# Generalizations of Two Sided Power Distributions and Their Convolution

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**Abstract** — A general form of a family of bounded two-sided continuous distributions is introduced. The uniform and triangular distributions are possibly the simplest and best known members of this family. We also describe families of continuous distribution on a bounded interval generated by convolutions of these two sided distributions. Examples of various forms of convolutions of triangular distributions are presented and analyzed.

## 1. INTRODUCTION

The Standard Two Sided Power (STSP) distributions introduced by Van Dorp and Kotz (1), (2) can be motivated from at least two different aspects. The original motivation is to extend the triangular distribution which has, inter alia, applications in various problems associated with risk analysis and uncertainty elicitation (see, e.g., Johnson (3)). This leads us to a two-parameter STSP distribution with the density

$$f(x|\theta,n)=\begin{cases} n\left(\frac{x}{\theta}\right)^{n-1} & 0<x\leq\theta \\ n\left(\frac{1-x}{1-\theta}\right)^{n-1} & \theta<x<1, \end{cases} \tag{1.1}$$

where $\theta \in$ [0,1], $n > 0$, not necessarily an integer. Nadarajah (4) (as noted in Van Dorp and Kotz (1)) arrived at a form similar form to $(1.1)$ and presented in Nadarajah (5) the following reparameterized version of $(1.1)$:

$$f(x|\theta,\alpha,\beta) = \begin{cases} c\left(\frac{x}{\theta}\right)^{\alpha-1} & 0 < x \leq \theta \\ c\left(\frac{1-x}{1-\theta}\right)^{\beta-1} & \theta < x < 1, \end{cases} \tag{1.2}$$

where $\alpha > 0$ and $\beta > 0$ are chosen so that

$$\theta^{\alpha-1} = (1-\theta)^{\beta-1}$$

and

$$c \ = \left\{ \frac{\theta^{\alpha}}{\alpha} + \frac{(1-\theta)^{\beta}}{\beta} \right\}^{-1}.$$

is the proportionality factor ensuring that $(1.2)$ integrates to one. Nadarajah's (5) two-parameter reparameterization $(1.2)$ results in additional flexibility of the distribution $(1.2)$ as compared with $(1.1)$.

A four-parameter Two Sided Power (TSP) distribution is obtained in Van Dorp and Kotz (2) by incorporating the location and scale parameters $a$ and $b$ in $(1.1)$ to yield the density

$$f(x|a,m,b,n) = \begin{cases} \frac{n}{(b-a)}\left(\frac{x-a}{m-a}\right)^{n-1} & a < x \leq m \\ \frac{n}{(b-a)}\left(\frac{b-x}{b-m}\right)^{n-1} & m \leq x < b\ , \end{cases} \tag{1.3}$$

where $a$ and $b$ are arbitrary real numbers with $a < b$ and $m = (b-a)\theta + a$. The mean value associated with $(1.3)$ equals

$$E[X] = \frac{a + (n-1)m + b}{n+1} \tag{1.4}$$

It follows from $(1.4)$ that the parameter $n$ assigns relative importance to the most likely value $m$ (or equivalently $\theta$) in $(1.4)$ and the denominator $(n+1)$ may be interpreted as a *virtual sample size.* Analogous interpretation can be attributed to the parameters $\alpha$ and $\beta$ in $(1.2)$ only when $\theta = \frac{1}{2}$.

As it was shown in Van Dorp and Kotz (1), the distributions $(1.1)$, $(1.3)$ and consequently $(1.2)$ serve as viable alternatives to the versatile and flexible two and four-parameter beta distributions. Properties and maximum likelihood estimators (MLE) of the STSP density $(1.1)$ are presented in Van Dorp and Kotz (1). These properties include inter alia i) quantiles given in a closed form, ii) a maximum likelihood estimation procedure that is algorithmically straightforward and efficient and iii) parameters that possess a clear cut and meaningful interpretation. Moment estimation of parameters in $(1.1)$ and four-parameter MLE estimation involving parameters in $(1.3)$ are discussed in Van Dorp and Kotz (2). The beta distribution does not enjoy the above mentioned properties, while the various forms of the STSP density $(1.1)$ (or $(1.2)$) are as encompassing as those of the two-parameter beta distribution in the unimodal and U-shaped domains (see Van Dorp and Kotz (1) and Nadarajah (5)). Van Dorp and Kotz (1) have also demonstrated that the TSP distribution is a useful model for describing uncertainty in financial data, among other applications.

A novel approach to STSP distributions is the realization that $(1.1)$ can be viewed as a particular case of the (general) two-sided continuous family with support $[0,1]$ given by the density

$$g\{x|\theta, p(\cdot|\Psi)\} = \begin{cases} p(\frac{x}{\theta}|\Psi) & 0 < x \leq \theta \\ p(\frac{1-x}{1-\theta}|\Psi) & \theta < x < 1, \end{cases} \qquad (1.5)$$

where $p(\cdot|\Psi)$ is an appropriately selected continuous pdf defined on $[0,1]$ with parameter(s) $\Psi$, which may in principle be vector-valued. The density $p(\cdot|\Psi)$ will be referred to as the <u>generating density</u> of the resulting two-sided family of distributions and the parameter $\theta$ is termed the <u>reflection parameter</u>. The simplest linear choice

$$p(y) = 2y, 0 \leq y \leq 1, \qquad (1.6)$$

generates the triangular distribution (see Figure 1A). A more general form of $p(y|\cdot)$ given by

$$p(y|n) = ny^{n-1},\ 0 \leq y \leq 1, n > 0,$$

generates via $(1.5)$ the STSP distribution $(1.1)$, extending the triangular distribution (see Figure 1B). The density of a normalized exponential distribution, i.e.

$$p(y|\lambda) = \ exp(-\lambda y)/\{1 - e^{-\lambda}\},\ 0 \leq y \leq 1,\ \lambda > 0,$$

generates the two sided truncated exponential distribution (see Figure 1C). The density

$$p(y|\alpha) = \ \frac{2}{\alpha + 1}\{\alpha(1 - y) + y\}, 0 \leq y \leq 1,\ \alpha \geq 0, \qquad (1.7)$$

to be called the *linear slope distribution* results in the two sided slope distribution (see, Figure 1D). The linear slope distribution (cf. $(1.7)$), which - to the best of our knowledge - has not been discussed in the literature, is a one-parameter distribution supported on $[0, 1]$ with the property that $p(0|\alpha) = \alpha p(1|\alpha)$. Hence, for $\alpha > 1$ the density $p(y|\alpha)$ (cf. $(1.7)$) is a downward sloping linear function and for $0 < \alpha < 1$ an upward sloping one. For $\alpha = 1$, the slope distribution reduces to the uniform distribution on $[0, 1]$, while $\alpha = 0$ leads us to the triangular distribution. Evidently, a *non linear slope* generating density can easily be constructed by modifying $(1.7)$. Finally, the density

$$p(y|m) = \frac{2m(m + 1)}{3m + 1}y^{\frac{m-1}{2}} + \frac{1 - m^2}{3m + 1}y^m, 0 \leq y \leq 1,\ m > 0,$$

to be referred to as the *ogive distribution*, generates a two sided ogive distribution (see, Figure 1E) which is a distribution worthy of a further investigation.

The five examples presented above portray a strictly increasing (decreasing) convex density, two increasing linear generating densities and a generating density possessing an inflection point. A large variety of bounded continuous distributions can be constructed using the outlined procedure.

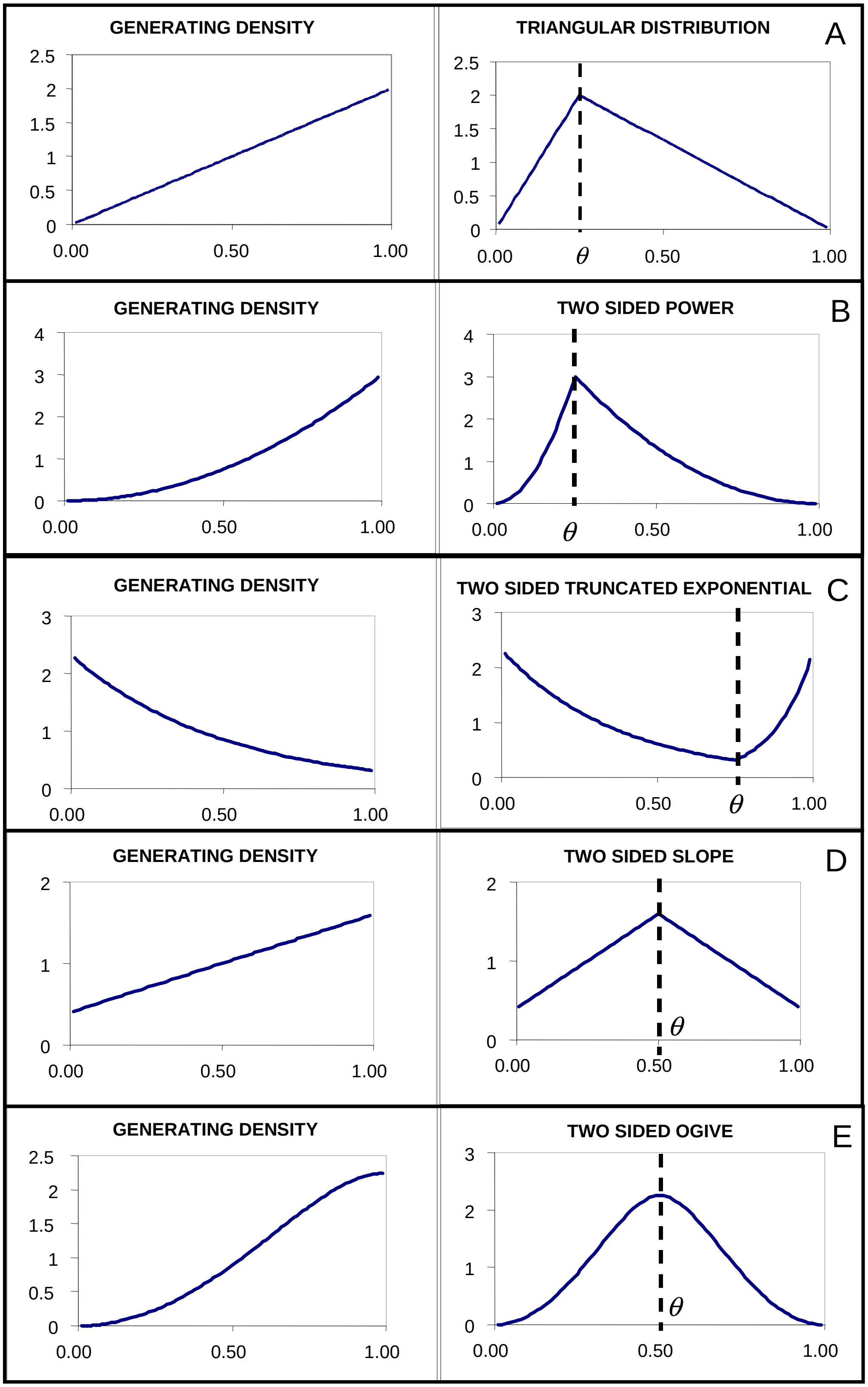


Figure 1. Examples of Two Sided distributions and their Generating Densities

We note that our model is different in structure (although similar in spirit) from the double Weibull distribution introduced in Balakrishnan and Kocherlakota (6).

The cumulative distribution function (cdf) associated with $(1.5)$ is

$$G\{x|\theta, P(\,\cdot\,\Psi)\} = \begin{cases} \theta P(\frac{x}{\theta}|\Psi) & 0 < x < \theta \\ 1-(1-\theta)P(\frac{1-x}{1-\theta}|\Psi) & \theta \leq x < 1, \end{cases} \tag{1.8}$$

where $P(\,\cdot\,|\Psi)$ is the cdf of the generating density $p(\,\cdot\,|\Psi)$. An important main and revealing property of the two sided family given by $(1.5)$, or alternatively by $(1.8)$, is that

$$G(\theta|\theta, \Psi) = \theta P(1|\Psi) = \theta \tag{1.9}$$

regardless of the functional form of the generating density $p(\,\cdot\,|\Psi)$ (or generating cdf $P(\,\cdot\,|\Psi)$). In other words, the cdf's of all the members of the two-sided family hinge at $\theta$, which can be interpreted as the pivotal point of the distribution. We emphasize the structural difference between the reflection (hinge) parameter $\theta$ which determines the "turning point" of the distribution under consideration and the parameters included in $\Psi$ which control the form of the two sides of the distribution to the left and the right of $\theta$.

If $X \sim g\{\,\cdot\,|\theta, p(\,\cdot\,|\Psi)\}$ (cf. $(1.5)$) and $Y \sim p(\,\cdot\,|\Psi)$, the following relationship between the moments around zero of $X$ and $Y$ can straightforwardly be derived by induction

$$E[X^k|\theta, \Psi] = \theta^{k+1}E[Y^k|\Psi] + \sum_{i=0}^{k}\binom{k}{i}(-1)^i(1-\theta)^{i+1}E[Y^i|\Psi]. \tag{1.10}$$

From $(1.10)$, utilizing the modern computational facilities (if necessary for large value of $k$), moments of two sided distributions can be calculated in the case when closed form expressions for the moments of the generating density exist. In particular, we have for the first two moments

$$E[X|\theta, \Psi] = (2\theta - 1)E[Y|\Psi] + (1-\theta) \tag{1.11}$$

and

$$Var(X|\theta,\Psi) = \{\theta^3 + (1-\theta)^3\}Var(Y|\Psi) + \theta(1-\theta)\{E[Y|\Psi]-1\}^2. \quad (1.12)$$

Note that it follows from $(1.11)$ for $\theta = 1, 0$ and $\frac{1}{2}$ that

$$E[X|1,\Psi] = E[Y|\Psi], \qquad E[X|0,\Psi] = 1 - E[Y|\Psi], \qquad E[X|\frac{1}{2},\Psi] = \frac{1}{2}, \quad (1.13)$$

and from $(1.12)$ we obtain

$$Var(X|1,\Psi) = Var(X|0,\Psi) = Var(Y|\Psi). \quad (1.14)$$

The density $(1.5)$ implies for $\theta = 1$ that $g\{x|\theta, p(\cdot|\Psi)\} = p(x|\Psi)$. Hence, the relations for $\theta = 1$ in $(1.13)$ and $(1.14)$ are verified. The second result $(\theta = 0)$ in $(1.13)$ and $(1.14)$ follows from the observation that $g\{x|0, p(\cdot|\Psi)\}$ represents the *mirror reflection* of the generating density, i.e. $p(1-x|\Psi)$. The third result in $(1.13)$ holds regardless of the form of $p(x|\Psi)$ due to the symmetry of $g\{x|\frac{1}{2}, p(\cdot|\Psi)\}$ around $\theta = \frac{1}{2}$.

In this paper we shall derive the structure of the densities of the various distributions on a bounded interval that can be constructed by adding two independent two sided variables $X$ and $Y$ both defined by $(1.5)$. (Note that these $X$ and $Y$ are unrelated to the specific $X$ and $Y$ utilized in equations $(1.10)$ - $(1.14)$ ). Such a scenario may be useful, for example, in applications to financial engineering when evaluating the uncertainty in the annual return of a diversified portfolio of two investments $X$ and $Y$, where the uncertainties in the individual annual returns of $X$ and $Y$ are adequately represented by two sided distributions under the reasonable assumption that the returns are independent. (The principle of diversification in investment analysis is indeed based on the assumption of essential independence or near independence between the annual returns of diverse investment options.) Alternatively, one may view $X$ and $Y$ as the uncertainty of two independent consecutive activity durations that are often modeled by triangular distributions, often used in the PERT Analysis (see, e.g., Johnson (3)), where the overall completion time $X + Y$ is the purpose of the analysis.

Results on convolution and, more generally, sums and linear combinations are by now available for most distributions commonly used in statistical, economics, engineering and medical applications, amongst others. They are scattered in numerous books on statistics and probability theory as well as publications dealing with order statistics and non-parametric methodology. A selective bibliography has been recently compiled by the authors and is available upon request. Applications in queuing theory, civil engineering and random number generations seems to be particularly prominent topics where these distributional structures are used in addition to large number of applications to statistical and reliability theories and methodologies.

Although the calculations presented below may seem to be somewhat involved due to the fact that the component distributions are defined by two different algebraic expressions which requires careful examination of various sub cases, the results seem to be rewarding by opening new avenues for additional investigation. In the next section we provide the basic theoretical framework for the convolution of two sided distributions. In Section 3 we present in some detail an example of the convolution of two triangular distributions in closed form within this framework.

## 2. CONVOLUTION OF TWO SIDED DISTRIBUTIONS

Let $X \sim g_X\{x|\theta_x, p(\cdot|\Psi)\}$, where $g_X\{x|\theta_x, p(\cdot|\Psi)\}$ (cf. $(1.5)$) be a two sided density with the reflection parameter $0 \leq \theta_x \leq 1$ and the generating density $p(\cdot|\Psi)$ and let $Y \sim g_Y\{y|\theta_y, q(\cdot|\Upsilon)\}$, where $g_Y\{y|\theta_y, q(\cdot|\Upsilon)\}$ be an analogous density. Let $Z = X + Y$, where $X$ and $Y$ are assumed to be independent. Finally, let $g_Z(z|\Theta)$ be the density of $Z$ (the convolution of $g_X\{x|\theta_x, p(\cdot|\Psi)\}$ and $g_Y\{y|\theta_y, q(\cdot|\Upsilon)\}$), i.e.

$$g_Z(z|\Theta) = \int_{x=0}^{x=1} g_X\{x|\theta_x, p(\cdot|\Psi)\} g_Y\{z-x|\theta_y, q(\cdot|\Upsilon)\} 1_{[0,1]}(z-x)dx \quad (2.1)$$

where $\Theta = \{\theta_x, \Psi, \theta_y, \Upsilon\}$ and $1_A(x)$ be the indicator function

$$1_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A. \end{cases}$$

We shall assume without loss of generality that $\theta_x \leq \theta_y$. Several equivalent expressions for the convolution $g_Z(z|\Theta)$ are available but we have found representation $(2.1)$ to be most convenient for our purposes. Utilizing the identities

$$\int_{x=0}^{x=1} k(x)dx = \int_{x=0}^{x=\theta} k(x)dx + \int_{x=\theta}^{x=1} k(x)dx,$$

$$\int_{x=a}^{x=b} k(x)1_{[0,1]}(z-x)dx = \int_{x=a}^{x=b} k(x)1_{[0,\theta]}(z-x)dx + \int_{x=a}^{x=b} k(x)1_{[\theta,1]}(z-x)dx,$$

$$1_{[a,b]}(z-x) = 1_{[z-b,z-a]}(x),$$

where $k(x)$ is an arbitrary integrable function on the interval $[0,1]$ , and the definitions $(2.1)$ and $(1.5)$, one can verify that $g_Z(\,\cdot\,|\Theta)$ can be expressed as a sum of four integrals. Specifically,

$$g_Z(z|\Theta) = \sum_{i=1}^{4} I_i(z|\Theta), \tag{2.2}$$

where

$$\begin{cases} I_1(z|\Theta) = \int_{x\in A_1(z)} p(\frac{x}{\theta_x}|\Psi) q(\frac{z-x}{\theta_y}|\Upsilon)dx \\ I_2(z|\Theta) = \int_{x\in A_2(z)} p(\frac{x}{\theta_x}|\Psi) q(\frac{1-z+x}{1-\theta_y}|\Upsilon)dx \\ I_3(z|\Theta) = \int_{x\in A_3(z)} p(\frac{1-x}{1-\theta_x}|\Psi) q(\frac{z-x}{\theta_y}|\Upsilon)dx \\ I_4(z|\Theta) = \int_{x\in A_4(z)} p(\frac{1-x}{1-\theta_x}|\Psi) q(\frac{1-z+x}{1-\theta_y}|\Upsilon)dx, \end{cases} \tag{2.3}$$

and the regions of integration are respectively

$$\begin{cases} A_1(z|\,\theta_x,\,\theta_y) = [0,\theta_x) \cap (z-\theta_y, z] \\ A_2(z|\,\theta_x,\,\theta_y) = [0,\theta_x) \cap [z-1, z-\theta_y] \\ A_3(z|\,\theta_x,\,\theta_y) = [\theta_x, 1] \cap (z-\theta_y, z] \\ A_4(z|\,\theta_x,\,\theta_y) = [\theta_x, 1] \cap [z-1, z-\theta_y]. \end{cases} \tag{2.4}$$

As an illustration, Figure 2 explains calculations of $A_i(z|\,\theta_x,\,\theta_y),\ i = 1,\dots,4$, for the case that $\theta_x \leq z \leq \theta_y$ under the assumptions that $\theta_x \leq \theta_y$ and $\theta_x + \theta_y \leq 1$.

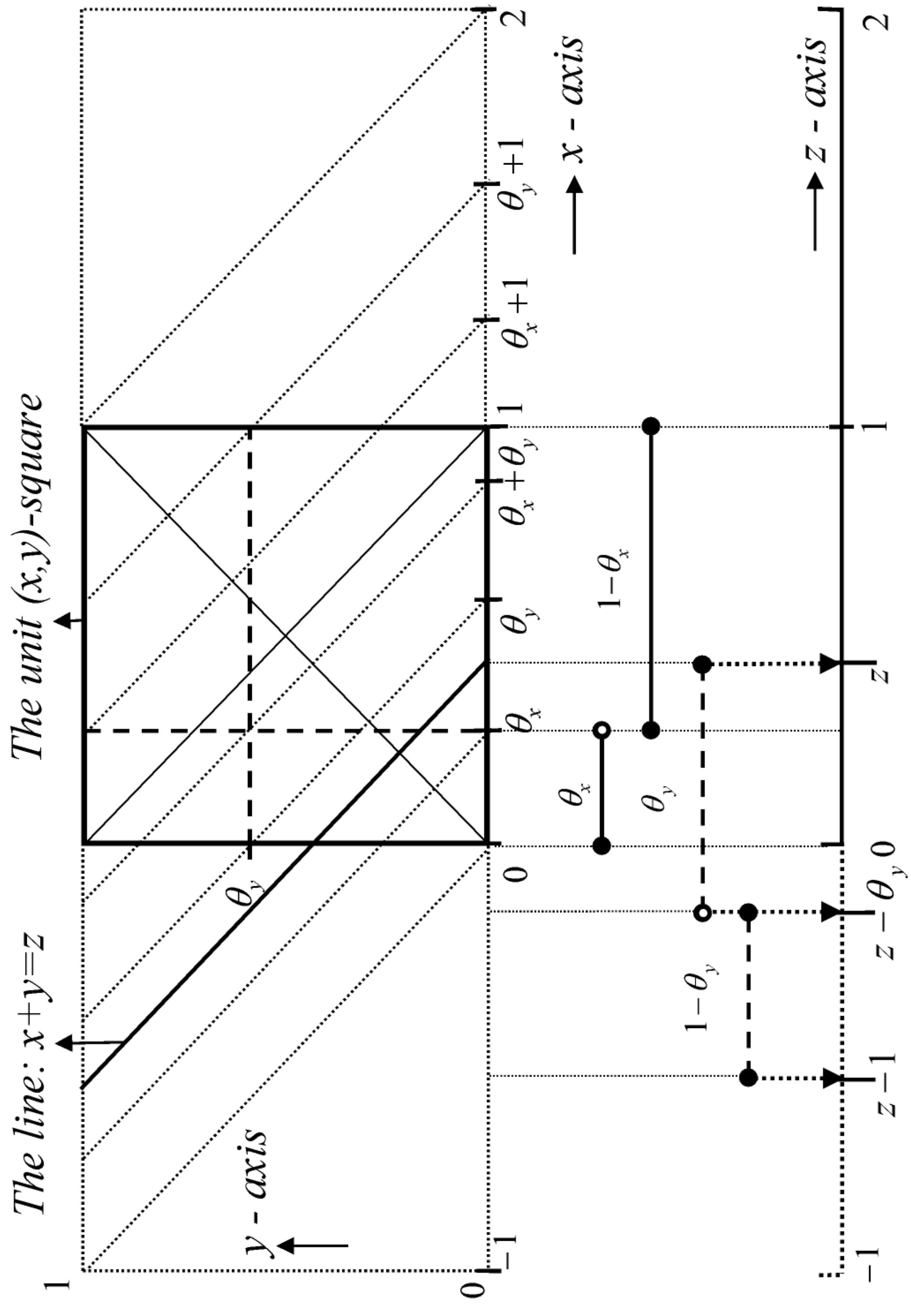


Figure 2. Calculation of $A_i(z|\,\theta_x,\,\theta_y)$, $i=1,\dots 4$, (cf. $(2.4)$ for $\theta_x \le z \le \theta_y$ under the assumption that $\theta_x \le \theta_y$ and $\theta_x+\theta_y \le 1$

From Figure 2 it follows that in this case

$$\begin{cases} A_1(z|\,\theta_x,\,\theta_y) = [0,\theta_x) & \theta_x \le z \le \theta_y \\ A_2(z|\,\theta_x,\,\theta_y) = \emptyset & \theta_x \le z \le \theta_y \\ A_3(z|\,\theta_x,\,\theta_y) = [\theta_x, z] & \theta_x \le z \le \theta_y \\ A_4(z|\,\theta_x,\,\theta_y) = \emptyset. & \theta_x \le z \le \theta_{y.} \end{cases} \tag{2.5}$$

Hence, from $(2.5)$ it follows that neither the integral $I_2(z|\Theta)$ nor the integral $I_4(z|\Theta)$(cf. $(2.3)$) contribute to $g_Z(z|\Theta)$ (cf. $(2.1)$). Also, it follows from $(2.5)$ and $(2.3)$ that for $\theta_x \le z \le \theta_y$ the contribution of, for example, the third integral $I_3(z|\Theta)$ to $g_Z(z|\Theta)$ equals

$$I_3(z|\Theta) = \int_{\theta_x}^{z} p(\frac{1-x}{1-\theta_x}|\Psi)q(\frac{z-x}{\theta_y}|\Upsilon)dx, \quad \theta_x \le z \le \theta_y \,.$$

By *sliding* the three dotted arrows pointing downward to the z-axis in Figure 2 associated with the points $z-1, z-\theta_y$ and $z$, respectively, to the left and right, while keeping $z \in [0,2]$, it follows from the definition of the regions in $(2.4)$ that the non-overlapping partition of the following consecutive seven subintervals ought to be considered when evaluating $A_i(z),\ i = 1,\ldots,4$:

$$\begin{array}{l} z \in [0,\theta_x), z \in [\theta_x,\theta_y), z \in [\theta_y, \theta_x+\theta_y),\ z \in [\theta_x+\theta_y,\ 1), \\ z \in [1,\theta_x+1),\ z \in [\theta_x+1,\theta_y+1), z \in [\theta_y+1,2]. \end{array} \tag{2.6}$$

Table 1 summarizes calculation of $A_i(z|\,\theta_x,\,\theta_y),\ i = 1,\ldots,4$, for partition $(2.6)$ on the support $[0,2]$ of $Z$ under the assumptions $\theta_x \le \theta_y$ and $\theta_x + \theta_y \le 1$. Recall that it was assumed without loss of generality that $\theta_x \le \theta_y$, otherwise one simply interchanges $x$ and $y$ in all the expressions. The corresponding partition of $[0,2|$ into seven consecutive subintervals under the assumption that $\theta_x + \theta_y > 1$ will be

$$\begin{array}{l} z \in [0,\theta_x), z \in [\theta_x,\theta_y), z \in [\theta_y,1),\ z \in [1,\theta_x+\theta_y), \\ z \in [1,\theta_x+1),\ z \in [\theta_x+1,\theta_y+1), z \in [\theta_y+1,2]. \end{array} \tag{2.7}$$

The calculation of $A_i(z|\theta_x,\theta_y),\ i = 1,\ldots,4$ defined in $(2.4)$ for partition $(2.7)$ is presented in Table 2.

Table 1. Calculation of $A_i(z|\,\theta_x,\,\theta_y)$, $i=1,\ldots 4,$ (cf. $(2.4)$ for the seven non-overlapping subintervals partitioning the support $[0,2]$ of $Z$ in $(2.6)$.

| | | $A_1(z)$ | $A_2(z)$ | $A_3(z)$ | $A_4(z)$ |
|---|---|---|---|---|---|
| 1 | $z\in[0,\theta_x)$ | $[0,z]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 2 | $z\in[\theta_x,\theta_y)$ | $[0,\theta_x)$ | $\emptyset$ | $[\theta_x,z]$ | $\emptyset$ |
| 3 | $z\in[\theta_y,\theta_x+\theta_y)$ | $(z-\theta_y,\theta_x)$ | $[0,z-\theta_y]$ | $[\theta_x,z]$ | $\emptyset$ |
| 4 | $z\in[\theta_x+\theta_y,\,1)$ | $\emptyset$ | $[0,\theta_x)$ | $(z-\theta_y,z]$ | $[\theta_x,z-\theta_y]$ |
| 5 | $z\in[1,\theta_x+1)$ | $\emptyset$ | $[z-1,\theta_x)$ | $(z-\theta_y,1]$ | $[\theta_x,z-\theta_y]$ |
| 6 | $z\in[\theta_x+1,\theta_y+1)$ | $\emptyset$ | $\emptyset$ | $(z-\theta_y,1]$ | $[z-1,z-\theta_y]$ |
| 7 | $z\in[\theta_y+1,2]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $[z-1,1]$ |

Table 2. Calculation of $A_i(z|\,\theta_x,\,\theta_y)$, $i=1,\ldots 4,$ (cf. $(2.4)$) for the seven non-overlapping subintervals partitioning the support $[0,2]$ of $Z$ in $(2.7)$.

| | | $A_1(z)$ | $A_2(z)$ | $A_3(z)$ | $A_4(z)$ |
|---|---|---|---|---|---|
| 1 | $z\in[0,\theta_x)$ | $[0,z]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 2 | $z\in[\theta_x,\theta_y)$ | $[0,\theta_x)$ | $\emptyset$ | $[\theta_x,z]$ | $\emptyset$ |
| 3 | $z\in[\theta_y,1)$ | $(z-\theta_y,\theta_x)$ | $[0,z-\theta_y]$ | $[\theta_x,z]$ | $\emptyset$ |
| 4 | $z\in[1,\theta_x+\theta_y)$ | $(z-\theta_y,\theta_x)$ | $[z-1,z-\theta_y]$ | $[\theta_x,1]$ | $\emptyset$ |
| 5 | $z\in[\theta_x+\theta_y,\theta_x+1)$ | $\emptyset$ | $[z-1,\theta_x)$ | $(z-\theta_y,1]$ | $[\theta_x,z-\theta_y]$ |
| 6 | $z\in[\theta_x+1,\theta_y+1)$ | $\emptyset$ | $\emptyset$ | $(z-\theta_y,1]$ | $[z-1,z-\theta_y]$ |
| 7 | $z\in[\theta_y+1,2]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $[z-1,1]$ |

We urge the reader to examine carefully Figure 2 and Tables 1 and 2 and to note the various symmetries involved therein. We emphasize that, without exception, out of the four different scenarios of $A_i(z)$, $i=1,\ldots,4$ appearing in Tables 1 and 2, at least one of them will always be the empty set $\emptyset$. Moreover, for the first interval in the partition of $[0,2]$ only $I_1(z|\Theta)$ contributes to $g_Z(z|\Theta)$, whereas in the last interval in the partition of $[0,2]$ only $I_4(z|\Theta)$ is involved in $g_Z(z|\Theta)$.

Under the assumption that $\theta_x+\theta_y\leq 1$ or its converse $\theta_x+\theta_y>1$, the explicit functional form of the generating density $p(\,\cdot\,|\Psi)$ of $X$ and that of the generating density $q(\,\cdot\,|\Upsilon)$ of $Y$, the density of $Z$ can straightforwardly be derived utilizing $(2.2)$, $(2.3)$ and Tables 1 or 2, respectively. As an illustration, the next section presents the convolution of two triangular distributions in a closed form under the assumption that $\theta_x + \theta_y \le 1$. In addition, graphs of the convolution density $g_Z(z|\Theta)$ (cf. $(2.1)$) and its summands $I_i(z|\Theta), i = 1, \ldots, 4$ (cf. $(2.2)$) are developed for these examples.

## 3. EXAMPLES

Let $X$ and $Y$ be triangular distributions on $[0,1]$ with the reflection parameters $\theta_x$ and $\theta_y$, respectively. Utilizing $(2.3)$ with the generating densities $p(x|\Psi) = 2x$ and $q(y|\Upsilon) = 2y$, respectively, it follows that

$$\begin{cases} I_1(z|\Theta) = \frac{4}{\theta_x\theta_y}\left[\frac{1}{2}zx^2 - \frac{1}{3}x^3\right]_{a_1(z)}^{b_1(z)} \\ I_2(z|\Theta) = \frac{4}{\theta_x(1-\theta_y)}\left[\frac{1}{2}x^2 - \frac{1}{2}zx^2 + \frac{1}{3}x^3\right]_{a_2(z)}^{b_2(z)} \\ I_3(z|\Theta) = \frac{4}{(1-\theta_x)\theta_y}\left[zx - \frac{1}{2}x^2 - \frac{1}{2}zx^2 + \frac{1}{3}x^3\right]_{a_3(z)}^{b_3(z)} \\ I_4(z|\Theta) = \frac{4}{(1-\theta_x)(1-\theta_y)}\left[x - zx + \frac{1}{2}zx^2 - \frac{1}{3}x^3\right]_{a_4(z)}^{b_4(z)} \end{cases} \tag{3.1}$$

where $a_i(z)$ and $b_i(z)$ are the endpoints of the intervals $A_i(z|\theta_x,\theta_y)$, $i = 1,\ldots,4$, specified in Table 1 and $x$ denotes the integration variable. From $(3.1)$ and Table 1 one deduces that

$$\frac{3\theta_x\theta_y}{2} I_1(z|\theta_x,\theta_y) = \tag{3.2}$$
$$\begin{cases} z^3 & z \in [0,\theta_x) \\ 3\theta_x^2 z - 2\theta_x^3 & z \in [\theta_x,\theta_y) \\ 3(\theta_x^2+\theta_y^2)z - z^3 + \; - 2(\theta_x^3+\theta_y^3) & z \in [\theta_y, \theta_x+\theta_y) \\ 0 & elsewhere, \end{cases}$$

$$\frac{3\theta_x(1-\theta_y)}{2} I_2(z|\theta_x,\theta_y) = \tag{3.3}$$
$$\begin{cases} 3z^2 - z^3 + 3\theta_y(\theta_y-2)z + \theta_y^2(3-2\theta_y) & z \in [\theta_y, \theta_x+\theta_y) \\ \theta_x^2(2\theta_x+3) - 3\theta_x^2 z & z \in [\theta_x+\theta_y,\, 1) \\ z^3 - 3z^2 + 3(1-\theta_x^2)z + \theta_x^2(2\theta_x+3) - 1 & z \in [1, \theta_x+1) \\ 0 & elsewhere, \end{cases}$$

$$\frac{3(1-\theta_x)\theta_y}{2} I_3(z|\theta_x,\theta_y) = \tag{3.4}$$
$$\begin{cases} 3z^2 - z^3 + 3\theta_x(\theta_x - 2)z + \theta_x^2(3-2\theta_x) & z \in [\theta_x, \theta_x+\theta_y) \\ \theta_y^2(2\theta_y+3) - 3\theta_y^2 z & z \in [\theta_x+\theta_y,\ 1) \\ z^3 - 3z^2 + 3(1-\theta_y^2)z + \theta_y^2(2\theta_y+3) - 1 & z \in [1, \theta_y+1) \\ 0 & elsewhere, \end{cases}$$

and

$$\frac{3(1-\theta_x)(1-\theta_y)}{2} I_4(z|\theta_x,\theta_y) = \tag{3.5}$$
$$\begin{cases} z^3 - 6z^2 + 3\{2 + 2(\theta_x+\theta_y) - \theta_x^2 - \theta_y\}z + & z \in [\theta_x+\theta_y,\ \theta_x+1) \\ 2(\theta_x^3+\theta_y^3) - 6(\theta_x+\theta_y) & \\ & \\ (-3 + 6\theta_y - 3\theta_y^2)z + 2\theta_y^3 - 6\theta_y + 4 & z \in [\theta_x+1,\ \theta_y+1) \\ (2-z)^3 & z \in [\theta_y+1, 2] \\ 0 & elsewhere. \end{cases}$$

(Observe that, $I_3(z|\theta_y,\theta_x) = I_2(z|\theta_x,\theta_y)$). Utilizing $(3.2)$ - $(3.5)$ and collecting the terms in $(2.2)$ we arrive at

$$\frac{3}{2} g_Z(z|\theta_x,\theta_y) = \begin{cases} \frac{z^3}{\theta_x\theta_y} & z \in [0,\theta_x) \\ \frac{3z^2 - z^3 - 3\theta_x z + \theta_x^2}{(1-\theta_x)\theta_y} & z \in [\theta_x,\theta_y) \\ \frac{g_{3Z}(z|\Theta)}{\theta_x(1-\theta_x)\theta_y(1-\theta_y)} & z \in [\theta_y, \theta_x+\theta_y) \\ \frac{z^3 - 6z^z + (6+3\theta_x+3\theta_y)z - \theta_x(3+\theta_x) - \theta_y(3+\theta_y)}{(1-\theta_x)(1-\theta_y)} & z \in [\theta_x+\theta_y,\ 1) \\ \frac{g_{5Z}(z|\Theta)}{\theta_x(1-\theta_x)\theta_y(1-\theta_y)} & z \in [1, \theta_x+1) \\ \frac{z^3 - 3z^2 + 3(1-\theta_y)z + \theta_y^2 + 4\theta_y - 1}{(1-\theta_x)\theta_y} & z \in [\theta_x+1, \theta_y+1) \\ \frac{(2-z)^3}{(1-\theta_x)(1-\theta_y)} & z \in [\theta_y+1, 2] \end{cases} \tag{3.6}$$

where in $(3.6)$

$$g_{3Z}(z|\theta_x,\theta_y) = (\theta_x\theta_y - 1)z^3 + 3(\theta_x+\theta_y-2\theta_x\theta_y)z^2 -$$

$$3\{\theta_x^2(1-\theta_y) + \theta_y^2(1-\theta_x)\}z + \theta_x^3(1-\theta_y) + \theta_y^3(1-\theta_x)$$

and

$$g_{5Z}(z|\theta_x,\theta_y) = (\theta_x+\theta_y-\theta_x\theta_y)z^3 - 3(\theta_x+\theta_y)z^2 +$$

$$3\{\theta_x(1+\theta_y^2)+\theta_y(1+\theta_x^2)\}z-(\theta_x+\theta_y)-\theta_x\theta_y(\theta_x^2+\theta_y^2+3\theta_x+3\theta_y-2).$$

It is straightforward, although somewhat tedious, to verify that the appropriate integrals of the terms in $g_Z(z|\theta_x,\theta_y)$ defined by $(3.6)$ add up to $1$.(In our calculations we have been aided by the newly available software which makes the task by far less time consuming).

Figure 3 represents the convolution of $Z = X+Y$, where $\theta_x=\frac{1}{4}$ and $\theta_y=\frac{1}{2}$. The specific values $\theta_x=\frac{1}{4}$ and $\theta_y=\frac{1}{2}$ result in seven subintervals partitioning the support $[0,2]$ of $Z$ indicated by the dotted lines in Figure 3G. Figures 3A and 3B represent the component distribution of $X$ and $Y$ in the convolution of $Z=X+Y$. Figures 3C, D, E and F display the functions $I_i(z|\theta_x,\theta_y),\ i=1,\ldots,4$ given by $(3.2)$ - $(3.5)$, respectively, for this situation. Finally, Figure 3G depicts the convolution pdf of $Z$ given by $(3.6)$. Note that the component graphs of $I_2(z|\theta_x,\theta_y)$ (Figure 3D) and $I_3(z|\theta_x,\theta_y)$ (Figure 3E) are peaked. The component graphs of $I_1(z|\theta_x,\theta_y)$ (Figure 3C) and $I_4(z|\theta_x,\theta_y)$ (Figure 3F) are, however, smooth. The smoothness of the resulting convolution pdf of $Z$ (Figure 3G) may be somewhat surprising given the peakedness of $I_2(z|\theta_x,\theta_y)$ and $I_3(z|\theta_x,\theta_y)$ in this case.

Figure 4 provides some additional examples of the form of the convolution pdf for different values of $\theta_x$ and $\theta_y$. In Figure 4A (4B) the values $\theta_y=\theta_x=0\,(\frac{1}{4})$ result in a skewed convolution pdf that appears to have the shape of a beta distribution, a feature that may merit further investigation. The closed form expression $(3.6)$ in case of Figure 4A simplifies to

$$g_Z(z|0,0)=\begin{cases}\frac{2}{3}\{z^3-6z^z+6z\} & z\in[0,1)\\ \frac{2}{3}(2-z)^3 & z\in[1,2].\end{cases}\qquad(3.7)$$

In Figure 4C (4D), $1-\theta_y=\theta_x=\frac{1}{2}\,(\frac{1}{4})$ resulting in a convolution pdf that appears to have the shape similar to that of a normal distribution.

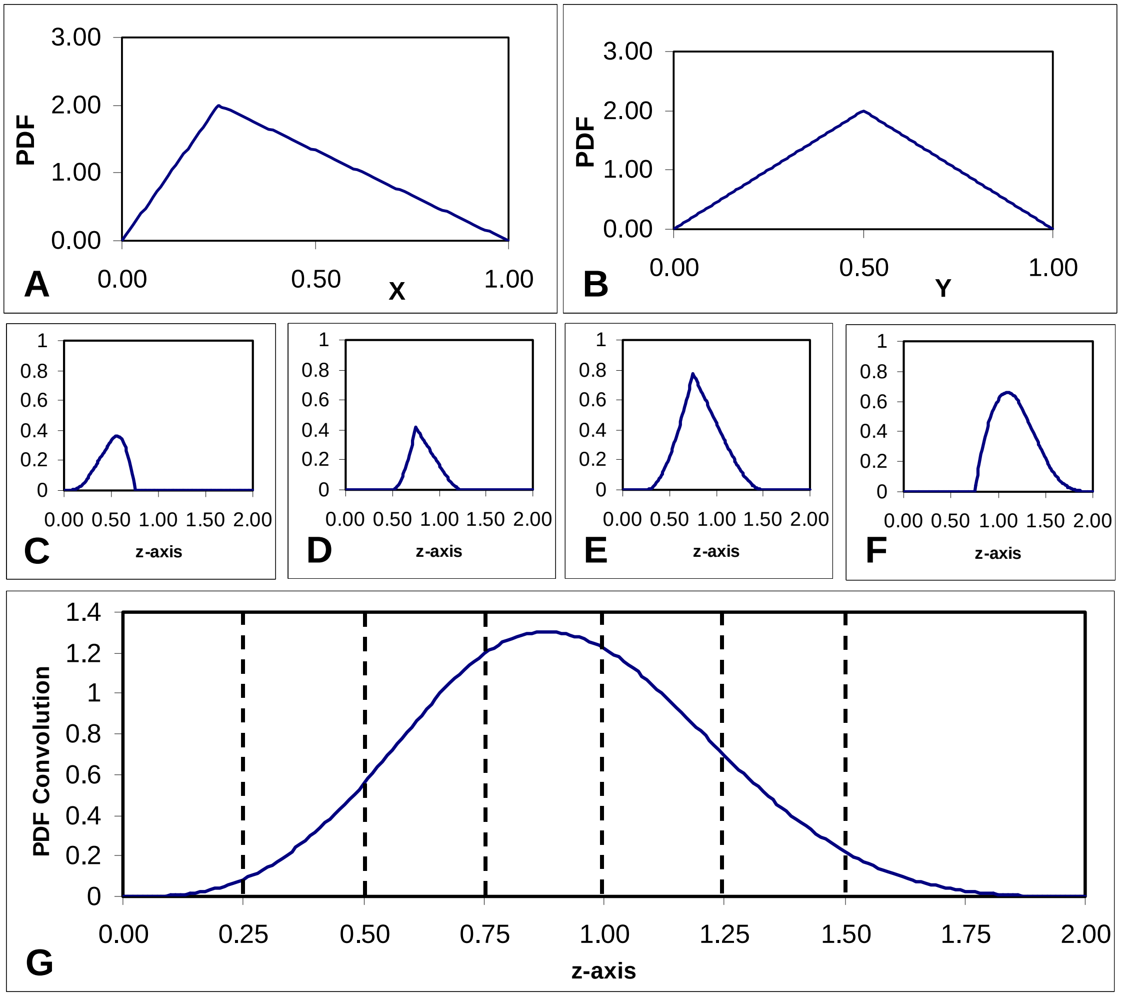


Figure 3. Convolution of an asymmetric triangular variable $X$ with $\theta_x = \frac{1}{4}$ and a symmetric triangular variable $Y$ with $\theta_y = \frac{1}{2}$ and support $[0, 1]$, such that

$$E[Z] = E[X + Y] < 1.$$

Recall that the sum of two symmetric triangular distributions are equivalent to the sum of four uniform random variables on $[0, \frac{1}{2}]$, which are often used as an approximation to the normal distribution. A closed form expression of the convolution of two symmetric triangular distributions (Figure 4C) could be of interest in this connection.

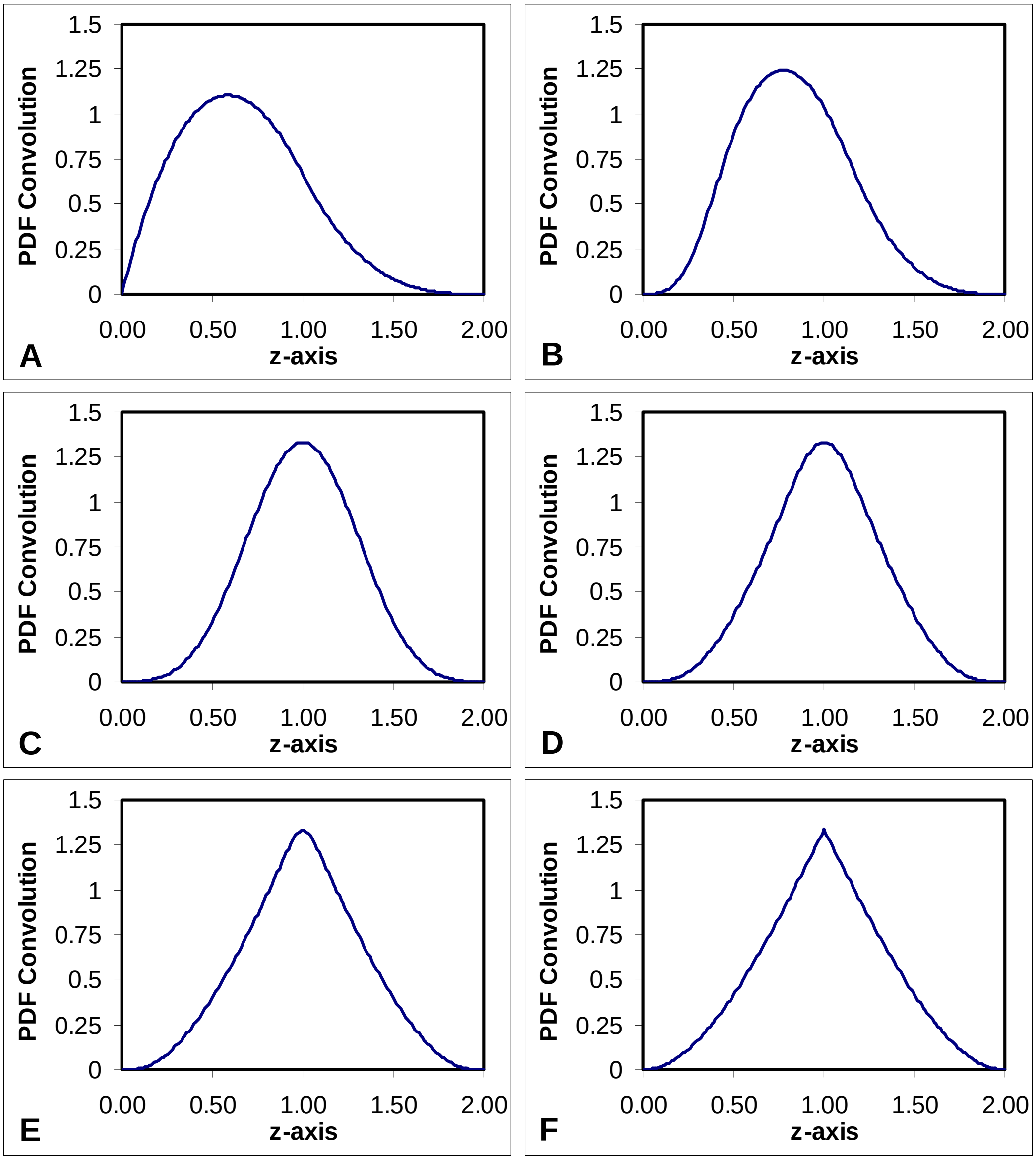


Figure 4. Examples of probability density function of $Z = X + Y$, where X and Y are triangular distribution on $[0, 1]$ with $\theta_x \le \theta_y$ and $\theta_x + \theta_y \le 1$;

A : $\theta_x = \theta_y = 0$; B : $\theta_x = \theta_y = 0.25$; C : $\theta_y = 1 - \theta_x = 0.5$;

D : $\theta_y = 1 - \theta_x = 0.75$; E : $\theta_y = 1 - \theta_x = 0.9$; F : $\theta_y = 1 - \theta_x = 1$ .

It is given by

$$g_Z(z|\frac{1}{2},\frac{1}{2}) = \begin{cases} \frac{8}{3}z^3 & z \in [0,\frac{1}{2}) \\ \frac{4}{3}\left\{1-6(z-1)^3-6(z-1)^2\right\} & z \in [\frac{1}{2},1) \\ \frac{4}{3}\left\{6(z-1)^3-6(z-1)^2+1\right\} & z \in [1,1\frac{1}{2}) \\ \frac{8}{3}(2-z)^3 & z \in [1\frac{1}{2},2]. \end{cases} \tag{3.8}$$

Finally, in Figure 4E (4F) $1-\theta_y = \theta_x = \frac{1}{10}$ (0) results in a more "peaked" convolution pdf. This is especially pronounced in Figure 4F where the convolution pdf appears to have the shape of a symmetric two sided power distribution. The closed form expression $(3.6)$ in the case of Figure $4$F simplifies to

$$g_Z(z|0,1) = \begin{cases} \frac{2}{3}\left\{3(z-1)-(z-1)^3+2\right\} & z \in [0,1) \\ \frac{2}{3}\left\{(z-1)^3-3(z-1)+2)\right\} & z \in [1,2]. \end{cases} \tag{3.9}$$

Note that the basic seven-interval partition of $[0,2]$ in $(3.6)$ reduces to a two-, four- and two-interval partitions in $(3.7)$ - $(3.9)$, respectively.

Although the explicit expression derived in $(3.6)$ is not needed to calculate the classical measures such as the coefficient of variation $(\beta_0)$, skewness $(\beta_1)$ and kurtosis $(\beta_2)$, it is natural to compare their values for the convolution $Z$ with those for the components $X$ and $Y$. These component measures for $X$ can easily be calculated for the general two-sided family by utilizing $(1.10)$ which provides the moments $\mu'_k = E[X^k]$ $k=1,\ldots,4$ in terms of the first four moments of its generating density $p(\cdot|\Psi)$ (cf. $(1.5)$), their definition

$$\beta_0 = \frac{\mu'_1}{\mu_2}, \beta_1 = \frac{\mu_3^2}{\mu_2^3}, \ \beta_2 = \frac{\mu_4}{\mu_2^2}, \tag{3.10}$$

the classical relationship between central moments $\mu_k = E[(X-E[X])^k]$ , $k=2,\ldots,4,$ and the moments around the origin $\mu'_k = E[X^k],\ k=1,\ldots,4,$ given by

$$\begin{cases} \mu_2 = \mu'_2 - {\mu'_1}^2 \\ \mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3 \\ \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2{\mu'_1}^2 - 3{\mu'_1}^4 \end{cases}$$

(see, e.g. Stuart and Ord (7)). In the case of $(3.6)$ the moments of the generating density $p(x) = 2x$ are $2/(2+k),\ k = 1, \ldots, 4$. With analogous calculations for $Y$ and the following relationship (in the obvious notation) between the moments of the convolution $Z = X + Y$ and that of its components, i.e.

$$\begin{cases} \mu'_{1,Z} = \mu'_{1,X} + \mu'_{1,Y} \\ \mu_{2,Z} = \mu_{2,X} + \mu_{2,Y} \\ \mu_{3,Z} = \mu_{3,X} + \mu_{3,Y} \\ \mu_{4,Z} = \mu_{4,X} + \mu_{4,Y} + \mu_{2,X} \cdot \mu_{2,Y}, \end{cases} \tag{3.11}$$

we may calculate the coefficient of variation ($\beta_0$), skewness ($\beta_1$) and kurtosis ($\beta_2$) for $Z$ from their definition $(3.10)$.

Figure 5 displays a comparison of the coefficient of variation ($\beta_{0,Z}$) for $Z$ corresponding to the convolution expression $(3.6)$ in the extreme cases that $\theta_x = \theta_y = \theta$ (identical triangular distribution) and $1 - \theta_y = \theta_x = \theta$ (reflected triangular distributions). In addition, Figure 5 depicts $\beta_0$ for a triangular distribution and a uniform distribution as well. It follows from Figure 5 that in the case of identical component triangular distributions the coefficient of variation $\beta_{0,Z}$ (with the minimum of $2$ and the maximum of $4$) is strictly larger than $\beta_{0,X}$. The coefficient $\beta_{0,Z} = 2\sqrt{3} \approx 3.46$ in the case when $X$ and $Y$ are symmetric triangular distributions ($\theta = 0.5$) is exactly twice the coefficient of variation of the uniform $[0,1]$ distribution, while the coefficient $\beta_{0,Z} = \sqrt{6} \approx 2.45$ in the case when $X$ and $Y$ are uniform (as it was already mentioned the convolution of two uniform distributions is a symmetric triangular distribution). Finally, $\beta_{0,Z}$ with identical positively (negatively) skewed component triangular distributions is strictly smaller (larger) than $\beta_{0,Z}$ with reflected component triangular distributions for $\theta < \frac{1}{2}$ ( $> \frac{1}{2}$). From $(1.10)$ and $(3.11)$ it can easily be derived that the variance of a convolution with identical triangular and reflected triangular distributions are identical. Hence, the latter observation follows solely from the difference in the mean of the convolution with reflected triangular distributions (which equals $1$ regardless of $\theta$) and that of the convolution with identical triangular distributions which is less (greater) than $1$ for $\theta < \frac{1}{2}$ $(\theta > \frac{1}{2})$.

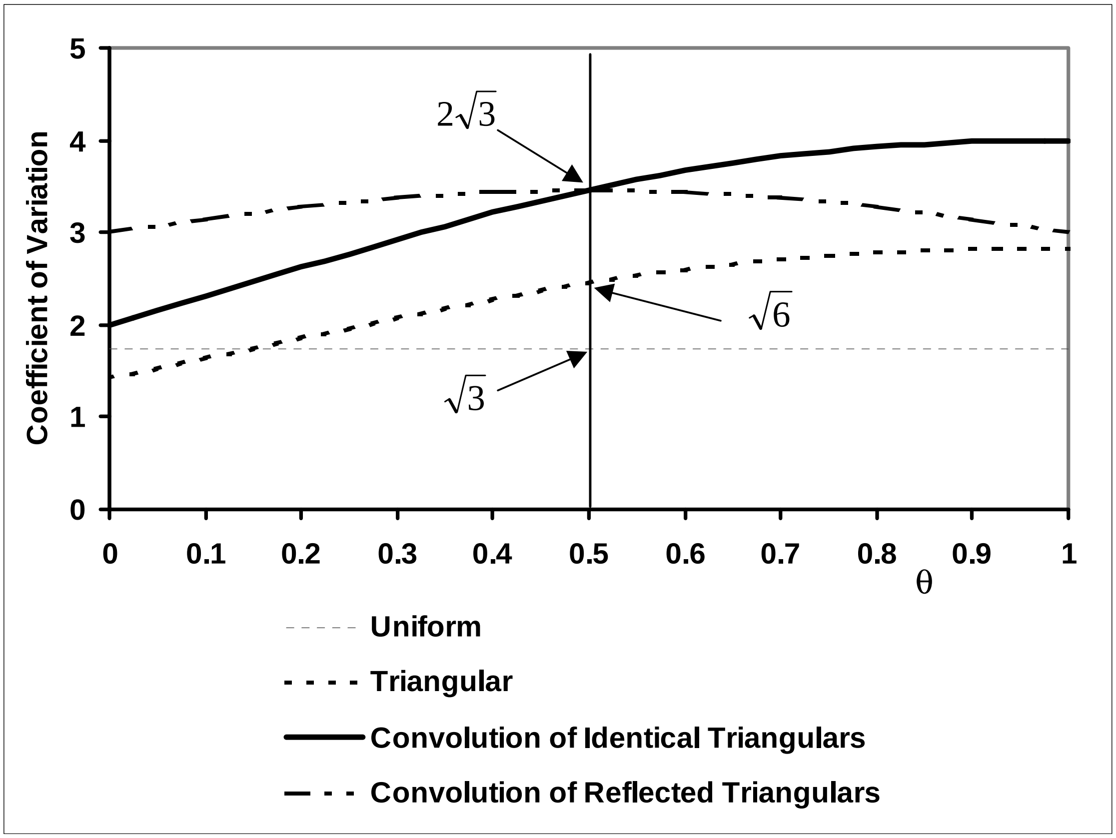


Figure 5. Comparison of the coefficients of variation $\beta_0$ between convolutions of triangular distributions (identical and reflected) and their components.

The convolution of reflected triangular distributions $(1 - \theta_y = \theta_x = \theta)$ is a symmetric distribution around its mean 1 and hence the skewness $\beta_1 = 0$. Figure 6 displays $\beta_1$ for the convolution of identical triangular distributions as well as that of its components. The sign of $\beta_1$ of the convolution of identical triangular distributions agrees with the sign of $\beta_1$ of the component triangular distributions. However, it follows from Figure 6 that $\beta_1$ of the convolution of identical triangular distributions is strictly less (larger) than that of its components for $\theta < \frac{1}{2}$ $(\theta > \frac{1}{2})$. The latter observation holds for the general form of two-sided distributions due to i) the fact that the total probability mass is split into $\theta$ and $1-\theta$ at the point $\theta$ ( cf. $(1.9)$) and ii) the independence assumption between $X$ and $Y$.

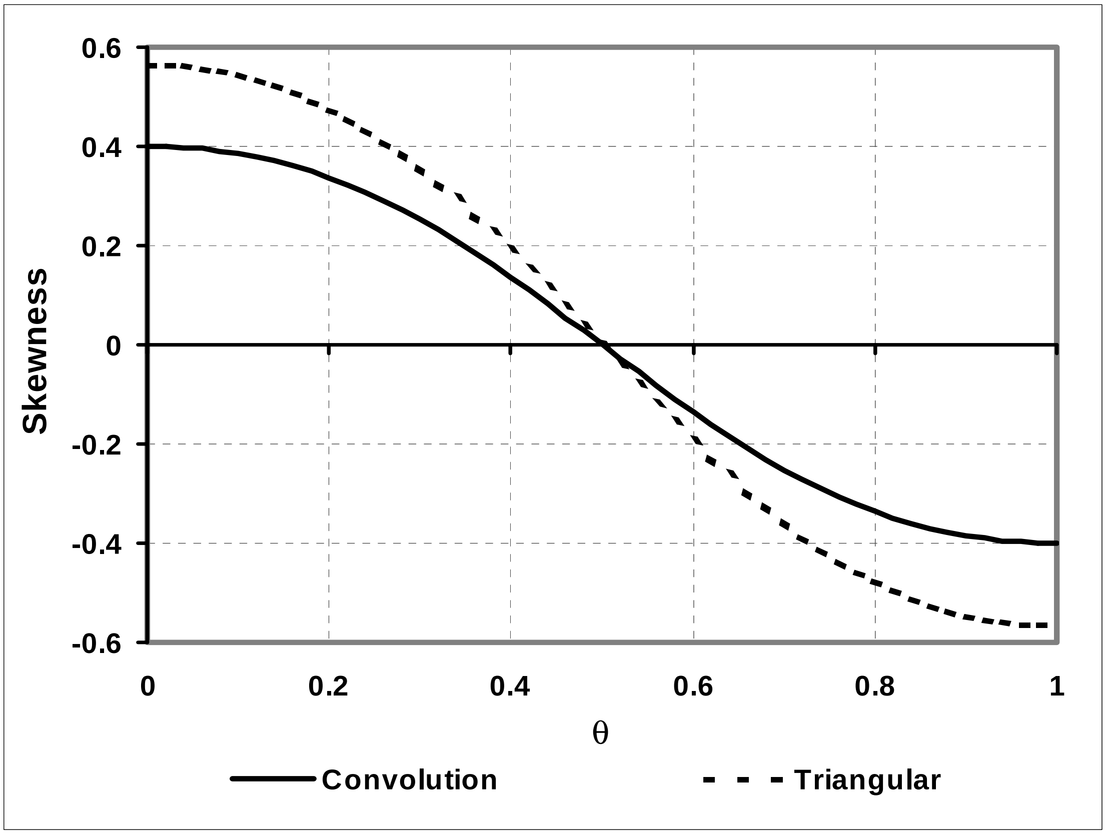


Figure 6. Comparison of skewness $\beta_1$ between convolution of identical triangular distributions and that of its components.

The kurtosis $\beta_2$ in case of the convolution of identical and reflected triangular distributions equals $2\frac{7}{10}$ regardless of the value of $\theta$. The kurtosis $\beta_2$ of a triangular distribution is also constant and equals $2\frac{4}{10}$ (see, e.g., Johnson and Kotz (8)). The latter result is perhaps more intuitive than the former. Indeed, the kurtosis in all the examples in Figure 4 equal $2\frac{7}{10}$, which indicates by now the well known fact that kurtosis is not just a measure of *peakedness*, but rather a measure of *peakedness and fatness of the tails* (see, e.g., Stuart and Ord (7)). The values $2\frac{7}{10}$ (kurtosis of a convolution of 4 uniform random variables), $2\frac{4}{10}$ (kurtosis of a convolution of two uniform random variables) and $1\frac{8}{10}$ ( kurtosis of a uniform random variable) are indicative of the rate at which sums of uniform random variables approach a normal distribution (with a kurtosis of $3.0$ ). It should however be noted, that skewness $0$ and a kurtosis $3$ are only necessary, but not sufficient conditions for normality of the distribution. Indeed, as a by product of our investigations one can easily verify utilizing the above setup that a "peaked" TSP distribution with $\theta = \frac{1}{2}$ and $n \approx 3.3723$ yields a continuous non-normal distribution with skewness $\beta_1 = 0$ and kurtosis $\beta_2 = 3.0000$.

## 4. CONCLUDING REMARKS

Two topics are discussed in this paper. Firstly, a mechanism of generating a wide variety of two-sided continuous distributions with bounded support characterized by a reflection parameter $\theta$ and secondly, a procedure for calculating convolutions of these distributions utilizing seven non-overlapping regions of integration is presented. We found this representation of the convolution procedure to be in tune with the available mathematical software but welcome comments and suggestions for possible alternative approaches which may perhaps be appropriate in more general cases.

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