \\ u^{n_2+1} & v^{n_2+1} & w^{n_2+1} \\ u^{n_3} & v^{n_3} & w^{n_3} \end{pmatrix}$$

In the case $K = \mathbb{C}$ and for $u = e^{i\sigma}$, $v = e^{-i\tau}$, $w = e^{i(-\sigma+\tau)}$ we obtain the special definition given by Koornwinder [10]. There it is shown that the polynomials

$P^{1/2}_{m,n}(z, \bar{z})$ in the complex conjugate variables $z = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)}$ and $\bar{z}$ are eigenfunctions of partial differential operators of orders 2 and 3 and are orthogonal on Steiner's hypocycloid with respect to the weight function $\mu(z, \bar{z})^{1/2}$ mentioned in the introduction. The polynomials $P^{1/2}_{m,0}(x, y)$, denoted by $f_m(x, y)$, over a field K, have been introduced by Lidl [15] as Chebyshev polynomials of the second kind. Definition 4.1 gives a more general class of polynomials of this type. The paper [15] also contains generating function, recurrence equation and partial differential equation for $P^{1/2}_{m,0}(x, y)$. More generally, we have

Theorem 4.2 (Generating Function).

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P^{1/2}_{m,n}(x, y)\, s^m t^n = \frac{1 - st}{(1 - xs + ys^2 - s^3)(1 - yt + xt^2 - t^3)}.$$

This can be shown by using Definition 4.1 for the polynomials $P^{1/2}_{m,n}(x, y)$ in terms of u, v and w, where $uvw = 1$. We leave out the rather lengthy but elementary calculations and note that in a forthcoming paper, we will study the k-dimensional case in more detail. In the case $n = 0$ we find

$$(4.1) \qquad \sum_{m=0}^{\infty} P^{1/2}_{m,0}(x, y)\, s^m = \frac{1}{1 - xs + ys^2 - s^3},$$

$$(4.2) \qquad \sum_{m=0}^{\infty} P^{1/2}_{0,m}(x, y)\, t^m = \frac{1}{1 - yt + xt^2 - t^3} \quad \text{for} \quad m > 0.$$

Compare with Lidl [15, p. 186].

Lemma 4.3 (i) $P^{1/2}_{m,n}(x, y) = P^{1/2}_{n,m}(y, x)$,

(ii) $P^{1/2}_{-m,-n}(x, y) = P^{1/2}_{n,m}(x, y)$,

(iii) $P^{1/2}_{m,n}(x, y) = P^{1/2}_{m,0}(x, y)\, P^{1/2}_{-n,0}(x, y) - P^{1/2}_{m-1,0}(x, y)\, P^{1/2}_{-(n-1),0}(x, y)$

for $m, n > 0$.

The proofs are simple, using Theorem 4.2 and the definition of these polynomials. Equating coefficients of $s^m t^n$ in Theorem 4.2 gives the following recurrence relations, we omit the variable x, y in writing down the polynomials.

Theorem 4.4 (Recurrence Relation)

$$P^{1/2}_{m,n} = xP^{1/2}_{m-1,n} - yP^{1/2}_{m-2,n} + P^{1/2}_{m-3,n}\,;$$

$$P^{1/2}_{m,n} = yP^{1/2}_{m,n-1} - xP^{1/2}_{m,n-2} + P^{1/2}_{m,n-3}$$

for $m > 2$ *and* $n > 2$ *respectively, with*

$$P^{1/2}_{00} = 1, \quad P^{1/2}_{10} = x, \quad P^{1/2}_{11} = xy - 1, \quad P^{1/2}_{20} = x^2 - y,$$

$$P^{1/2}_{21} = x^2y - y^2 - x, \quad P^{1/2}_{22} = x^2y^2 - x^3 - y^3.$$

Theorem 4.5 (Explicit Expression)

$$P^{1/2}_{m,0}(x, y) = \sum_{i=0}^{[m/2]} \sum_{j=0}^{[m/3]} (-1)^i \binom{m - i - 2j}{i + j} \binom{i + j}{i} x^{m-2i-3j} y^i \quad for \quad m \geqq 0 .$$

Proof.

$$\sum_{m=0}^{\infty} P^{1/2}_{m,0}(x, y)\, s^m = (1 - xs + ys^2 - s^3)^{-1} = \sum_{p=0}^{\infty} (xs - ys^2 + s^3)^p =$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{p} \sum_{j=0}^{q} (-1)^{q-j} \binom{p}{q} \binom{q}{j} x^{p-q} y^{q-j} s^{p+q+j} .$$

For $p = m - i - 2j$, $q = i + j$ we obtain the given explicit expression. □

Using Lemma 4.3 we have $P^{1/2}_{-m,0}(x, y) = P^{1/2}_{m,0}(y, x)$, so we are able to calculate explicit expressions for $P^{1/2}_{-m,0}(x, y)$, $m > 0$, and also for $P^{1/2}_{m,n}(x, y)$ by property (iii) of Lemma 4.3. We close this section by establishing a simple relationship between our Chebyshev polynomials of the first and second kind in two variables, which follow from (4.1).

Lemma 4.6.

$$P^{-1/2}_{m,0}(x, y) = \begin{cases} 6\, P^{1/2}_{m,0}(x, y) & if \quad m = 0 , \\ 6\, P^{1/2}_{m,0}(x, y) - 4x\, P^{1/2}_{m-1,0}(x, y) & if \quad m = 1 , \\ 6\, P^{1/2}_{m,0}(x, y) - 4x\, P^{1/2}_{m-1,0}(x, y) + 2y\, P^{1/2}_{m-2,0}(x, y) & if \quad m \geqq 2 . \end{cases}$$

5. FURTHER GENERALIZATIONS

Danković [2] defines a class of polynomials, in 2 variables over the complex numbers (see also [5]), by giving a generating function for these polynomials. This is a Chebyshev type class of polynomials with weight function $(1 - x^2 - y^2)^{-1/2}$.

Definition 5.1. The polynomials $D^{-1/2}_{m,n}(x, y)$ are given by the generating function

(5.1)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} D^{-1/2}_{m,n}(x, y)\, s^m t^n = \frac{1 + (y^2 - 1)\, s^2 + (x^2 - 1)\, t^2}{1 - 2xs - 2yt + 2xyst - (y^2 - 1)\, s^2 - (x^2 - 1)\, t^2}$$

The form of the generating function is such that the polynomials can be easily generalized to k-dimensional polynomials; this will be done elsewhere. Comparing the coefficients of $s^m t^n$ in (5.1) we can find a recurrence relation. Note that $D^{-1/2}_{m,n}(x, y) = D^{-1/2}_{n,m}(y, x)$.

Theorem 5.2 (Recurrence Relation)

$$D_{m,n}^{-1/2} = 2x\, D_{m-1,n}^{-1/2} + 2y\, D_{m,n-1}^{-1/2} - 2xy\, D_{m-1,n-1}^{-1/2} + (y^2 - 1)\, D_{m-2,n}^{-1/2} + (x^2 - 1)\, D_{m,n-2}^{-1/2} \quad \textit{for} \quad m > 2\,, \quad n > 2$$

where $D_{00}^{-1/2} = 1$, $D_{10}^{-1/2} = 2x$, $D_{11}^{-1/2} = 6xy$, $D_{20}^{-1/2} = 4x^2 + 2y^2 - 2$, $D_{21}^{-1/2} = 16x^2y + 6y^3 - 6y$, $D_{22}^{-1/2} = 16x^4 + 16y^4 + 56x^2y^2 - 20x^2 - 20y^2 + 4$.

Theorem 5.3 (Explicit Expression)

$$D_{m,n}^{-1/2}(x, y) = \sum_{i=0}^{[m/2]} \sum_{j=0}^{[n/2]} a_{i,j}^{m,n} x^{m-2i}(y^2 - 1)^i\, y^{n-2j}(x^2 - 1)^j$$

where

$$a_{i,j}^{m,n} = 2^{p-i-j} \sum_{r=0}^{p-1} \frac{(m + n - r)(p - r - 1)!}{(-2)^r\, r!\, i!\, j!\, (n - 2j - r)!\, (m - 2i - r)!}\,,$$

with $p = m + n - i - j$.

Proof. We denote the r.h.s. of (5.1) by N/D and let $(\partial/\partial s)^m =: \partial_s^m$. Then $\partial_s^m\, \partial_t^n(N/D)\big|_{(0,0)} = m!\, n!\, D_{m,n}^{-1/2}(x, y)$. Using Leibniz's formula we have

$$\partial_s^m\, \partial_t^n(N/D) = N(\partial_s^m\, \partial_t^n D^{-1}) + m\, \partial_s N(\partial_s^{m-1}\, \partial_t^n D^{-1}) + n\, \partial_t N(\partial_s^m\, \partial_t^{n-1} D^{-1}) + \frac{m(m-1)}{2}\, \partial_s^2 N(\partial_s^{m-2}\, \partial_t^n D^{-1}) + \frac{n(n-1)}{2}\, \partial_t^2 N(\partial_s^m\, \partial_t^{n-2} D^{-1})\,. \tag{5.2}$$

We use $\partial_s^3 D = 0$ and induction on n to obtain

$$\partial_s^n D^{-k} = \sum_{i=0}^{[n/2]} (-1)^{n-i} \frac{n!\, (k + n - i - 1)!}{2^i i!\, (n - 2i)!}\, D^{-(k+n-i)}(\partial_s D)^{n-2i}\, (\partial_s^2 D)^i \tag{5.3}$$

and

$$\partial_s^n(\partial_t D)^k = \frac{k!}{(k - n)!}\, (\partial_t D)^{k-n}\, (\partial_s \partial_t D)^n \quad \text{for} \quad k \geqq 0\,. \tag{5.4}$$

Now using (5.3) and (5.4) and Leibniz's formula we have

$$\partial_s^m \partial_t^n(D^{-1}) = \sum_{i=0}^{[m/2]} \sum_{j=0}^{[n/2]} \sum_{k=2i}^{m} (-1)^{n+k-i-j}\, B_{ijk}^{mn} D^{-(n-j+k-i+1)} (\partial_t D)^{n-2j-m+k}(\partial_s D)^{k-2i}(\partial_s^2 D)^i\, (\partial_t^2 D)^j\, (\partial_s \partial_t D)^{m-k} \tag{5.5}$$

where

$$B_{ijk}^{mn} = \frac{m!\, n!\, (n - j + k - i)!}{2^{i+j} i!\, j!\, (m - k)!\, (k - 2i)!\, (n - 2j - m + k)!}$$

Substituting (5.5) in (5.2) and evaluating at $s = 0$ and $t = 0$ gives the result. □

We note that we could define polynomials $D^{1/2}_{m,n}(x, y)$ by a generating function which is similar to (5.1), but where the numerator is just 1. The polynomials $P_n^{-1/2}(x)$ and $P_n^{1/2}(x)$ over a field K, given in (2.3) and (2.4), respectively, can be generalized in the following way (see Schur [19, pp. 446–447]. Let u and v be roots of $z^2 - xz + b = 0$ over K then we define

$$(5.5) \qquad P_n^{-1/2}(x; b) = u^n + v^n ,$$

$$(5.6) \qquad P_n^{1/2}(x; b) = (u - v)^{-1} (u^{n+1} - v^{n+1}), \quad \text{if} \quad n \in \mathbb{Z} ,$$

where $x = u + v$ and $uv = b \in K$. In other words, the polynomials $P_n^{-1/2}(x; b)$ are power sums of the roots of $z^2 - xz + b = 0$. The polynomials (5.5) and (5.6) are called Dickson polynomials, they have several interesting algebraic and arithmetic properties, see e.g. [12; pp. 209]. The results of section 2 can alo be obtained for the Dickson polynomials. As an example we state the generating functions

$$(5.7) \qquad \frac{2 - xs}{1 - xs + bs^2} \quad \text{and} \quad \frac{1}{1 - xs + bs^2}$$

for $P_n^{-1/2}(x; b)$ and $P_n^{1/2}(x; b)$, respectively. The definitions of the polynomials in sections 3 and 4 can also be given in this more general form. Instead of the condition $uvw = 1$ in Definitions 3.1 and 4.1 the condition $uvw = b \in K$ is used. Details for the polynomials in this generalized form will be given for the k-dimensional case in a continuation of this paper.

We can derive in the next theorem an algebraic property for the polynomials $D^{-1/2}_{m,0}(x, y)$. By comparing the explicit expressions for $D^{-1/2}_{m,0}(x, y)$ and $P_n^{-1/2}(x; b)$, see, e.g. [12; p. 209], we obtain

Lemma 5.4.

$$D^{-1/2}_{m,0}(x, y) = P_m^{-1/2}(2x; 1 - y^2), \quad m > 0 ;$$

$$D^{-1/2}_{0,n}(x, y) = P_n^{-1/2}(2y; 1 - x^2), \quad n > 0 .$$

Let $GF(q)$ be a finite field with $q \neq 2$ elements, then a polynomial $f(x)$ over $GF(q)$ is called a *permutation polynomial*, if $f(x) = a$ has exactly one solution in $GF(q)$ for each $a \in GF(q)$. A polynomial $f(x, y)$ over $GF(q)$ is called a *permutation polynomial*, if $f(x, y) = a$ has exactly q solutions in $GF(q)^2$ for each $a \in GF(q)$. Permutation polynomials for finite fields have been studied extensively (see, e.g. [12; Chapter 4] and [14]. One particular result for the Dickson polynomials $P_n^{-1/2}(x; b)$ over $GF(q)$ given there is

Lemma 5.5. $P_n^{-1/2}(x; b) \in GF(q)[x]$ *is a permutation polynomial for* $GF(q)$, *iff* $(n, q^2 - 1) = 1$ *for* $b \in GF(q)$.

Theorem 5.6. $D_{m,n}^{-1/2}(x, y)$ *is a permutation polynomial for* $GF(q)$, char $GF(q) \neq 2$, *iff either* $n = 0$ *and* $(m, q^2 - 1) = 1$, *or* $m = 0$ *and* $(n, q^2 - 1) = 1$.

Proof. Let m and n be nonzero.

Case 1: m odd. Then Theorem 5.3 shows that $D_{m,n}^{-1/2}(0, y) = 0$ and $D_{m,n}^{-1/2}(1, 0) = 0$, thus the number of solutions of $D_{m,n}^{-1/2}(x, y) = 0$ in $GF(q)^2$ is $\geqq q + 1$ and $D_{m,n}^{-1/2}$ cannot be a permutation polynomial.

Case 2: n odd, is treated similarly.

Case 3: m and n are even. Then $D_{m,n}^{-1/2}(x, y) = D_{m,n}^{-1/2}(-x, -y)$ and for $a \neq D_{m,n}^{-1/2}(0, 0)$, $D_{m,n}^{-1/2}(x, y) = a$ has an even number of solutions. Therefore $D_{m,n}^{-1/2}(x, y)$ is not a permutation polynomial.

Case 4: $n = 0$, $(m, q^2 - 1) \neq 1$. Then $D_{m,0}^{-1/2}(x, y) = P_m^{-1/2}(2x; 1 - y^2)$ by Lemma 5.4. But $P_m^{-1/2}(2x; 1)$ is not a permutation polynomial by Lemma 5.5, therefore $P_m^{-1/2}(2x; 1) \neq a$ for some $a \in GF(q)$. Since $-y \neq y$ for $y \neq 0$, the equation $P_m^{-1/2}(2x; 1 - y^2) = a$ has an even number of solutions, consequently $D_{m,0}^{-1/2}(x, y)$ is not a permutation polynomial.

Case 5: $m = 0$, $(n, q^2 - 1) \neq 1$ is treated similarly.

Conversely, assume $n = 0$ and $(m, q^2 - 1) = 1$ or $m = 0$ and $(n, q^2 - 1) = 1$, then $P_m^{-1/2}(2x; 1 - y^2)$ and $P_n^{-1/2}(2y; 1 - x^2)$ are permutation polynomials, and so are $D_{m,0}^{-1/2}(x, y)$ and $D_{0,n}^{-1/2}(x, y)$. □

A different type of generalized Chebyshev polynomials was introduced by Hays [7]. We mention here one of his two-dimensional generalizations. Let $T_n(x)$ be as in (2.1) and replace x by $x - (y + y^{-1})/2$ in the original generating function

$$\frac{1 - s^2}{1 - 2xs + s^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, s^n,$$

then two-dimensional Chebyshev polynomials over $\mathbb{C}$ are defined by

$$T_n(x - (y + y^{-1})/2) = \sum_{r=-n}^{n} T_{n,r}(x)\, y^r, \tag{5.8}$$

where

$$T_{n,r}(x) = \frac{n}{2}\sum_{m=0}^{[n/2]} \frac{(-1)^{m+r}(n - m - 1)!}{m!} \sum_{k=0}^{[l]} \frac{K(2x)^q H_q}{q!}, \quad T_{00}(x) = 1$$

with $l = (n - |r| - 2m)/2$, $K = 1/k!\,(k + |r|)!$, $q = n - |r| - 2m - 2k$ and $H_q = 0$ if $q < 0$ and $H_q = 1$ if $q \geqq 0$. The polynomials $U_n(x)$ of (2.2) are generalized in a similar way.

We conclude by mentioning an approach by Ricci [17] to define the polynomials

$P^{1/2}_{m,0}(x, y)$ over $\mathbb{C}$ from section 4. The paper [17] also includes the definition of the polynomials $P^{-1/2}_{m,0}(x, y)$ from section 3 and the results given in [15] for these polynomials. Let A be a complex 3×3 matrix and let J_1 denote the trace of A, J_2 be the sum of the principal minors of order 2 and J_3 be the det A, $\neq 0$. We define

$$x = J_1 J_3^{-1/3}, \quad y = J_2 J_3^{-2/3}$$

and a polynomial $U_m(x, y)$ by

$$(5.9) \qquad U_m(x, y) = x\, U_{m-1}(x, y) - y\, U_{m-2}(x, y) + U_{m-3}(x, y)$$

with

$$U_0 = 0, \quad U_1 = 1, \quad U_2 = x.$$

Then

$$A^m = J_3^{(m-2)/3}\, U_{m-1}(x, y)\, A^2 + J_3^{(m-1)/3}(-y\, U_{m-2}(x, y) + U_{m-3}(x, y))\, A + J_3^{m/3}\, U_{m-2}(x, y)\, I,$$

where I is the identity matrix. The polynomials $U_m(x, y)$ are equal to the polynomials $P^{1/2}_{m-1,0}(x, y)$ as given in Definition 4.1, or equivalent to $2 f_{m-1}(x, y)$ in the notation of [15].

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