QUARTERLY OF APPLIED MATHEMATICS
VOLUME LVII, NUMBER 2
JUNE 1999, PAGES 381-400

GENERALIZATIONS OF THE GREENBERG-RASCLE CONSTRUCTION OF PERIODIC SOLUTIONS TO QUASILINEAR EQUATIONS OF 1-D ELASTICITY

By

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1. Introduction. In this paper we continue the work, started in [2], on generalizations of the Greenberg-Rascle construction of spatially and temporally periodic solutions to quasilinear wave equations. We consider the system of 1-D elasticity equations

$$u_t - v_x = 0 \quad \text{and} \quad v_t - \sigma(u)_x = 0, \tag{WE1}$$

where $\sigma(u)$ is an odd function satisfying $\sigma'(u) > 0$ and $\sigma''(u) > 0$ for $u > 0$.

We let $c(u) = \sqrt{\sigma(u)}$ denote the propagation speed and write (WE1) as

$$u_t - v_x = 0 \quad \text{and} \quad v_t - c^2(u)u_x = 0. \tag{WE2}$$

We are interested in propagation speeds, c, that satisfy

$$c(-u) = c(u) \geq c_0 \stackrel{\text{def}}{=} c(0) > 0 \quad \text{and} \quad c'(u) \geq c_1 > 0, \quad u > 0, \tag{1.1}$$

and, thus, have jump discontinuity in $\frac{dc}{du}$ at $u = 0$.

Following the famous papers of Lax [5] and MacCamy and Mizel [6], researchers have collected a large body of evidence supporting the belief that solutions of systems of conservation laws develop shocks.[1] This paper deals with shock-free solutions that have a very remarkable property: they are spatially and temporally periodic!

The mathematical importance of having a periodic solution is amplified by the following observation. Let (v, u) be a bounded, nonconstant periodic solution of (WE1) and let

$$(v^\varepsilon, u^\varepsilon)(x,t) = (v, u)\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right). \tag{1.2}$$

Received April 15, 1997.
1991 *Mathematics Subject Classification.* Primary 35L65, 35L67; Secondary 35B10, 35B35.
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[1]The Lax paper deals with genuinely nonlinear versions of (WE1) in which the condition $\sigma''(u) > 0$ is required for all u and, thus, σ cannot be an odd function. MacCamy and Mizel, however, deal with 1-D elasticity equations that are not genuinely nonlinear and have an inflection point. Their analysis admits odd functions σ.

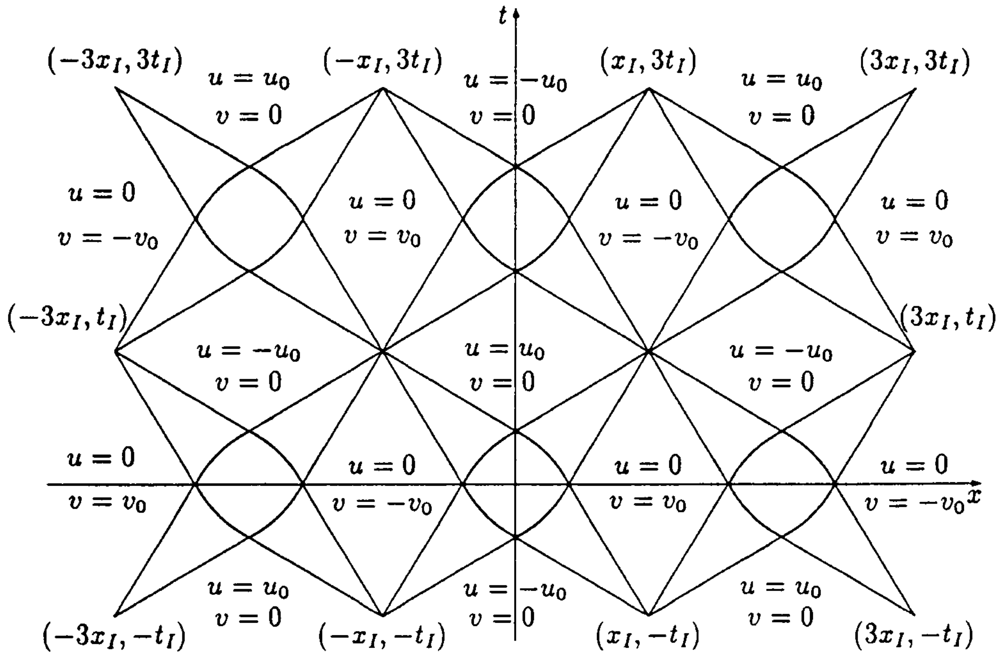

FIG. 1

It is easy to show that all $(v^\varepsilon, u^\varepsilon)$ satisfy the same equation (WE1) and that they constitute a bounded sequence without any strongly converging subsequences. This observation shows, in particular, that the regularity assumption $\sigma \in C^2$ made in DiPerna's compensated compactness result (see [3]) cannot be relaxed.

Greenberg and Rascle [1] were first to observe that for the special choice of sound speed relation given by

$$c(u) = \begin{cases} \frac{c_0 U^2}{(U-u)^2}, & 0 \le u < U, \\ \frac{c_0 U^2}{(U+u)^2}, & -U < u \le 0, \end{cases} \tag{1.3}$$

Eqs. (WE2) admit spatially and temporally periodic solutions. A schematic representation of their solutions is shown in Figure 1.

The Greenberg-Rascle construction exploits the fact that the non-constant interaction $v = x$ and $u = t$ could be matched to expanding and focusing simple waves that connect constant states

$$(v, u) = (0, u_0), (0, -u_0), (v_0, 0), (-v_0, 0), \tag{1.4}$$

where

$$v_0 = \int_0^{u_0} c(s)\, ds. \tag{1.5}$$

The Greenberg-Rascle solution is obtained by superposing odd and even reflections of the same solution defined on the rectangle K_{x_I, t_I} (see Figs. 1, 2). The graph of the solution inside K_{x_I, t_I} resembles a butterfly and for this reason the Greenberg-Rascle

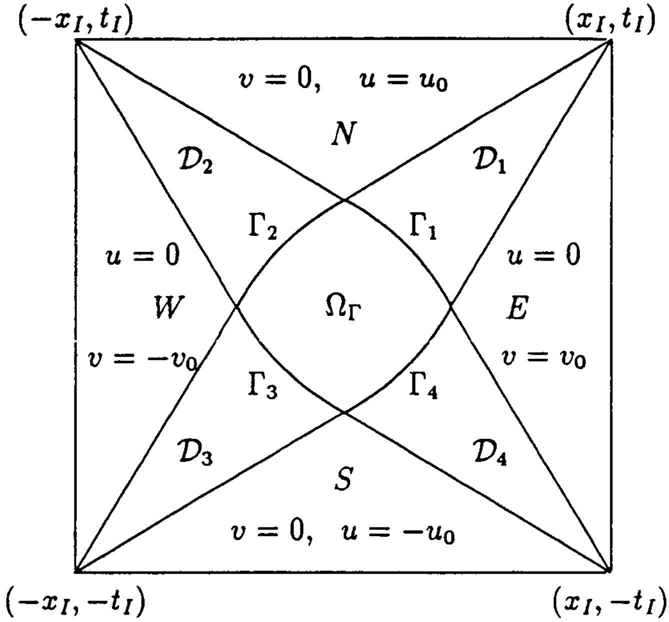

FIG. 2. Rectangle K_{x_I,t_I}

solutions are sometimes referred to as the butterfly solutions. The solution consists of constant states (1.4) on triangles E, S, W, and N, simple expanding waves in regions D_3 and D_4, and simple focusing waves in regions D_1 and D_2. In the diamond-shaped interaction region Ω_Γ the solution is given by $v = x$ and $u = t$, and along boundary curves $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 is matched to appropriate simple waves. The width and height of K_{x_I,t_I} and amplitudes of the solution are characterized by four parameters (x_I, t_I, u_0, v_0) related by (1.5) and

$$x_I = c_0 t_I + v_0. \tag{1.6}$$

The spatial period of their solution is $4x_I$ and the temporal period is $4t_I$.

The sequence $(v^\varepsilon, u^\varepsilon)$, obtained by applying dilatation (1.2) to the Greenberg-Rascle solution provides important limitations to the conjecture of Serre [4]. Serre has conjectured that the Young Measure associated with the sequence of solutions to certain systems of two conservation laws is a tensor product, $v^r_{x.t} \otimes v^w_{x.t}$, in Riemann invariant coordinates r, w. Explicit calculations show that $(v^\varepsilon, u^\varepsilon)$ produces a Young Measure that is not a product measure. The same can be shown about solutions constructed in this paper.

Greenberg and Peszek [2] allowed more complicated interactions than $v = x$ and $u = t$. They used a hodograph transformation inside the interaction region Ω_Γ and regarded x and t as functions of v and u. They proved that for sufficiently small u_0 and for any sufficiently small $W = \{w_i\}_{i=1}^{\infty} \in l^{\infty}$ there exist a sound speed relation, c, and a one-to-one map $(v,u) \to (x,t)$ that defines a smooth solution in the interaction region satisfying

$$t(v,0) = 0, \quad t_u(v,0) = g(v) \overset{\text{def}}{=} 1 + \sum_{n=1}^{\infty} \frac{w_n v^{2n}}{(2n)!}. \tag{1.7}$$

The Greenberg-Peszek result shows that there are sound speed relations other than (1.3) that admit spatially and temporally periodic solutions of Greenberg-Rascle type. However, no attempt was made to characterize the class of wave equations of the form (WE1) that admit such solutions.

We note that the class of all admissible (WE1) is closed with respect to taking the inverse of σ. More precisely, if (v,u) is a shock-free solution of (WE1) then $U(x,t) = \sigma(u(t,x))$ and $V(x,t) = v(t,x)$ satisfy

$$U_t - V_x = 0, \quad V_t - \sigma^{-1}(U)_x = 0.$$

This "nonlinear 90° rotation" of Greenberg-Rascle solutions produces a class of c's that are continuous, even, positive and satisfy $dc/du < 0$ for $u > 0$. In this paper, however, we restrict attention to sound speed functions that satisfy (1.1).

Our goal is to obtain some descriptions of the class of admissible c's. The first result reduces the problem of constructing shock-free spatially and temporally periodic solutions to solving a linear, hyperbolic boundary value problem. We show that if p is defined by

$$p\left(\int_0^u c(s)\,ds\right) = \frac{c'(u)}{c^2(u)}, \quad u \geq 0$$

and if the boundary value problem

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p(\overline{u})t_{\overline{u}} = 0 & \text{in } \Omega_{++} = \{(v,\overline{u}) : v \geq 0, 0 \leq \overline{u} \leq v_0 - v\}, \\ t(v,0) = 0, & 0 \leq v \leq v_0, \\ t_v(0,\overline{u}) = 0, & 0 \leq \overline{u} \leq v_0, \\ t(v_0 - \overline{u}, \overline{u}) = t_I(1 - e^{-\frac{1}{2}\int_0^{\overline{u}} p(s)\,ds}), & 0 \leq \overline{u} \leq v_0 \end{cases} \tag{1.8}$$

has a solution such that

$$t_{\overline{u}} > |t_v|, \tag{1.9}$$

then (WE2) admits a Greenberg-Rascle type solution with parameters (x_I, t_I, u_0, v_0), where $x_I = c(0)(t_I + \int_0^{v_0} t_{\overline{u}}(s,0)\,ds)$, u_0 is such that $v_0 = \int_0^{u_0} c(s)\,ds$, and v_0 and t_I are as in (1.8). The proof of this result utilizes ideas developed in [2]. We note that the above linear problem is much more complicated than standard hyperbolic initial value problems; in particular, it has a solution only for a certain class of functions p.

Our interest is in determining the class of functions $p > 0$ for which (1.8) has a solution satisfying (1.9). We assume that $p \in C^1(R_+)$ is an arbitrarily given function and that $p \geq \delta > 0$. Our basic observation is that, for small v_0, the solution of (1.8) on the characteristic triangle L, bounded by the lines $\overline{u} = 0, \overline{u} = v$, and $\overline{u} = v_0 - v$, can be uniquely determined from the data

$$t(v_0 - \overline{u}, \overline{u}) = t_I(1 - t^{-\frac{1}{2}\int_0^{\overline{u}} p(s)\,ds}), \qquad 0 \leq \overline{u} \leq v_0/2. \tag{1.10}$$

Thus, the initial data

$$t_{\overline{u}}(v,0) = g(v), \qquad 0 \le v \le v_0 \tag{1.11}$$

is defined uniquely in terms of (1.10). We replace (1.8) with the initial value problem

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p(\overline{u})t_{\overline{u}} = 0 & \text{in } \Omega_{++}, \\ t(v,0) = 0, & 0 \le v \le v_0, \\ t_{\overline{u}}(v,0) = g(v), & 0 \le v \le v_0, \\ t_v(0,\overline{u}) = 0, & 0 \le \overline{u} \le v_0 \end{cases} \tag{1.12}$$

and ask whether (1.12) has a solution satisfying (1.9) and

$$t(v_0 - \overline{u}, \overline{u}) = t_I(1 - e^{-\frac{1}{2}\int_0^{\overline{u}} p\,ds}), \quad v_0/2 \le \overline{u} \le v_0. \tag{1.13}$$

It turns out that in most cases such a solution does not exist; however, one can modify p on the set $(v_0/2, \infty)$ in such a way that $p > 0$ and (1.12) has a solution satisfying (1.9) and (1.13). The modified $p|_{[v_0/2, v_0]}$ is determined uniquely in terms of its values on the set $[0, v_0/2]$. Summarizing, we show that any function $p_G \in C^1(R)$ satisfying $p_G \ge \delta > 0$ can be modified in a unique way to a function $p \in C([0, v_0])$ satisfying $p(\overline{u}) = p_G(\overline{u})$ for $\overline{u} \in [0, v_0/2]$ and such that (1.8) has a solution satisfying (1.9).

Now, let us assume that c_G is a given function satisfying (1.1), t_I and v_0 are given positive parameters, and v_0 is sufficiently small. We let $u_{1/2}$ be such that

$$\frac{v_0}{2} = \int_0^{u_{1/2}} c_G(s)\,ds.$$

Our main result states that there exist positive parameters u_0 and x_I, and a unique sound speed function c satisfying $c(u) = c_G(u)$ for $-u_{1/2} \le u \le u_{1/2}$ and such that (WE2) admits a Greenberg-Rascle construction of spatially and temporally periodic solutions with parameters (x_I, t_I, u_0, v_0).

2. Reformulation of the problem. We assume that the propagation speed function c satisfies (1.1) and is continuous and C^1 away from $u = 0$. In this section we state and prove sufficient conditions for (WE2) to admit a Greenberg-Rascle type construction of spatially and temporally periodic solutions.

We let Π denote the primitive function of c,

$$\Pi(u) \stackrel{\text{def}}{=} \int_0^u c(s)\,ds, \qquad u \ge 0 \tag{2.1}$$

and regard Π as a new variable which we call $\overline{u}$,

$$\overline{u} = \Pi(u) = \int_0^u c(s)\,ds, \qquad u \ge 0. \tag{2.2}$$

We note that Π satisfies

$$\Pi, \Pi', \Pi'' > 0 \quad \text{for } u > 0. \tag{2.3}$$

We define a new function

$$p(\overline{u}) = p\left(\int_0^u c(s)\,ds\right) = \frac{c'(u)}{c^2(u)}, \qquad \overline{u}, u \ge 0. \tag{2.4}$$

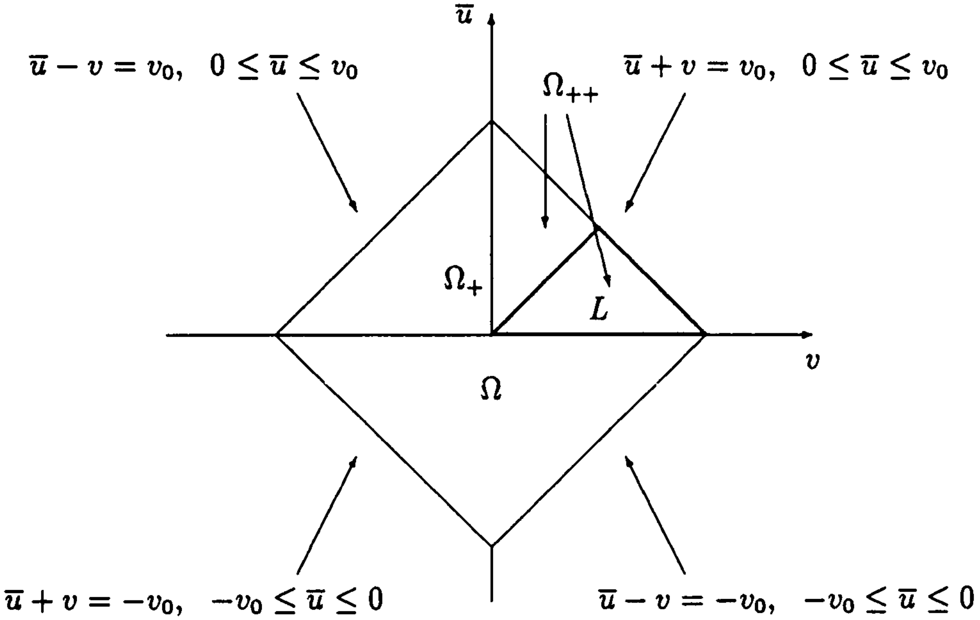

FIG. 3. Sets $\Omega, \Omega_+, \Omega_{++}$, and L

In this section we follow the ideas developed in [2]. The basic change from the previous paper is the use of variables $(\overline{u}, v)$, which will prove more convenient in reformulating the problem.

THEOREM 2.1. Consider the following boundary value problem (see Fig. 3):

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p(\overline{u})t_{\overline{u}} = 0 & \text{in } \Omega_{++} = \{(v, \overline{u}) : v \ge 0, 0 \le \overline{u} \le v_0 - v\}, \\ t(v, 0) = 0, & 0 \le v \le v_0, \\ t_v(0, \overline{u}) = 0, & 0 \le \overline{u} \le v_0, \\ t(v_0 - \overline{u}, \overline{u}) = t_I(1 - e^{-\frac{1}{2}\int_0^{\overline{u}} p(s)\,ds}), & 0 \le \overline{u} \le v_0 \end{cases} \tag{2.5}$$

and assume that $p > 0$ is continuous and that (2.5) has a solution $t \in C^1$ satisfying

$$t_{\overline{u}} > |t_v|. \tag{2.6}$$

Then (WE2) admits a Greenberg-Rascle type solution with parameters (x_I, t_I, u_0, v_0), where

$$x_I = c(0)\left(t_I + \int_0^{v_0} t_{\overline{u}}(s, 0)\,ds\right), \tag{2.7}$$

u_0 is such that $v_0 = \int_0^{u_0} c(s)\,ds$, and v_0 and t_I are as in (2.5).

Proof. To prove the theorem, we assume that t_I, v_0 are given, that $t \in C^1$ satisfies (2.5) and (2.6), and that x_I and u_0 are as described in the theorem. Our goal is to construct the interaction between expanding and focusing waves shown in Figure 2.

Define the variable u by the identity (2.2), let

$$x(v,\overline{u}) \stackrel{\text{def}}{=} \int_0^v c(u)t_{\overline{u}}(s,\overline{u})\,ds, \tag{2.8}$$

and extend variables x and t on the whole closed square

$$\Omega = \{(v,\overline{u}) : -v_0 - v \le \overline{u} \le v_0 - v, -v_0 \le v \le v_0\}$$

shown in Figure 3 by requiring that

(i) $t(v,-\overline{u}) = -t(v,\overline{u})$, $t(-v,\overline{u}) = t(v,\overline{u})$,

(ii) $x(v,-\overline{u}) = x(v,\overline{u})$, $x(-v,\overline{u}) = -x(v,\overline{u})$.

We note that x and t satisfy

$$x_v(v,\overline{u}) - c(u)t_{\overline{u}}(v,\overline{u}) = 0 \quad \text{and} \quad x_{\overline{u}}(v,\overline{u}) - c(u)t_v(v,\overline{u}) = 0 \quad \text{in } \Omega \tag{2.9}$$

and

$$t_{\overline{u}} > |t_v| \quad \text{in } \Omega. \tag{2.10}$$

We have the following lemma.

LEMMA 2.1. Suppose (x,t) satisfy (2.9) for $(v,\overline{u}) \in \Omega$. If, in addition, t satisfies (2.10) then the map $(v,u) \to (x,t)$ is one-to-one on Ω.

Proof of the lemma. Suppose the map is not one-to-one. Then there are points $(v_1,\overline{u}_1)$ and $(v_2,\overline{u}_2)$ in Ω with $(v_1,\overline{u}_1) \ne (v_2,\overline{u}_2)$ such that $t(v_1,\overline{u}_1) = t(v_2,\overline{u}_2)$ and $x(v_1,\overline{u}_1) = x(v_2,\overline{u}_2)$. Since Ω is convex, the identity

$$\begin{aligned}0 = t(v_1,\overline{u}_1) - t(v_2,\overline{u}_2) &= (v_1 - v_2)\int_0^1 t_v(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds \\ &+ (\overline{u}_1 - \overline{u}_2)\int_0^1 t_{\overline{u}}(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds,\end{aligned} \tag{2.11}$$

obtained by integrating the derivative, implies that

$$\frac{\overline{u}_1 - \overline{u}_2}{v_1 - v_2} = -\frac{\int_0^1 t_v(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds}{\int_0^1 t_{\overline{u}}(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds}, \tag{2.12}$$

while

$$\begin{aligned}0 = x(v_1,\overline{u}_1) - x(v_2,\overline{u}_2) &= (v_1 - v_2)\int_0^1 x_v(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds \\ &+ (\overline{u}_1 - \overline{u}_2)\int_0^1 x_{\overline{u}}(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds,\end{aligned} \tag{2.13}$$

together with the fact that x satisfies (2.9), implies that

$$\begin{aligned}0 = x(v_2,\overline{u}_2) - x(v_1,\overline{u}_1) &= (v_1 - v_2)\int_0^1 ct_{\overline{u}}(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds \\ &+ (\overline{u}_1 - \overline{u}_2)\int_0^1 ct_v(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds\end{aligned} \tag{2.14}$$

and (2.14) in turn yields

$$\frac{v_1 - v_2}{\overline{u}_1 - \overline{u}_2} = -\frac{\int_0^1 ct_v(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds}{\int_0^1 ct_{\overline{u}}(sv_1 + (1-s)v_2, s\overline{u}_1 + (1-s)\overline{u}_2)\,ds}. \tag{2.15}$$

The identity (2.12), together with (2.6), implies that

$$\left|\frac{\overline{u}_1 - \overline{u}_2}{v_1 - v_2}\right| < 1, \tag{2.16}$$

while (2.15) and (2.6) imply that

$$\left|\frac{v_1 - v_2}{\overline{u}_1 - \overline{u}_2}\right| < 1 \tag{2.17}$$

and (2.16) and (2.17) provide the desired contradiction. □

We note that the condition (2.10) also implies that the Jacobian of $(v, \overline{u}) \to (x, t)$ is bounded away from zero on Ω and, thus, the map $(v, \overline{u}) \to (x, t)$ constitutes a C^1 diffeomorphism.[2]

We write x and t in terms of variables (v, u), where u is defined by (2.2). The set Ω is now transformed onto a diamond-shaped set Ω_γ, in the (v, u)-plane, bounded by curves

$$v = \int_u^{u_0} c(s)\,ds, \qquad 0 \le u \le u_0, \tag{2.18}$$

$$v = -\int_u^{u_0} c(s)\,ds, \quad 0 \le u \le u_0, \tag{2.19}$$

$$v = -\int_{-u_0}^{u} c(s)\,ds, \quad -u_0 \le u \le 0, \tag{2.20}$$

$$v = \int_{-u_0}^{u} c(s)\,ds, \qquad -u_0 \le u \le 0. \tag{2.21}$$

Finally, we use the inverse hodograph transformation and regard v and u as functions of x and t. The set Ω_γ is now transformed onto a new set which we call Ω_Γ. We use the notation from Figure 2. Set Ω_Γ is bounded by curves $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 defined by

$$\Gamma_1 : (x(v_0 - \overline{u}, \overline{u}), t(v_0 - \overline{u}, \overline{u})), \qquad 0 \le \overline{u} \le v_0, \tag{2.22}$$

$$\Gamma_2 : (x(v_0 + \overline{u}, \overline{u}), t(v_0 + \overline{u}, \overline{u})), \qquad -v_0 \le \overline{u} \le 0, \tag{2.23}$$

$$\Gamma_3 : (x(-v_0 - \overline{u}, \overline{u}), t(-v_0 - \overline{u}, \overline{u})), \quad -v_0 \le \overline{u} \le 0, \tag{2.24}$$

$$\Gamma_4 : (x(-v_0 + \overline{u}, \overline{u}), t(-v_0 + \overline{u}, \overline{u})), \quad 0 \le \overline{u} \le v_0, \tag{2.25}$$

and simple calculations show that v and u satisfy (WE2) on Ω_Γ.

As we will show, Ω_Γ is the region of interaction between forward and backward simple waves shown in Figure 3.

The symmetries (i), (ii) guarantee that the triangle

$$\Omega_{++} = \Omega \cap \{(\overline{u}, v) : \overline{u} \ge 0, v \ge 0\}$$

is mapped onto $\Omega_\Gamma \cap \{(x,t) : x \ge 0, t \ge 0\}$.

We observe that the solution $(x,t) \to (v,u)$ defined on Ω_Γ and satisfying (i) *and* (ii) *represents the interaction of forward and backward simple waves emanating from points*

[2]The fact that the Jacobian is bounded away from zero follows from the compactness of Ω.

$(-x_I,-t_I)$ and $(x_I,-t_I)$ and reconverging at points $(-x_I,t_I)$ and (x_I,t_I) and bounded by the constant states shown in Figure 2 if and only if the rays emanating from points $(x(v_0-\overline{u},\overline{u}),t(v_0-\overline{u},\overline{u}))\in\Gamma_1$ with speed $c(u)$ pass through (x_I,t_I), that is, if

$$c(u)=\frac{x(v_0-\overline{u},\overline{u})-x_I}{t(v_0-\overline{u},\overline{u})-t_I},\qquad 0\le\overline{u}\le v_0. \tag{2.26}$$

To show (2.26) we rewrite $(2.5)_4$,

$$-\frac{1}{2}\int_0^{\overline{u}}p(s)\,ds=\ln(t_I-t(v_0-\overline{u},\overline{u}))-\ln(t_I), \tag{2.27}$$

and differentiate (2.27) using the identity $p(\overline{u})=\frac{c'(u)}{c^2(u)}$ to get

$$2c^2(u)(t_{\overline{u}}-t_v)(v_0-\overline{u},\overline{u})=\frac{dc}{du}(t_I-t(v_0-\overline{u},\overline{u})). \tag{2.28}$$

Equations (2.9) yield

$$c'(u)(\overline{t}(\overline{u})-t_I)+c^2(u)\frac{d\overline{t}}{d\overline{u}}=c\frac{d\overline{x}}{d\overline{u}}, \tag{2.29}$$

where $\overline{t}(\overline{u})=t(v_0-\overline{u},\overline{u})$ and $\overline{x}(\overline{u})=x(v_0-\overline{u},\overline{u})$. Note that

$$\overline{x}(0)=x(v_0,0)=c(0)\int_0^{v_0}t_{\overline{u}}(s,0)\,ds$$

and that the condition (2.7) can be written as

$$\overline{x}(0)=x_I-t_Ic(0). \tag{2.30}$$

We integrate (2.29) with respect to u and utilize (2.30) and the fact that $\overline{t}(0)=0$ and $d\overline{u}=c\,du$ to obtain

$$(t(v_0-\overline{u},\overline{u})-t_I)c(u)=x(v_0-\overline{u},\overline{u})-x_I. \tag{2.31}$$

This proves (2.26) and show that, along the curve Γ_1, the constructed functions (u,v) match a simple backward wave focusing at (x_I,t_I) and connecting constant states $(v_0,0)$ and $(0,u_0)$.

We also note that, if (2.26) holds, then the fact that $c(-u)=c(u)$ and the symmetry conditions (i), (ii) guarantee that the rays emanating from $\begin{pmatrix}(x,t)\in\Gamma_2\\(x,t)\in\Gamma_3\\(x,t)\in\Gamma_4\end{pmatrix}$ with slope $\begin{pmatrix}-c(u)\\c(u)\\-c(u)\end{pmatrix}$ pass through $\begin{pmatrix}(-x_I,t_I)\\(-x_I,-t_I)\\(x_I,-t_I)\end{pmatrix}$. Thus, constructed interaction matches, along curves Γ_2,Γ_3 and Γ_4, appropriate simple waves defined on D_2,D_3 and D_4 and connecting constant states

$$(v,u)=\begin{cases}(0,u_0), & (x,t)\in N\\(0,-u_0), & (x,t)\in S\\(v_0,0), & (x,t)\in E\\(-v_0,0), & (x,t)\in W\end{cases} \tag{2.32}$$

(see Fig. 3). This concludes our construction of (v,u) inside the rectangle K_{x_I,t_I}.

To construct spatially and temporally periodic solutions we superpose odd and even reflections of the solution defined inside K_{x_I,t_I} (as shown in Figure 1). The obtained solution is spatially and temporally periodic with periods $4x_I$ and $4t_I$.

3. Technical lemmas. Let L denote the characteristic triangle in the $(v,\overline{u})$-plane bounded by the lines $\overline{u}=0, \overline{u}=v$ and $\overline{u}=v_0-v$ (see Figure 3). Let $\gamma, h \in C^1([0,v_0/2])$ be such that $h(0)=0$.

We prove the following.

LEMMA 3.1. Assume $v_0^2\|\gamma\|_{L^\infty[0,v_0/2]} \leq 16$. The boundary value problem

$$\begin{cases} \hat{t}_{\overline{u}\overline{u}} - \hat{t}_{vv} + \gamma(\overline{u})\hat{t} = 0 & \text{in } L, \\ \hat{t}(v,0) = 0 & \text{for } 0 \leq v \leq v_0, \\ \hat{t}(v_0-\overline{u},\overline{u}) = h(\overline{u}) & \text{for } 0 \leq \overline{u} \leq v_0/2 \end{cases} \tag{3.1}$$

has a unique weak solution $\hat{t} \in C^1(L)$.

Proof. For convenience, we transform the problem to diagonal (ξ,η)-coordinates defined by

$$\eta = \frac{\overline{u}+v}{\sqrt{2}}, \qquad \xi = \frac{v-\overline{u}}{\sqrt{2}} \tag{3.2}$$

and let

$$\eta_0 = \frac{v_0}{\sqrt{2}}, \quad \hat{h}(s) = h\left(-\frac{s}{\sqrt{2}} + \frac{v_0}{2}\right).$$

Note that $\hat{t}$ satisfies (3.1) if and only if

$$\hat{t}(\xi,\eta) = \hat{h}(\xi) - \hat{h}(\eta) + \frac{1}{2}\int_\eta^{\eta_0}\int_\xi^\eta \gamma\hat{t}(\overline{\xi},\overline{\eta})\,d\overline{\xi}\,d\overline{\eta}. \tag{3.3}$$

The reader is advised to consult Figure 4.

We let

$$T(t)(\xi,\eta) = \hat{h}(\xi) - \hat{h}(\eta) + \frac{1}{2}\int_\eta^{\eta_0}\int_\xi^\eta \gamma t(\overline{\xi},\overline{\eta})\,d\overline{\xi}\,d\overline{\eta} \tag{3.4}$$

and our goal is to find $\hat{t}$ such that $T(\hat{t}) = \hat{t}$.

We note that for any $\hat{t}_1$ and $\hat{t}_2 \in C(L)$

$$|(T(\hat{t}_1) - T(\hat{t}^2))(\xi,\eta)| = \frac{1}{2}\left|\int_\eta^{\eta_0}\int_\xi^\eta \gamma(\hat{t}_1-\hat{t}_2)\,d\overline{\xi}\,d\overline{\eta}\right| \leq \|\gamma\|_{L^\infty[0,v_0/2]}\frac{\eta_0^2}{8}\|\hat{t}_1-\hat{t}_2\|_{L^\infty(L)} \tag{3.5}$$

and, thus, T is a contraction on $C(L)$. Let $\hat{t}$ be the unique fixed point of $T : T(\hat{t}) = \hat{t}$. From the definition of T, $\hat{t}$ satisfies (3.3). The fact that $\hat{t} \in C^1(L)$ follows immediately from (3.3). □

LEMMA 3.2. Assume

$$\|\gamma\|_{L^\infty[0,v_0/2]}v_0^2 \leq 8 \tag{3.6}$$

and that $h \in C^2([0,v_0/2])$ satisfies $h,\ h' \geq 0$.

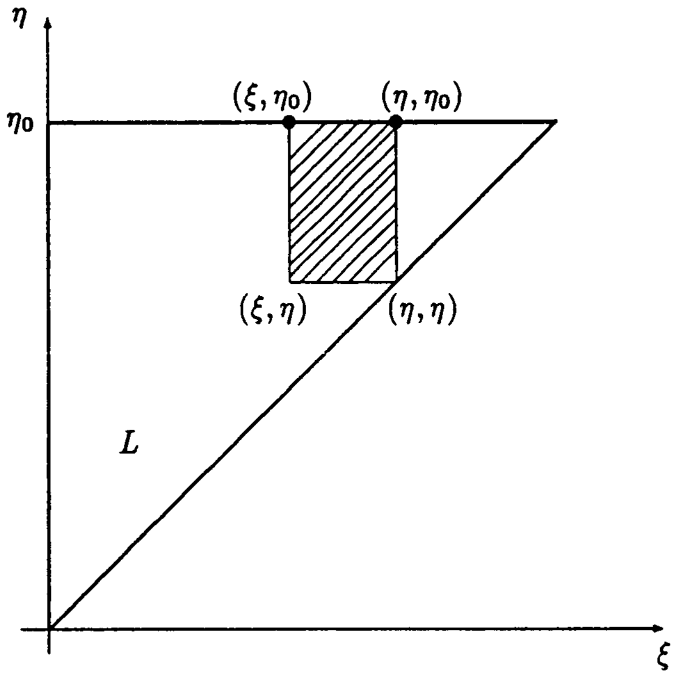

FIG. 4

Then the solution $\hat{t}$, constructed in the previous lemma, satisfies $\hat{t}_{\overline{u}}(\cdot, 0) \in C^1$ and

$$\|\hat{t}\|_{L^\infty(L)} \le v_0 \|h'\|_{L^\infty[0,v_0/2]}, \tag{3.7}$$

$$\hat{t}_{\overline{u}}(v,0) \ge \min_{s\in[0,v_0/2]} h'(s) - v_0^2 \|\gamma\|_{L^\infty[0,v_0/2]} \|h'\|_{L^\infty[0,v_0/2]}, \tag{3.8}$$

$$\hat{t}_{\overline{u}}(v,0) \le \|h'\|_{L^\infty[0,v_0/2]} (1 + v_0^2 \|\gamma\|_{L^\infty[0,v_0/2]}), \tag{3.9}$$

$$\begin{aligned} |\hat{t}_{\overline{u}v}(v,0)| \le \frac{\|h''\|_{L^\infty[0,v_0/2]}}{2} &+ \frac{v_0}{4} \|h'\|_{L^\infty[0,v_0/2]} (\|\gamma\|_{L^\infty[0,v_0/2]} \\ &+ \frac{v_0^2}{2} \|\gamma\|^2_{L^\infty[0,v_0/2]} + v_0 \|\gamma'\|_{L^\infty[0,v_0/2]}). \end{aligned} \tag{3.10}$$

Proof. Again, it is more convenient to work in (ξ, η) coordinates. We observe that $\hat{h}(\xi) - \hat{h}(\eta) \le \eta_0 \|\hat{h}'\|_\infty$ and apply identity (3.3) to obtain

$$\|\hat{t}\|_{L^\infty(L)} \le \eta_0 \|\hat{h}'\|_\infty + \frac{\eta_0^2}{8} \|\gamma\|_\infty \|\hat{t}\|_{L^\infty(L)} \le \frac{v_0}{2} \|h'\|_{L^\infty[0,v_0/2]} + \frac{v_0^2}{16} \|\gamma\|_\infty \|\hat{t}\|_{L^\infty(L)}. \tag{3.11}$$

Inequalities (3.11) and (3.6) yield (3.7).

Differentiating (3.3) and using $\hat{t}_{\overline{u}} = \frac{1}{\sqrt{2}}(t_\eta - t_\xi)$ gives

$$\frac{\partial}{\partial \overline{u}} \hat{t}|_{(\xi,\eta)=(s,s)} = \frac{1}{\sqrt{2}} \left[-2\hat{h}'(s) + \int_s^{\eta_0} (\gamma \hat{t})(s,r)\, dr \right] \tag{3.12}$$

which, in turn, implies (3.8), (3.9) and the fact that $\hat{t}_{\overline{u}}(\cdot, 0) \in C^1$.

Inequality (3.10) is obtained by differentiating the identity (3.12) in the v direction

$$\frac{\partial^2}{\partial \overline{u}\,\partial v}\hat{t}|_{(\xi,\eta)=(s,s)} = \frac{1}{2}\left[-2\hat{h}''(s) + \int_s^{\eta_0} (\gamma\hat{t})_\xi(s,r)\,dr\right]$$

and using the fact that $\hat{t}_\xi(s,r) = \hat{h}'(s) + \frac{1}{2}\int_r^{\eta_0}(\gamma\hat{t})(s,\overline{r})\,d\overline{r}$. □

Let Ω_+ be the characteristic triangle bounded by the lines $\overline{u} = 0, \overline{u} + v = v_0$, and $\overline{u} - v = v_0$ (see Fig. 3). Consider the following initial value problem:

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p(\overline{u})t_{\overline{u}} = Q(v,\overline{u}) & \text{in } \Omega_+, \\ t(v,0) = 0, & -v_0 \le v \le v_0, \\ t_u(v,0) = g(v), & -v_0 \le v \le v_0, \end{cases} \tag{3.13}$$

where $Q \in L^\infty(\Omega_+)$, $p \in C([0,v_0])$ is nonnegative, and $g \in C([-v_0,v_0])$.

LEMMA 3.3. The solution of (3.13) satisfies

$$\|t_{\overline{u}}(\cdot,\overline{u})\|_\infty \le e^{\int_0^{\overline{u}} p(s)\,ds}\left(\|g\|_{L^\infty([-v_0,v_0])} + \int_0^{\overline{u}} e^{-\int_0^s p(r)\,dr}\|Q(\cdot,s)\|_\infty\,ds\right). \tag{3.14}$$

Proof. We differentiate the identity

$$t(v,\overline{u}) = \frac{1}{2}\int_{v-\overline{u}}^{v+\overline{u}} g(s)\,ds + \frac{1}{2}\int_0^{\overline{u}}\int_{v-s}^{v+s} Q(r,\overline{u}-s) - pt_{\overline{u}}(r,\overline{u}-s)\,dr\,ds \tag{3.15}$$

to obtain

$$\begin{aligned} t_{\overline{u}}(v,\overline{u}) &= \tfrac{1}{2}(g(v+\overline{u}) + g(v-\overline{u})) \\ &\quad + \frac{1}{2}\int_0^{\overline{u}}\Big(Q(v-\overline{u}+s,s) + Q(v+\overline{u}-s,s) \\ &\quad - pt_{\overline{u}}(v-\overline{u}+s,s) - pt_{\overline{u}}(v+\overline{u}-s,s)\Big)\,ds \end{aligned} \tag{3.16}$$

and

$$\|t_{\overline{u}}(\cdot,\overline{u})\|_\infty \le \|g\|_{L^\infty([-v,v_0])} + \int_0^{\overline{u}} \|Q(\cdot,s)\|_\infty + p(s)\|t_{\overline{u}}(\cdot,s)\|_\infty\,ds. \tag{3.17}$$

The generalized Gronwall inequality yields (3.14). □

COROLLARY 3.1. Assume that t satisfies (3.13) and that $Q \equiv 0$. Then

$$\|t_{\overline{u}v}(\cdot,\overline{u})\|_\infty \le e^{\int_0^{\overline{u}} p(s)\,ds}\|g'\|_{L^\infty([-v_0,v_0])}, \tag{3.18}$$

$$\|t_v(\cdot,\overline{u})\|_\infty \le \overline{u}e^{\int_0^{\overline{u}} p(s)\,ds}\|g'\|_{L^\infty([-v_0,v_0])}, \tag{3.19}$$

and

$$\|t_{\overline{u}\overline{u}}(\cdot,\overline{u})\|_\infty \le \left(1 + \int_0^{\overline{u}} p(s)e^{\int_0^s p(r)\,dr}ds\right)\|g'\|_\infty + e^{\int_0^{\overline{u}} p(r)\,dr}\|g\|_\infty. \tag{3.20}$$

If, in addition, $\mu = \min_v g(v)$ satisfies

$$\mu - v_0\|p\|_{L^\infty([0,v_0])}e^{v_0\|p\|_{L^\infty([0,v_0])}}\|g\|_{L^\infty([-v_0,v_0])} \ge 0, \tag{3.21}$$

then

$$\|t_{\overline{u}}(\cdot,\overline{u})\|_\infty \le \|g\|_{L^\infty([-v_0,v_0])}, \tag{3.22}$$

$$\|t(\cdot,\overline{u})\|_\infty \le \overline{u}\|g\|_{L^\infty([-v_0,v_0])}, \tag{3.23}$$

and

$$\min_{v\in[-v_0+\overline{u},v_0-\overline{u}]} t_{\overline{u}}(v,\overline{u}) \ge \mu - \|g\|_{L^\infty([-v_0,v_0])}\int_0^{\overline{u}} p(s)\,ds. \tag{3.24}$$

Proof. Inequalities (3.19) and (3.18) are a direct consequence of (3.14) once the initial data g in $(3.13)_3$ is replaced by g'. We calculate $t_{\overline{u}\overline{u}}$ from (3.16),

$$\begin{aligned} t_{\overline{u}\overline{u}}(v,\overline{u}) &= \tfrac{1}{2}(g'(v+\overline{u}) - g'(v-\overline{u})) - p(\overline{u})t_{\overline{u}}(v,\overline{u}) \\ &\quad + \frac{1}{2}\int_0^{\overline{u}} p(s)(t_{\overline{u}v}(v-\overline{u}+s,s) - t_{\overline{u}v}(v+\overline{u}-s,s))\,ds, \end{aligned} \tag{3.25}$$

and use (3.18) and (3.14) to obtain (3.20). Inequality (3.23) follows immediately from (3.22). To prove the remaining inequalities we observe that (3.16) implies

$$\begin{aligned} \min_v t_{\overline{u}}(v,\overline{u}) &\ge -\frac{1}{2}\int_0^{\overline{u}} pt_{\overline{u}}(v-\overline{u}+s,s) + pt_{\overline{u}}(v+\overline{u}-s,s))\,ds \\ &\ge \mu - \int_0^{\overline{u}} pe^{\int_0^s p}\|g\|_{L^\infty([-v_0,v_0])}\,ds \\ &\ge \mu - v_0\|p\|_{L^\infty([0,v_0])}e^{v_0\|p\|_{L^\infty([0,v_0])}}\|g\|_{L^\infty([-v_0,v_0])} \end{aligned} \tag{3.26}$$

and that the condition (3.21) implies that $t_{\overline{u}} \ge 0$ on Ω_+. This together with (3.16) and the fact that $p > 0$ give (3.22). Finally, the first inequality in (3.26) together with (3.22) yield (3.24). □

COROLLARY 3.2. If $p \in C([0,v_0])$, $Q \equiv 0$, and $g \in C([-v_0,v_0])$, then the solution t of (3.13) satisfies $t \in C^1(\Omega_+)$.

If, in addition, g satisfies

$$|g'(v)| \le \text{Const a.e.}$$

then $t_{\overline{u}}$ and t_v are Lipschitz continuous on Ω_+.

Proof. Inequality (3.14) can be used to show that $t_{\overline{u}} \in C(\Omega_+)$. This and the identity (3.15) imply that $t_v \in C(\Omega_+)$. The Lipschitz continuity of the derivatives follows from (3.18), (3.20), and (3.14). □

4. The main result. Assume we are given positive parameters v_0 and t_I and a function $p_G \in C^2(R_+)$ satisfying

$$p_G \ge \delta > 0. \tag{4.1}$$

Let $r_0 > 0$ be an arbitrary constant and define

$$\gamma(\overline{u}) \overset{\text{def}}{=} -\tfrac{1}{4}p_G^2(\overline{u}) - \tfrac{1}{2}p_G'(\overline{u}). \tag{4.2}$$

Our concern is in verifying whether the boundary value problem (2.5) admits a solution satisfying (2.6). We will prove the following.

THEOREM 4.1. There exists a constant

$$\varepsilon = \varepsilon(\|p_G\|_{L^\infty(0,r_0)}, \|p'_G\|_{L^\infty[0,r_0]}, \|p''_G\|_{L^\infty[0,r_0]}, \delta, t_I) \le 2r_0$$

such that if $v_0 \le \varepsilon$, then there exists a unique Lipschitz continuous function p_N satisfying $p_N(\overline{u}) = p_G(\overline{u})$ for $\overline{u} \in [0, v_0/2]$ and such that

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p_N(\overline{u})t_{\overline{u}} = 0 & \text{in } \Omega_{++} = \{(v,\overline{u}) : v \ge 0, 0 \le \overline{u} \le v_0 - v\}, \\ t(v,0) = 0, & 0 \le v \le v_0, \\ t_v(0,\overline{u}) = 0, & 0 \le \overline{u} \le v_0, \\ t(v_0 - \overline{u}, \overline{u}) = t_I(1 - e^{-\frac{1}{2}\int_0^{\overline{u}} p_N(s)\,ds}), & 0 \le \overline{u} \le v_0 \end{cases} \tag{4.3}$$

has a solution $t \in C^1$ satisfying

$$t_{\overline{u}} > |t_v|. \tag{4.4}$$

REMARKS. We explain the meaning of the word "unique" used in the above theorem. As will become clear in the proof, if $p \in C([0, v_0])$ is such that $p|_{[0,v_0/2]} \equiv p_G|_{[0,v_0/2]}$ and if there exists $v_0/2 < v \le v_0$ for which $p(v) \ne p_N(v)$, then (2.5) has NO SOLUTIONS.

We note that, if B is a bounded subset of $C^2([0, v_0/2])$ satisfying

$$B \subset \{p \in C^2([0, v_0/2]) : p \ge \delta > 0\}$$

and if v_0 is sufficiently small, then Theorem 4.1 defines a map $p_G|_{[0,v_0/2]} \in B \to p_N|_{[v_0/2,v_0]} \in C([v_0/2, v_0])$. It can be shown using estimates similar to those developed in the last section that, for small v_0, the map $p_G|_{[0,v_0/2]} \in B \to p_N|_{[v_0/2,v_0]} \in C([v_0/2, v_0])$ is continuous (see how p_N is constructed in the proof of Theorem 4.1).

We delay the proof of the theorem to explicitly list the smallness conditions on v_0. To simplify our notation we define

$$P_0 = \frac{t_I}{2}\|p_G\|_{L^\infty[0,v_0/2]} e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}}(1 + v_0^2\|\gamma\|_{L^\infty[0,v_0/2]}), \tag{4.5}$$

$$\begin{aligned} P_1 = \frac{t_I}{2} e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}} \Big\{ &\|\gamma\|_{L^\infty[0,v_0/2]} + \frac{v_0}{4}\|p_G\|_{L^\infty[0,v_0/2]}(\|\gamma\|_{L^\infty[0,v_0/2]} \\ &+ \frac{v_0^2}{2}\|\gamma\|_{L^\infty[0,v_0/2]} + v_0\|\gamma'\|_{L^\infty[0,v_0/2]})\Big\}, \end{aligned} \tag{4.6}$$

$$K = \frac{t_I}{2}(\delta - v_0^2\|\gamma\|_{L^\infty[0,v_0/2]}\|p_G\|_{L^\infty[0,v_0/2]} e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}}). \tag{4.7}$$

Following are the sufficient conditions on the parameters t_I and v_0 to construct a Greenberg-Rascle type solution:

$$\|\gamma\|_{L^\infty[0,v_0/2]}v_0^2 \leq 8, \tag{4.8}$$

$$v_0 P_0 \geq t_I/2, \tag{4.9}$$

$$v_0 P_1 e^{8v_0P_0/t_I} \leq P_0, \tag{4.10}$$

$$K \geq 8v_0 P_0^2 e^{8v_0P_0/t_I}/t_I, \tag{4.11}$$

$$K \geq v_0\left(P_0^2\frac{8}{t_I} + \exp(8v_0P_0/t_I)P_1\right), \tag{4.12}$$

$$\lambda = v_0 t_I\left\{P_0 + v_0P_1 + v_0\frac{t_I}{2}(P_0^2 + P_0P_1e^{8v_0P_0/t_I})\right\}e^{8v_0P_0/t_I} \leq 1. \tag{4.13}$$

Proof of Theorem 4.1. Our goal is to construct functions p_N and t that satisfy (4.3), (4.4) and such that $p_N(\overline{u}) = p_G(\overline{u})$ for $\overline{u} \in [0, v_0/2]$. First we construct solution t in the characteristic triangle, L, bounded by the lines $\overline{u} = 0, \overline{u} = v$, and $\overline{u} = v_0 - v$. Define

$$h(\overline{u}) = t_I(e^{\frac{1}{2}\int_0^{\overline{u}} p_G(s)\,ds} - 1), \qquad 0 \leq \overline{u} \leq v_0/2, \tag{4.14}$$

and let γ be such as in (4.2). We note that t is a solution of $(4.3)_{1,2,4}$ on L if and only if

$$\hat{t} \stackrel{\text{def}}{=} te^{\frac{1}{2}\int_0^{\overline{u}} p_G(s)\,ds} \tag{4.15}$$

satisfies

$$\begin{cases} \hat{t}_{\overline{u}\overline{u}} - \hat{t}_{vv} + \gamma(\overline{u})\hat{t} = 0 & \text{in } L, \\ \hat{t}(v,0) = 0, & 0 \leq v \leq v_0, \\ \hat{t}(v_0 - \overline{u}, \overline{u}) = h(\overline{u}), & 0 \leq \overline{u} \leq v_0. \end{cases} \tag{4.16}$$

Lemma 3.1 together with assumption (4.8) yield the existence of solution $\hat{t}$ to (4.16) in L. Thus, $(4.3)_{124}$ has a unique solution t defined on L by (4.15).

To construct the solution t of (4.3) in the whole region Ω_{++} we define

$$g(v) = \begin{cases} t_{\overline{u}}(v,0) = \hat{t}_{\overline{u}}(v,0), & \text{if } v > 0, \\ t_{\overline{u}}(-v,0) = \hat{t}_{\overline{u}}(-v,0), & \text{if } v < 0. \end{cases} \tag{4.17}$$

Function g is Lipschitz continuous on $[-v_0, v_0]$ and C^1 away from $v = 0$. Lemma 3.2 and (4.8) imply that g satisfies the following conditions:

$$\mu = \min_{v\in[0,v_0/2]} g(v) \geq \frac{t_I}{2}(\delta - v_0^2\|\gamma\|_{L^\infty[0,v_0/2]}\|p_G\|_{L^\infty[0,v_0/2]}e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}}), \tag{4.18}$$

$$\|g\|_{L^\infty[-v_0/2,v_0/2]} \leq \frac{t_I}{2}\|p_G\|_{L^\infty[0,v_0/2]}e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}}(1 + v_0^2\|\gamma\|_{L^\infty[0,v_0/2]}), \tag{4.19}$$

$$\begin{aligned}\|g'\|_{L^\infty[-v_0/2,v_0/2]} \leq \frac{t_I}{2}e^{\frac{1}{4}v_0\|p_G\|_{L^\infty[0,v_0/2]}}\{\|\gamma\|_{L^\infty[0,v_0/2]} + \frac{v_0}{4}\|p_G\|_{L^\infty[0,v_0/2]} \\ \times (\|\gamma\|_{L^\infty[0,v_0/2]} + \frac{v_0^2}{2}\|\gamma\|_{L^\infty[0,v_0/2]} + v_0\|\gamma'\|_{L^\lambda[0,v_0/2]})\}.\end{aligned} \tag{4.20}$$

These conditions can be simplified using (4.5)–(4.7),

$$\mu = \min_{v_0 \in [0, v_0/2]} g(v) \geq K, \tag{4.21}$$

$$\|g\|_{L^\infty[-v_0/2, v_0/2]} \leq P_0, \tag{4.22}$$

$$\|g'\|_{L^\infty[-v_0/2, v_0/2]} \leq P_1. \tag{4.23}$$

We write (4.3) in the following form:

$$\begin{cases} t_{\overline{u}\overline{u}} - t_{vv} + p_N(\overline{u})t_{\overline{u}} = 0 & \text{in } \Omega_{++} = \{(v, \overline{u}) : v \geq 0, 0 \leq \overline{u} \leq v_0 - v\}, \\ t(v, 0) = 0, & 0 \leq v \leq v_0, \\ t_v(0, \overline{u}) = 0, & 0 \leq \overline{u} \leq v_0, \\ p_N(\overline{u}) = 2[(t_{\overline{u}} - t_v)(t_I - t)^{-1}](v_0 - \overline{u}, \overline{u}), & 0 \leq \overline{u} \leq v_0. \end{cases} \tag{4.24}$$

Let $t^{gp} \in C^1(\Omega_+)$ denote the solution[3] to the following initial value problem:

$$\begin{cases} \frac{\partial^2}{\partial \overline{u}^2} t^{gp} - \frac{\partial^2}{\partial v^2} t^{gp} + p(\overline{u}) \frac{\partial}{\partial \overline{u}} t^{gp} = 0 & \text{in } \Omega_+ \\ t^{gp}(v, 0) = 0, & -v_0 \leq v \leq v_0, \\ \frac{\partial}{\partial \overline{u}} t^{gp}(v, 0) = g(v), & -v_0 \leq v \leq v_0, \end{cases} \tag{4.25}$$

where p is a given continuous, nonnegative function, g is as in (4.17), and Ω_+ is the characteristic triangle bounded by lines $\overline{u} = 0, \overline{u} + v = v_0$, and $\overline{u} - v = v_0$ (see Fig. 3). We observe that (4.17) implies

$$\frac{\partial}{\partial v} t^{gp}(0, \overline{u}) = 0, \qquad 0 \leq \overline{u} \leq v_0. \tag{4.26}$$

We define the mapping S_g,

$$S_g(p)(\overline{u}) = 2 \frac{\frac{\partial}{\partial \overline{u}} t^{gp}(v_0 - \overline{u}, \overline{u}) - \frac{\partial}{\partial v} t^{gp}(v_0 - \overline{u}, \overline{u})}{t_I - t^{gp}(v_0 - \overline{u}, \overline{u})}, \tag{4.27}$$

and observe that if p_f is a fixed-point of S_g (i.e., if $S_g(p_f) = p_f$) then t^{gp_f} satisfies (4.24). Our goal is to show that S_g has a fixed-point p_f such that $p_f = p_G$ on $[0, v_0/2]$.

Let

$$D_M = \{p \in C([0, v_0]) : p = p_G \text{ on } [0, v_0/2] \text{ and } 0 \leq p \leq M\}, \tag{4.28}$$

where

$$M = 8P_0/t_I$$

and note that D_M is a convex, closed subspace of $C([0, v_0])$. We will show that $S_g : D_M \to D_M$ and that S_g is a contraction on D_M.

Let $p \in D_M$; the identities

$$S_g(p)(\overline{u}) = p_G(\overline{u}) \quad \text{for } \overline{u} \in [0, v_0/2]$$

and

$$t^{gp} = t \quad \text{on } L$$

follow directly from the definition of g.

[3] See Corollary 3.2.

Condition (4.11) implies that the assumption (3.21) of Corollary 3.1 is satisfied for $p \in B_M$ and g defined in (4.17). We need to show that $0 \leq S_g(p) \leq M$. To show the second inequality we apply Corollary 3.1 and use the assumption (4.9) which, together with (3.23), implies that $t^{gp} \leq \frac{t_I}{2}$, $\frac{1}{t_I - t^{gp}} \leq \frac{2}{t_I}$, and

$$S_g(p) = 2\frac{\frac{\partial}{\partial \overline{u}} t^{gp}(v_0 - \overline{u}, \overline{u}) - \frac{\partial}{\partial v} t^{gp}(v_0 - \overline{u}, \overline{u})}{t_I - t^{gp}(v_0 - \overline{u}, \overline{u})} \leq \frac{4}{t_I}(\|g\|_\infty + v_0 e^{v_0 M}\|g'\|_\infty). \tag{4.29}$$

Inequalities (4.29), (4.22), (4.23), and (4.10) imply that $S_g(p) \leq \frac{4}{t_I}(\|g\|_\infty + P_0) \leq M$.

The inequality $S_g(p) \geq 0$ follows from the fact that $\frac{\partial}{\partial \overline{u}} t^{gp} \geq |\frac{\partial}{\partial v} t^{gp}|$ which will also imply the bijectivity condition (4.4). To see that, we apply Corollary 3.1, (4.21)–(4.23), and (4.12) to obtain

$$\begin{aligned} \frac{\partial}{\partial \overline{u}} t^{gp} &\geq \mu - v_0\|g\|_\infty M \geq K - v_0 \frac{8}{t_I} P_0^2 \\ &\geq v_0 \exp(8 v_0 P_0 / t_I) P_1 \geq v_0 e^{v_0 M} \|g'\|_{L^\infty([-v_0, v_0])} \geq |t_v(\cdot, \overline{u})|. \end{aligned} \tag{4.30}$$

We still need to show that S_g is a contraction on D_M. To do that let $p_1, p_2 \in D_M$ and $t^1 = t^{gp_1}$, $t^2 = t^{gp_2}$ and define $w = t^1 - t^2$. We note that w satisfies

$$\begin{cases} w_{\overline{u}\overline{u}} - w_{vv} + p_1(\overline{u}) w_{\overline{u}} = (p_2 - p_1) t^2_{\overline{u}} & \text{in } \Omega_+, \\ w_{\overline{u}}(v, 0) = w(v, 0) = 0, & -v_0 \leq v \leq v_0. \end{cases} \tag{4.31}$$

Applying Lemma 3.3 and Corollary 3.1 gives

$$\begin{aligned} \|w_{\overline{u}}\|_\infty &\leq \|p_1 - p_2\|_{L^\infty[0, v_0]} \|g\|_{L^\infty([-v_0, v_0])} \int_0^{\overline{u}} e^{\int_s^{\overline{u}} p_1(r)\, dr} \\ &\leq v_0 P_0 e^{8 v_0 P_0 / t_I} \|p_1 - p_2\|_{L^\infty[0, v_0]}, \end{aligned} \tag{4.32}$$

$$\|w\|_\infty \leq v_0^2 P_0 e^{8 v_0 P_0 / t_I} \|p_1 - p_2\|_{L^\infty[0, v_0]} \tag{4.33}$$

and

$$\|w_v\|_\infty \leq v_0^2 \|g'\|_\infty e^M \|p_1 - p_2\|_\infty \leq v_0^2 P_1 e^{8 v_0 P_0 / t_I} \|p_1 - p_2\|_{L^\infty[0, v_0]}. \tag{4.34}$$

Assumption (4.13) yields

$$\begin{aligned} \frac{1}{2}|(S_g(p_1) - S_g(p_2))| &= \left| \left(\frac{t^1_{\overline{u}} - t^1_v}{t_I - t^1} - \frac{t^2_{\overline{u}} - t^2_v}{t_I - t^2} \right) (v_0 - \overline{u}, \overline{u}) \right| \\ &= \left| \left(\frac{w_{\overline{u}} - w_v}{t_I - t^1} + (t^2_{\overline{u}} - t^1_v) \left(\frac{w}{(t_I - t^1)(t_I - t^2)} \right) \right) (v_0 - \overline{u}, \overline{u}) \right| \\ &\leq \frac{t_I}{2}(|w_{\overline{u}}| + |w_v|) + \frac{t_I^2}{4}(|t^2_{\overline{u}}| + |t^1_v|)|w| \\ &\leq v_0\{P_0 + v_0 P_1 + v_0 \frac{t_I}{2}(P_0^2 + P_0 P_1 e^{8 v_0 P_0 / t_I})\} \frac{t_I}{2} e^{8 v_0 P_0 / t_I} \|p_1 - p_2\|_{L^\infty[0, v_0]} \\ &\leq \frac{\lambda}{2} \|p_1 - p_2\|_{L^\infty[0, v_0]} \end{aligned}$$

and, thus, S_g is a contraction as stated. We have constructed functions $p_N \stackrel{\text{def}}{=} p_f$ and $t \stackrel{\text{def}}{=} t^{g,p_f}$ satisfying (4.3). Estimates (4.30) show that t satisfies (4.4). We note that (4.27) and Corollary 3.2 imply that p_N is Lipschitz continuous.

To prove that function p_N is determined uniquely we let p_1 and t_1 satisfy (4.3) and (4.4), and let $p_1(\overline{u}) = p_G(\overline{u})$ for $\overline{u} \in [0, v_0/2]$. We note that t_1 has to satisfy (4.16) on L and, thus, $t_1 = t^{gp_1}$. We also note that Corollary 3.1 and conditions (4.8)–(4.12) imply that $p_1 \in B_M$.[4] Thus, $p_1 \in B_M$ is a fixed point of S_g. Since S_g is a contraction on B_M, we conclude that $p_1 \equiv p_N$. This observation ends the proof of Theorem 4.1. □

Now we return our attention to elasticity equations (WE2). Assume we have a given $c_G \in C(R) \cap C^3(R_+)$ satisfying

$$c_G(-u) = c_G(u) \geq c_{G0} \stackrel{\text{def}}{=} c_G(0) > 0 \quad \text{and} \quad c_G'(u) > 0, \qquad u > 0 \tag{4.35}$$

and let Π_G denote the primitive function of c_G,

$$\Pi_G(u) \stackrel{\text{def}}{=} \int_0^u c_G(s)\,ds, \qquad u > 0. \tag{4.36}$$

Function Π_G satisfies

$$\Pi_G, \Pi_G', \Pi_G'' > 0 \quad \text{for } u > 0. \tag{4.37}$$

We assume that t_I, v_0, and $u_{1/2}$ are positive parameters related by

$$\Pi_G(u_{1/2}) = v_0/2. \tag{4.38}$$

The following is an immediate consequence of Theorems 2.1 and 4.1 and is the main result of this paper.

THEOREM 4.2. Assume that v_0 is sufficiently small. There exist $x_I > 0$ and a unique function $c_N \in C(R) \cap C^1(R_+)$ satisfying (1.1) and

$$c_N(u) = c_G(u) \quad \text{for } u \in [-u_{1/2}, u_{1/2}],$$

and such that

$$u_{,t} - v_{,x} = 0 \quad \text{and} \quad v_{,t} - c_N^2(u)u_{,x} = 0 \tag{WE}$$

admit a Greenberg-Rascle type solution with parameters (x_I, t_I, u_0, v_0), where u_0 is such that $v_0 = \int_0^{u_0} c_N(s)\,ds$.

Proof. Let p_G be such that

$$p_G(\Pi_G(u)) \stackrel{\text{def}}{=} \frac{c_G'(u)}{c_G^2(u)}. \tag{4.39}$$

Equation (4.39) is equivalent to

$$\Pi_G''(u) = p_G(\Pi_G(u))(\Pi_G'(u))^2. \tag{4.40}$$

[4]To prove that $p_1 \in B_M$ one needs to repeat estimates (4.29) and (4.30).

Theorem 4.1 implies that there exists a Lipschitz continuous function $p = p_N$ satisfying $p_N(\overline{u}) = p_G(\overline{u})$ for $\overline{u} \in [0, v_0/2]$, and such that (2.5) admits a solution satisfying (2.6). Theorem 2.1, in turn, shows that if $c = c_N$ is such that

$$p_N\left(\int_0^u c_N(s)\,ds\right) = \frac{c_N'(u)}{c_N^2(u)}$$

then (WE2) admits a Greenberg-Rascle construction of spatially and temporally periodic solutions. We define

$$c_N(u) = \Pi_N'(u),\ u \geq 0 \quad \text{and} \quad c_N(u) = \Pi_N'(-u),\ u \leq 0$$

where Π_N is the solution of the following ODE:

$$\begin{cases} \Pi_N''(u) = p_N(\Pi_N(u))(\Pi_N'(u))^2, \\ \Pi_N(0) = 0, \\ \Pi_N'(0) = c_G(0). \end{cases} \tag{4.41}$$

We show that if $\frac{8}{t_I} P_0 c_G(0) v_0 < 1$, then (4.41) has a unique solution defined on $[0, v_0]$. This follows from standard results on local existence and uniqueness of ODEs, and from the fact that

$$0 \leq \Pi_N''(u) \leq \|p\|_\infty (\Pi_N'(u))^2$$

implies the uniform bound on Π_N',

$$0 \leq c_N(u) = \Pi_N'(u) \leq \frac{c_G(0)}{1 - \|p\|_\infty c_G(0) u} \leq \frac{c_G(0)}{1 - \frac{8}{t_I} P_0 c_G(0) v_0}$$

for $0 \leq u \leq v_0$. Function Π_N satisfies

$$\Pi_N, \Pi_N', \Pi_N'' > 0 \quad \text{for } u > 0. \tag{4.42}$$

We observe that $\Pi_N = \Pi_G$ as long as $\Pi_G \leq v_0/2$. Inequalities (4.37) and (4.42) show that $\Pi_N = \Pi_G$ for $0 \leq u \leq u_{1/2}$ and, thus, that $c_N(u) = c_G(u)$ for $-u_{1/2} \leq u \leq u_{1/2}$. □

We conclude this paper with the following remark. All admissible sound speed relations c_N constructed in the above theorem satisfy $c_N \in C(R) \cap C^1(R_+)$. One can construct (nonuniquely) more regular $c_N \in C(R) \cap C^m(R_+)$, $1 < m \leq \infty$. We note that the basic obstacle in improving on regularity of c_N (away from $u = 0$) is the fact that the function g constructed in (4.17) has, in general, a singularity in the first derivative at $v = 0$. One can bypass this problem by redefining g smooth for $-\varepsilon_1 < v < \varepsilon_1$ and repeating our construction. This procedure yields smoother c_N satisfying $c_N(u) = c_G(u)$ on a smaller set $u \in [-u_{1/2} - \varepsilon_2, u_{1/2} + \varepsilon_2]$, where ε_2 is such that $\Pi_G(u_{1/2} - \varepsilon_2) = \frac{v_0}{2} - \varepsilon_1$. We note, however, that c_N is no longer uniquely defined in terms of $c_G|_{[0, u_{1/2}]}$.

Acknowledgment. I would like to thank my advisor J. M. Greenberg for introducing me to this problem, providing me with many intuitions and insights. I would also like to thank the members of the Department of Mathematics, Rutgers University for their hospitality during the summer of 1994, when I conducted this research.

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