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# GENERALIZATIONS OF THE FIBONACCI AND LUCAS POLYNOMIALS

**Gospava B. Djordjević***

**Abstract**

In this note we consider two sequences of polynomials, which are denoted by $\{U_{n,m}^{(k)}\}$ and $\{V_{n,m}^{(k)}\}$, where $k, m, n$ are nonnegative integers, and $m \geq 2$. These sequences represent generalizations of the well-known Fibonacci and Lucas polynomials. For example, if $m = 2$, then we obtain exactly the Fibonacci and Lucas polynomials. If $m = 3$, then polynomials $U_{n,3}^{(k)}$ and $V_{n,3}^{(k)}$ were considered in papers (G. B. Djordjević, Fibonacci Quart. 39.2(2001), and G. B. Djordjević, Fibonacci Quart. 43.4(2005)).

## 1 Introduction

The Fibonacci and Lucas polynomials are well-known and widely investigated. In this paper we consider a more general situation, by investigating polynomials $U_{n,m}$ and $V_{n,m}$, where all polynomials are polynomials in a real variable $x$, and $m, n$ are nonnegative integers, $m \geq 2$. Recall that polynomials $U_{n,m}$ and $V_{n,m}$, respectively, are defined by recurrence relations (see [1, 2]):

$$U_{n,m} = xU_{n-1,m} + U_{n-m,m}, \quad n \geq m, \tag{1.1}$$

with $U_{0,m} = 0,\ U_{n,m} = x^{n-1},\ n = 1, 2, \ldots, m-1$, and

$$V_{n,m} = xV_{n-1,m} + V_{n-m,m}, \quad n \geq m, \tag{1.2}$$

with $V_{0,m} = 2,\ V_{n,m} = x^n,\ n = 1, \ldots, m-1,\ m \geq 2$ and $x$ is a real variable. In this case corresponding generating functions are given by:

$$U^m(t) = \frac{t}{1 - xt - t^m} = \sum_{n=0}^{\infty} U_{n,m}\, t^n \tag{1.3}$$

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$$V^m(t) = \frac{2 - xt}{1 - xt - t^m} = \sum_{n=0}^{\infty} V_{n,m}\, t^n. \tag{1.4}$$

It is easy to obtain the equality

$$V_{n,m} = U_{n+1,m} + U_{n+1-m,m}, \quad n \geq m - 1.$$

We denote by $U_{n,m}^{(k)}$ and $V_{n,m}^{(k)}$, respectively, derivatives of the $k^{th}$ order of polynomials $U_{n,m}$ and $V_{n,m}$, i.e.

$$U_{n,m}^{(k)} = \frac{d^k}{dx^k}\{U_{n,m}\} \quad \text{and} \qquad V_{n,m}^{(k)} = \frac{d^k}{dx^k}\{V_{n,m}\}.$$

For given real $x$, we take complex numbers $\alpha_1$, $\alpha_2$, $\ldots, \alpha_m$, such that they satisfy:

$$\sum_{i=1}^{m} \alpha_i = x, \ \sum_{i<j} \alpha_i\alpha_j = 0, \ \sum_{i<j<k} \alpha_i\alpha_j\alpha_k = 0, \ldots, \alpha_1 \cdots \alpha_m = (-1)^{n-1}, \tag{1.5}$$

where $i, j, k \in \{1, 2, \ldots, m\}$. For $m = 4$, equalities (1.5) yield:

$$\sum_{i=1}^{4} \alpha_i = x, \ \sum_{i<j} \alpha_i\alpha_j = 0, \ \sum_{i<j<k} \alpha_i\alpha_j\alpha_k = 0, \ \alpha_1\alpha_2\alpha_3\alpha_4 = -1, \tag{1.6}$$

for $i, j, k \in \{1, 2, 3, 4\}$.

If $m = 2$, then we obtain exactly the Fibonacci and Lucas polynomials. If $m = 3$, then polynomials $U_{n,3}^{(k)}$ and $V_{n,3}^{(k)}$ were considered in papers [1] and [2]. In Section 2 we investigate polynomials $U_{n,4}^{(k)}$, and in Section 3 we consider the general case of polynomials $U_{n,n}^{(k)}$. In Section 4 we prove some related identities.

## 2 Polynomials $U_{n,4}^{(k)}$

In this section we investigate polynomials $U_{n,4}^{(k)}$, which are a special case of polynomials $U_{n,m}^{(k)}$. From (1.1), for $m = 4$, we get

$$U_{n,4} = xU_{n-1,4} + U_{n-4,4}, \quad n \geq 4, \tag{2.1}$$

with initial values $U_{0,4} = 0$, $U_{1,4} = 1$, $U_{2,4} = x$, $U_{3,4} = x^2$. Hence, by (1.3), we have that $U^4(t)$ is the corresponding generating function

$$U^4(t) = \frac{t}{1 - xt - t^4} = \sum_{n=0}^{\infty} U_{n,4}\, t^n. \tag{2.2}$$

Differentiating both sides of (2.2) $k$ times with respect to $x$, we obtain

$$U_k^4(t) = \frac{k!\, t^{k+1}}{(1 - xt - t^4)^{k+1}} = \sum_{n=0}^{\infty} U_{n,4}^{(k)}\, t^n. \tag{2.3}$$

Now, we prove the following result.

**Theorem 2.1.** *For a nonnegative integer $k$ the following holds:*

$$U_k^4(t) = \frac{k!}{(\alpha_1 A_{10}^1)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} \tag{2.4}$$

$$+\frac{k!}{(\alpha_2 A_{10}^2)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}^2}{(1-\alpha_2 t)^{k+1-i}} \tag{2.5}$$

$$+\frac{k!}{(\alpha_3 A_{10}^3)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}^3}{(1-\alpha_3 t)^{k+1-i}} \tag{2.6}$$

$$+\frac{k!}{(\alpha_4 A_{10}^4)^{k+1}} \sum_{i=0}^{k} \frac{d_{k,i}}{(1-\alpha_4 t)^{k+1-i}}, \tag{2.7}$$

*where*

$$A_{10}^r = A_{10}^r(\alpha_r) = \frac{3\alpha_r^4 - 2\alpha_r^3 x + 1}{\alpha_r^4},\ A_{11}^r = A_{11}^r(\alpha_r) = \frac{3\alpha_r^3 x - 3\alpha_r^4 - 3}{\alpha_r^4},$$
$$A_{12}^r = A_{12}^r(\alpha_r) = \frac{\alpha_r^4 - \alpha_r^3 x + 3}{\alpha_r^4},\ A_{13}^r = A_{13}^r(\alpha_r) = -\frac{1}{\alpha_r^4},$$

$$a_{k,i}^r = (-1)^i (A_{10}^r)^i \binom{k+1}{i} -$$
$$\sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} (A_{10}^r)^{l+s} (A_{11}^r)^{j-2l} (A_{12}^r)^{l-s} (A_{13}^r)^s a_{k,i-j},$$

$r = 1, 2, 3, 4.$

*Proof.* Using the equality (1.6), we get

$$\frac{t^{k+1}}{(1-xt-t^4)^{k+1}} \tag{2.8}$$

$$= \frac{t^{k+1}}{(1-\alpha_1 t)^{k+1}(1-\alpha_2 t)^{k+1}(1-\alpha_3 t)^{k+1}(1-\alpha_4 t)^{k+1}} \tag{2.9}$$

$$= \sum_{i=0}^{k} \frac{a_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \sum_{i=0}^{k} \frac{a_{k,i}^2}{(1-\alpha_2 t)^{k+1-i}} \tag{2.10}$$

$$+ \sum_{i=0}^{k} \frac{a_{k,i}^3}{(1-\alpha_3 t)^{k+1-i}} + \sum_{i=0}^{k} \frac{a_{k,i}^4}{(1-\alpha_4 t)^{k+1-i}}. \tag{2.11}$$

Multiplying both sides of (2.8)– (2.11) with

$$\alpha_1^{k+1}(1-\alpha_2 t)^{k+1}(1-\alpha_3 t)^{k+1}(1-\alpha_4 t)^{k+1} \tag{2.12}$$

we get the following equality

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}} = \alpha_1^{k+1}\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2\right. \tag{2.13}$$

$$\left.+A_{13}^1(1-\alpha_1 t)^3\right)^{k+1}\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t), \tag{2.14}$$

($\Phi_1(t)$ is an analytic function at the point $t=\alpha_1^{-1}$, $t$ is a complex variable and $x$ is a real constant.) On the other hand, we see that:

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}}\left((1-\alpha_1 t)^{-1}-1\right)^{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i(1-\alpha_1 t)^{-(k+1-i)}, \tag{2.15}$$

so

$$\begin{aligned}
&\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i(1-\alpha_1 t)^{-(k+1-i)}\\
&=\alpha_1^{k+1}\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2 + A_{13}^1(1-\alpha_1 t)^3\right)^{k+1}\times\\
&\times\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t)\\
&=\alpha_1^{k+1}\sum_{j=0}^{k+1}\sum_{l=0}^{j}\sum_{s=0}^{l}\binom{k+1}{j}\binom{j}{l}\binom{l}{s}(A_{10}^1)^{k+1-j}(A_{11}^1)^{j-l}(A_{12}^1)^{l-s}A_{13}^s\times\\
&\times(1-\alpha_1 t)^{l+j+s}\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t).
\end{aligned}$$

Because the Laurent series is unique at the point $t=\alpha_1^{-1}$ for the function $(\alpha_1 t)^{-(k+1)}(1-\alpha_1 t)^{-(k+1)}$, from the last equality, and $l+j+s := j$, $j-l := j-2l-s$, we get:

$$\begin{aligned}
&\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(1-\alpha_1 t)^{-(k+1-i)}\\
&=\alpha_1^{k+1}\sum_{j=0}^{k+1}\sum_{l=0}^{j}\sum_{s=0}^{j-2l}\binom{k+1}{i}\binom{j-l-s}{l}\binom{l}{s}(A_{10}^1)^{k+1-j+l+s}(A_{11}^1)^{j-2l-s}\times\\
&\times(A_{12}^1)^{l-s}(A_{13}^1)^s\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t).
\end{aligned}$$

Comparing coefficients with respect to $(1-\alpha_1 t)^{-(k+1-i)}$, we find that:

$$(-1)^i (A^1_{10})^i \binom{k+1}{i} = \alpha_1^{k+1} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} \times$$
$$\times (A^1_{10})^{k+1+i-j} (A^1_{10})^{l+s} (A^1_{11})^{j-2l-s} (A^1_{12})^{l-s} (A^1_{13})^s A^1_{k,i-j}.$$

Hence, for

$$\alpha_1^{k+1} (A^1_{10})^{k+1+i-j} A^1_{k,i-j} = a^1_{k,i-j},$$

we get

$$(-1)^i (A^1_{10})^i \binom{k+1}{i} =$$
$$\sum_{j=0}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} (A^1_{10})^{l+s} (A^1_{11})^{j-2l} (A^1_{12})^{l-s} (A^1_{13})^s a^1_{k,i-j}.$$

It follows that

$$a^1_{k,i} = (-1)^i (A^1_{10})^i \binom{k+1}{i} -$$
$$\sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} (A^1_{10})^{l+s} (A^1_{11})^{j-2l} (A^1_{12})^{l-s} (A^1_{13})^s a^1_{k,i-j}.$$

In a similar way, we find the remaining coefficients $a^r_{k,i},\ r = 1, 2, 3, 4$:

$$a^r_{k,i} = (-1)^i (A^r_{10})^i \binom{k+1}{i}$$
$$-\sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} (A^r_{10})^{l+s} (A^r_{11})^{j-2l} (A^r_{12})^{l-s} (A^r_{13})^s a^r_{k,i-j}.$$

Coefficients $A^1_{10},\ A^1_{11},\ A^1_{12},\ A^1_{13}$ can be computed from the following equalities

$$A^1_{10} + A^1_{11}(1-\alpha_1 t) + A^1_{12}(1-\alpha_1 t)^2 + A^1_{13}(1-\alpha_1 t)^3 = (1-\alpha_2 t)(1-\alpha_3 t)(1-\alpha_4 t) \quad (2.16)$$

and using (1.6).

In a similar way, we find the remaining coefficients $A^r_{10}, A^r_{11},\ A^r_{12},\ A^r_{13}$, $r = 2, 3, 4$.

□

# 3 Polynomials $U^{(k)}_{n,m}$

In this section we investigate polynomials $U^{(k)}_{n,m}$. Differentiating (1.3), $k$-times with respect to $x$, we obtain

$$U^k_m(t) = \frac{k! t^{k+1}}{(1 - xt - t^m)^{k+1}} = \sum_{n=0}^{\infty} U^{(k)}_{n,m}\, t^n. \quad (3.1)$$

**Theorem 3.1.** *Let $k$ be a nonnegative integer, and let $m$ be a positive integer, $m \geq 2$. Then*

$$U_k^m(t) = \sum_{j=1}^{m} \frac{k!}{(\alpha_j A_{10}^j)^{k+1}} \sum_{i=0}^{k} \frac{a_{k,i}^j}{(1-\alpha_j t)^{k+1-i}}, \tag{3.2}$$

*where:*

$$\begin{aligned} A_{10}^j + \quad & A_{11}^j(1-\alpha_j t) + A_{12}^j(1-\alpha_j t)^2 + \cdots + A_{1,m-1}^j(1-\alpha_j t)^{m-1} \\ & = (1-\alpha_1 t)(1-\alpha_2 t)\cdots(1-\alpha_{j-1}t)(1-\alpha_{j+1}t)\cdots(1-\alpha_m t), \end{aligned}$$

*and $\alpha_1, \ldots, \alpha_m$ satisfy equalities* (1.5)*;*

$$a_{k,i}^j = (-1)^i (A_{10}^j)^i \binom{k+1}{i} - \tag{3.3}$$

$$\sum_{j_1=1}^{i} \sum_{j_2=0}^{j_1} \cdots \sum_{j_{m-1}=0}^{j_{m-2}} \binom{k+1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{m-2}}{j_{m-1}} (A_{10}^j)^{j_2+\cdots+j_{m-1}} \times \tag{3.4}$$

$$(A_{11}^j)^{j_1-j_2} \cdots \times (A_{1,m-1}^j)^{j_{m-1}} a_{k,i-j_1}^j, \;\; j = 1, 2, \ldots, m. \tag{3.5}$$

*Proof.* From (3.1) and (1.5) we obtain:

$$\frac{t^{k+1}}{(1-xt-t^m)^{k+1}} = \frac{t^{k+1}}{(1-\alpha_1 t)^{k+1}\cdots(1-\alpha_m)^{k+1}} \tag{3.6}$$

$$= \sum_{i=0}^{k} \frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \sum_{i=0}^{k} \frac{A_{k,i}^2}{(1-\alpha_2 t)^{k+1-i}} + \ldots \tag{3.7}$$

$$+ \sum_{i=0}^{k} \frac{A_{k,i}^m}{(1-\alpha_m t)^{k+1}}. \tag{3.8}$$

Multiplying (3.6)–(3.8) with $\alpha_1^{k+1}(1-\alpha_2 t)^{k+1}\cdots(1-\alpha_m t)^{k+1}$, we have the following equality

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}} = \alpha_1^{k+1}\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2 + \ldots \right. \tag{3.9}$$

$$\left. + A_{1,m-1}^1(1-\alpha_1 t)^{m-1}\right)^{k+1} \sum_{i=0}^{k} \frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t), \tag{3.10}$$

($\Phi_1(t)$ is an analytic function at $t = \alpha_1^{-1}$; $t$ is a complex variable; $x$ is a real constant.) The left side of the equality (3.9) can be rewritten in the following form:

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}} = \left((1-\alpha_1 t)^{-1} - 1\right)^{k+1} = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (1-\alpha_1 t)^{-(k+1-i)}. \tag{3.11}$$

The right side of the same equality is

$$\alpha_1^{k+1}\sum_{j_1=0}^{k+1}\sum_{j_1=0}^{j_1}\cdots\sum_{j_{m-1}}^{j_{m-2}}\binom{k+1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{m-1}}{j_{m-2}}(A_{10}^1)^{k+1-j_1}(A_{11}^1)^{j_1-j_2}\cdots \tag{3.12}$$

$$\times(A_{1,m-1}^1)^{j_{m-1}}(1-\alpha_1 t)^{j_1+\cdots+j_{m-1}}\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}}+\Phi_1(t). \tag{3.13}$$

First taking

$$\alpha_1^{k+1}(A_{10}^1)^{k+1+i-j_1}A_{k,i-j_1}^1=a_{k,i-j_1}^1,\ \text{and}\ j_1+j_2+\cdots+j_{m-1}:=j_1,$$

comparing coefficients with respect to $(1-\alpha_1 t)^{-(k+1-i)}$, and then using (3.11) and (3.12), we obtain coefficients $a_{k,i}^1$. Similarly, we compute other coefficients, $a_{k,i}^j$, $j=1,2,\ldots,j_{m-1}$. □

# 4 Some identities

In this section we prove some identities, for generalized polynomials $U_{n,m}^{(k)}$ and $V_{n,m}^{(k)}$. For $m=2$, these identities correspond to the Fibonacci and Lucas polynomials. For $m=3$, these identities correspond to generalized polynomials, which are considered in [1] and [2].

**Lemma 4.1.** *For positive integers $m,n$, such that $n\geq m\geq 2$, the following hold:*

$$\sum_{i=0}^{n}U_{i,m}=\frac{1}{x}\left(\sum_{j=0}^{m-1}U_{n+2-m+j,m}-1\right), \tag{4.1}$$

$$\sum_{i=0}^{n}V_{i,m}=\frac{1}{x}\left(\sum_{j=0}^{m-1}V_{n+2-m+j,m}-1\right), \tag{4.2}$$

$$\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}=h_{r+mn,m}, \tag{4.3}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i h_{r+mi,m}=(-1)^n x^n h_{r+(m-1)n,m}, \tag{4.4}$$

*where* $h_{n,m}=U_{n,m}$, *or* $h_{n,m}=V_{n,m}$.

*Proof.* We use the induction on $n$. It is easy to see that (4.1) is satisfied for $n=1$. Suppose that the equality (4.1) is valid for $n$, then (for $n:=n+1$):

$$\begin{aligned}\sum_{i=0}^{n+1}U_{i,m} &= \frac{1}{x}\left(\sum_{j=0}^{m-1}U_{n+2-m+j,m}-1\right)+U_{n+1,m}\\ &=\frac{1}{x}\left(\sum_{j=0}^{m-1}U_{n+2-m+j,m}-1+xU_{n+1,m}\right)\quad (\text{by }\ (1.1))\\ &=\frac{1}{x}\left(\sum_{j=0}^{m-1}U_{n+3-m+j,m}-1\right).\end{aligned}$$

Hence, the equality (4.1) holds for any positive integer $n$.

The equality (4.2) can be proved in a similar way, using the recurrence relation (1.2).

Suppose that (4.3) holds for $n$. Then, taking the value $n+1$ instead of $n$, from (1.1) and (1.2), we get:

$$\begin{aligned}&h_{r+m(n+1),m}=xh_{r+mn+m-1,m}+h_{r+mn,m}\\ &=\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}+xh_{r+mn+m-1,m}\\ &=\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}+x\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+m-1+(m-1)i,m}\\ &=\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}+\sum_{i=1}^{n+1}\binom{n}{i-1}x^i h_{r+(m-1)i,m}=\\ &\sum_{i=1}^{n}\left(\binom{n}{i}+\binom{n}{i-1}\right)x^i h_{r+(m-1)i,m}+h_{r,m}+x^{n+1}h_{r+(m-1)(n+1),m}\\ &=\sum_{i=1}^{n}\binom{n+1}{i}x^i h_{r+(m-1)i,m}+\binom{n+1}{0}h_{r,m}+\binom{n+1}{n+1}h_{r+(m-1)(n+1),m}\\ &=\sum_{i=0}^{n+1}\binom{n+1}{i}x^i h_{r+(m-1)i,m}.\end{aligned}$$

Now, we have proved the equality (4.3).

Suppose that (4.4) is correct for $n$. Then

$$\begin{aligned}
&(-1)^{n+1}x^{n+1}h_{r+(m-1)(n+1),m} = (-1)^{n+1}x^n(xh_{r+m-1+(m-1)n,m})\\
&= (-1)^{n+1}x^n(h_{r+m+(m-1)n,m} - h_{r+(m-1)n,m})\\
&= (-1)^{n+1}x^n h_{r+m+(m-1)n,m} + (-1)^n x^n h_{r+(m-1)n,m}\\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}h_{r+m(i+1),m} + \sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{r+mi,m}\\
&= \sum_{i=1}^{n}(-1)^i\left(\binom{n}{i-1}+\binom{n}{i}\right)h_{r+mi,m} + h_{r,m} + (-1)^{n+1}h_{r+m(n+1),m}\\
&= \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}h_{r+mi,m}.
\end{aligned}$$

□

**Theorem 4.1.** *For positive integers $m, n$, such that $n \geq m \geq 2$, the following equalities hold:*

$$x\sum_{i=0}^{n}U_{i,m}^{(k)} = \sum_{j=0}^{m-1}U_{n+2-m+j,m}^{(k)} - k\sum_{i=0}^{n}U_{i,m}^{(k-1)},\ \ k \geq 1. \tag{4.5}$$

$$x\sum_{i=0}^{n}V_{i,m}^{(k)} = \sum_{j=0}^{m-1}V_{n+2-m+j,m}^{(k)} - k\sum_{i=0}^{n}V_{i,m}^{(k-1)},\ \ k \geq 1. \tag{4.6}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^i)^{(j)}h_{r+(m-1)i,m}^{(k-j)} = h_{r+mn,m}^{(k)}, \tag{4.7}$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{r+mi,m}^{(k)} = (-1)^n\sum_{j=0}^{k}\binom{k}{j}(n-j+1)_j x^{n-j}h_{r+(m-1)n,m}^{(k-j)}. \tag{4.8}$$

*where $h_{r,m} = U_{r,m}$ or $h_{r,m} = V_{r,m}$.*

*Proof.* Differentiating both sides of equalities (4.1) and (4.2), on $x$, $k$–times, we obtain equalities (4.5) and (4.6). Using the induction on $k$, we prove (4.7). If $k = 0$, then (4.7) becomes

$$h_{r+mn,m} = \sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m},$$

so, we get the equality (4.7). Suppose that (4.7) holds for $k$ ($k \geq 0$). Then, for

$k := k+1$, we get

$$h^{(k+1)}_{r+mn,m} = \sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\frac{d}{dx}\Big((x^i)^{(j)}h^{(k-j)}_{r+(m-1)i,m}\Big) =$$

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\Big((x^i)^{(j+1)}h^{(k-j)}_{r+(m-1)i,m} + (x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}\Big)$$

$$\sum_{i=0}^{n}\sum_{j=1}^{k+1}\binom{n}{i}\binom{k}{j-1}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m} + \sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}$$

$$= \sum_{i=0}^{n}\sum_{j=1}^{k}\binom{n}{i}\binom{k+1}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m} + \sum_{i=0}^{n}\binom{n}{i}x^i h^{(k+1)}_{r+(m-1)i,m} +$$

$$\sum_{i=0}^{n}\binom{n}{i}(x^i)^{(k+1)}h_{r+(m-1)i,m} = \sum_{i=0}^{n}\sum_{j=0}^{k+1}\binom{n}{i}\binom{k+1}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}.$$

So, we have proved the equality (4.7). Similarly, we can get the equality (4.8). □

Further, we prove some equalities, using generating functions (1.3) and (1.4). Precisely, if we differentiate (1.4) $k$-times with respect to $x$, then we obtain

$$V^m_k(t) = \frac{k!t^k(1+t^m)}{(1-xt-t^m)^{k+1}} = \sum_{n=0}^{\infty} V^{(k)}_{n,m}\, t^n. \tag{4.9}$$

Using (3.1) and (4.9), we can easily prove the following theorem.

**Theorem 4.2.** *For integers $m, k, r$, such that $m \ge 2$, and $k, r \ge 0$, the following hold:*

$$U^m_k(t)U^m_r(t) = \frac{k!r!}{(k+r+1)!}U^m_{k+r+1}(t), \tag{4.10}$$

$$U^m_k(t)V^m(t) = \frac{2t^{-1}-x}{k+1}U^m_{k+1}(t), \tag{4.11}$$

$$V^m_k(t)V^m_r(t) = \frac{k!r!}{(k+r+1)!}V^m_{k+r+1}(t^{-1}+t^{m-1}),\ (r,k\ge 1), \tag{4.12}$$

$$U^m_k(t)V^m_r(t) = \frac{k!r!}{(k+r+1)!}V^m_{k+r+1}(t),\ (r,k\ge 1), \tag{4.13}$$

$$V^m_k(t)V(t) = \frac{1}{k+1}(2t^{-1}-x)V^m_{k+1}(t), \tag{4.14}$$

$$V^m(t)V^m(t) = (2t^{-1}-x)^2U^m_1(t). \tag{4.15}$$

The following result is an immediate consequence of Theorem 4.2:

**Theorem 4.3.** *Let $m, n, k$ be integers, such that $n \geq m \geq 2$ and $k \geq 0$. Then*

$$\sum_{i=0}^{n} U_{i,m}^{(k)} U_{n-i,m}^{(r)} = \frac{k!r!}{(k+r+1)!} U_{n,m}^{(k+r+1)}, \tag{4.16}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k)} V_{n-i,m} = \frac{1}{k+1} \left( 2U_{n+1,m}^{(k+1)} - xU_{n,m}^{(k+1)} \right), \tag{4.17}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k)} V_{n-i,m}^{(r)} = \frac{k!r!}{(k+r+1)!} \left( V_{n+1,m}^{(k+r+1)} + V_{n+1-m,m}^{(k+r+1)} \right), \tag{4.18}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k)} V_{n-i,m}^{(r)} = \frac{k!r!}{(k+r+1)!} V_{n,m}^{(k+r+1)}, \ (r \geq 1), \tag{4.19}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k)} V_{n-i,m} = \frac{1}{k+1} \left( 2V_{n+1,m}^{(k+1)} - xV_{n,m}^{(k+1)} \right), \tag{4.20}$$

$$\sum_{i=0}^{n} V_{i,m} V_{n-i,m} = 4U_{n+2,m}^{(1)} - 4xU_{n+1,m}^{(1)} + x^2 U_{n,m}^{(1)}. \tag{4.21}$$

*Proof.* Comparing coefficients with respect to $t^n$ in equalities (4.10)–(4.15), respectively, we obtain equalities (4.16)-(4.21). □

**Corollary 4.1.** *Equalities* (4.10)–(4.21) *for $m = 2$ and $m = 3$ correspond to the Fibonacci and Lucas polynomials, and to those considered in* [1] *and* [2].

# References

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University of Nis, Faculty of Technology, 16000 Leskovac, Serbia
*E-mail*: `gospava48@ptt.yu`