Research Article

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Generalized 4-connectivity of hierarchical star networks

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Abstract: The connectivity is an important measurement for the fault-tolerance of a network. The generalized connectivity is a natural generalization of the classical connectivity. An S-tree of a connected graph G is a tree $T = (V', E')$ that contains all the vertices in S subject to $S \subseteq V(G)$. Two S-trees T and T' are internally disjoint if and only if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. Denote by $\kappa(S)$ the maximum number of internally disjoint S-trees in graph G. The generalized k-connectivity is defined as $\kappa_k(G) = \min\{\kappa(S)|S \subseteq V(G)$ and $|S| = k\}$. Clearly, $\kappa_2(G) = \kappa(G)$. In this article, we show that $\kappa_4(HS_n) = n - 1$, where HS_n is the hierarchical star network.

Keywords: hierarchical star networks, fault-tolerance, generalized connectivity, disjoint S-trees

MSC 2020: 05C05, 05C40, 05C76

1 Introduction

The graphs considered in this article are simple undirected finite graphs. For graph theory symbols and terms that are covered but not mentioned in this article, please refer to [1]. Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $u \in V(G)$, and $N_G(u) = \{v \in V(G)\backslash u | uv \in E(G)\}$ be the neighbour set of u in the graph G, and $N_G[u] = N_G(u) \cup \{u\}$. Let $d_G(u) = |N_G(u)|$ be the degree of u in G. If $d_G(v) = k$ for any vertex $v \in V(G)$, then the graph G is k-regular. For any vertices $u, v \in V(G)$, an (u, v)-path starts at u and ends at v. Any two (u, v)-paths P and Q are internally disjoint if and only if $V(P) \cap V(Q) = \{u, v\}$.

The *classic connectivity* of graph G, denoted as $\kappa(G)$, is an important parameter to measure the reliability and fault-tolerance of the network. The connectivity $\kappa(G)$ is larger, the reliability of the network is higher. There are two versions to define $\kappa(G)$. The version definition of "cut" is that deleting the minimum number of vertices disconnects the graph G. The version of "path" is defined as follows: for any vertex set $S = \{u, v\}$, $\kappa_G(S)$ represents the maximum number of internally disjoint paths joining u and v in G, and $\kappa(G) = \min\{\kappa_G(S)|S \subseteq V(G)$ and $|S| = 2\}$.

Although there are fruitful research results in the study of classical connectivity, classical connectivity itself has certain limitations, which lead to large defects in evaluating the reliability of the network. For example, in the actual application of an interconnection network, all processors connected to the same processor are less likely to fail at the same time, so this parameter is not accurate enough to measure network reliability and fault tolerance. In view of this, Chartrand et al. [2] generalized classical connectivity

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and proposed the concept of *generalized connectivity*. Let G be a connected graph with $S \subseteq V(G)$. An S-tree of graph G is a tree $T = (V', E')$ that contains all the vertices in S subject to $S \subseteq V(G)$. Two S-trees T and T' are internally disjoint if and only if $E(T) \cap E(T') = \varnothing$ and $V(T) \cap V(T') = S$. Denote $\kappa(S)$ as the maximum number of internally disjoint S-trees in graph G. The *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa(S) | S \subseteq V(G) \text{ and } |S| = k\}$, clearly, $\kappa_2(G) = \kappa(G)$.

The bounds of generalized connectivity and the relationship between connectivity and generalized connectivity have been extensively investigated [3–5]. In addition, the generalized 3-connectivity of many networks has been studied, for example, complete graphs [6], card product graphs [7], product graphs [8], complete bipartite graph [9], star graphs, and alternating group graphs [10]. However, there are few results about generalized 4-connectivity, such as hypercubes [11], exchanged hypercubes [12], dual cube [13], and hierarchical cubic networks [14]. For more works and results on generalized connectivity, please refer to [15–18].

In this article, we study the generalized 4-connectivity of the hierarchical star networks.

2 Hierarchical star networks and their properties

In this section, we give the definition, structure of the hierarchical star networks, and some lemmas.

Definition 1. [19] An n-dimensional star graph, denoted by the graph S_n, is defined as an undirected graph with each vertex representing a distinct permutation of $[n]$ and two vertices are adjacent if and only if their labels differ only in the first and another position, that is, two vertices $u = u_1u_2 \dots u_n$, $v = v_1v_2 \dots v_n$ are adjacent if and only if $v = u_iu_2u_3 \dots u_{i-1}u_1u_{i+1} \dots u_n$ for some $i \in [n]\backslash\{1\}$, where $[n] = 1, 2, \dots, n$. In this situation, (u, v) is an i-edge.

The star graph S_4 is shown in Figure 1.

Let Γ_n be a permutation group on the set $[n]$ and $S = \{(1, 2), (1, 3), \dots, (1, n)\}$, where $(1, i)$ is a transposition of Γ_n. Then S_n is the undirected Cayley graph Cay(Γ_n, S). For a permutation $x \in \Gamma_n$, the permutation by interchanging the first element with ith element of x is denoted as $x(1, i)$ for $i \in [n]\backslash\{1\}$.

For two integers $i, j \in [n]$, denoted by $S_n^{j:i}$ the subgraph of S_n induced by all the vertices with the jth element being i. For a fixed dimension $j \in [n]\backslash 1$, S_n can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $i \in [n]$.

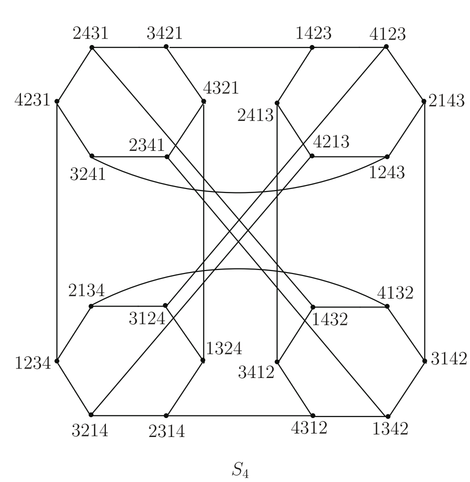

Figure 1: The graph of S_4.

Lemma 2.1. [19] *For any integer $n \geq 3$, S_n is $(n-1)$-regular and $(n-1)$-connected, vertex transitive, edge transitive, bipartite graph with girth 6. Any two vertices have at most one common neighbor in S_n.*

Definition 2. [20] An n-dimensional hierarchical star network, HS_n, is made of $n!$ n-dimensional star graphs S_n, called copies. Each vertex of HS_n is denoted by a two-tuple address $\langle a, b\rangle$, where both a and b are arbitrary permutation of n distinct symbols. The first n-bit permutation a identifies the copy of a and the second n-bit permutation b identifies the position of b inside its copy. Two vertices $\langle a, b\rangle$ and $\langle \hat{a}, \hat{b}\rangle$ in HS_n are adjacent, if one of the following three conditions holds:

(1) $a = \hat{a}$ and $(b, \hat{b}) \in E(S_n)$. That is, $\langle a, b\rangle$ is adjacent to $\langle a, \hat{b}\rangle$ if $(b, \hat{b}) \in E(S_n)$;
(2) $a \neq \hat{a}$, $a = b$, and $\hat{a} = \hat{b} = a(1, n)$. That is, $\langle a, a\rangle$ is adjacent to $\langle a(1, n), a(1, n)\rangle$;
(3) $a \neq \hat{a}$, $a \neq b$, $a = \hat{b}$, and $b = \hat{a}$. That is, $\langle a, b\rangle$ is adjacent to $\langle b, a\rangle$ if $a \neq b$.

The hierarchical star networks HS_2 and HS_3 are shown in Figure 2.

Remark 2.1. [21] Each node in HS_n is assigned a label $\langle a, b\rangle = \langle a_1a_2 \dots a_n, b_1b_2 \dots b_n\rangle$, where $a_1a_2 \dots a_n$ and $b_1b_2 \dots b_n$ are permutations of n distinct symbols (not necessarily distinct from each other). The edges of the HS_n are defined by the following n generators:

$$h_1(\langle a, b\rangle) = \begin{cases} \langle a(1, n), b(1, n)\rangle, & a = b; \\ \langle b, a\rangle, & a \neq b, \end{cases}$$

and $h_i(\langle a, b\rangle) = \langle a, b(1, i)\rangle$ for $i \in [n]\backslash\{1\}$.

Let $\langle a, b\rangle$ be a vertex of HS_n. The neighbor set of $\langle a, b\rangle$ is exactly $\{h_i(\langle a, b\rangle) | i \in [n]\}$. Furthermore, $h_1(\langle a, b\rangle)$ is called the external neighbor of $\langle a, b\rangle$ and $h_i(\langle a, b\rangle)$ is called the internal neighbor of $\langle a, b\rangle$ for $i \in [n]\backslash\{1\}$. We denote by HS_n^a the subgraph induced by the vertex set $\{\langle a, b\rangle \in V(HS_n) | b \in V(S_n)\}$, which is isomorphic to an n-dimensional star graph S_n identified by a. Moreover, we define $HS_n^{a,b}$ as a subgraph of HS_n induced by the vertex set $V(HS_n^a) \cup V(HS_n^b)$ and $HS_n^{\Gamma_n - \{a\}}$ as a subgraph of HS_n induced by the vertex set $V(HS_n)\backslash V(HS_n^a)$. For a vertex $x \in V(HS_n)$, we use x' to denote the external neighbor of x.

Remark 2.2. [21] Any vertex has exactly one external neighbor in HS_n, that is, every vertex $\langle a, b\rangle$ in HS_n^a is exactly incident one cross edge $(\langle a, b\rangle, h_1(\langle a, b\rangle))$. There is one or two cross edges between any pair of copies. Moreover, for a fixed copy HS_n^a, there are two cross edges between HS_n^a and $HS_n^{a(1,n)}$; there is only one cross edge between HS_n^a and HS_n^b, where $b \in \Gamma_n \backslash \{a, a(1, n)\}$.

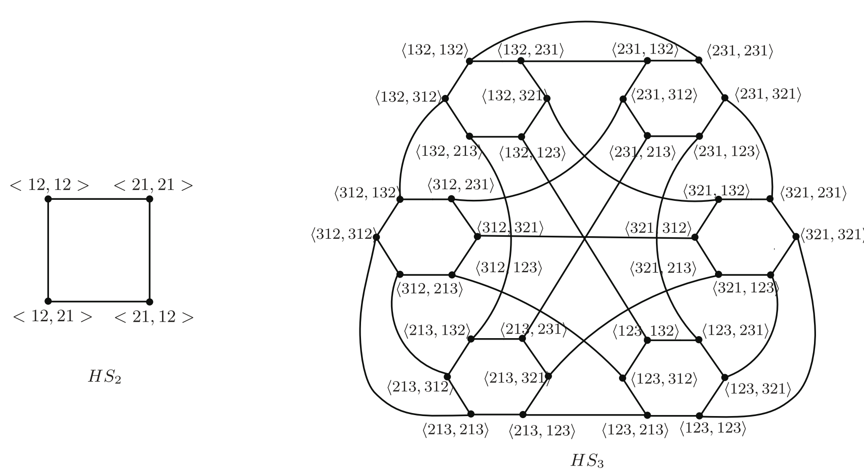

Figure 2: The graph of HS_2 and HS_3.

Lemma 2.2. [20] *For any integer $n \geq 3$, HS_n is an n-regular n-connected graph, and its girth is 4. Any two vertices have at most two common neighbors in HS_n.*

Lemma 2.3. *Let $HS_n^1, HS_n^2, \dots, HS_n^{n!}$ be the $n!$ copies of HS_n, and $H = HS_n[\bigcup_{j_i=1}^{k} V(HS_n^{j_i})]$ for $j_i \in [n!]$, $k \geq 1$, and $n \geq 3$, $1 \leq i \leq n!$, then H is connected.*

Proof. Without loss of generality, suppose $H = HS_n[\bigcup_{j_i=1}^{k} V(HS_n^{j_i})]$. By Remark 2.2, there is at least one cross edge between any two distinct copies of HS_n. Thus, H is connected. □

Lemma 2.4. *For $j_i \in [n!]$, $1 \leq i \leq n!$, let $v \in V(HS_n^{j_i})$ with $v = \langle x, y\rangle$ and $x = y$. The external neighbors of different vertices in $V(HS_n^{j_i})\backslash\{v\}$ belong to different copies of HS_n. Moreover, if $u \in V(HS_n^{j_i})\backslash\{v\}$ with $u = \langle x, y(1, n)\rangle$, then v', u' belong to the same copy of HS_n.*

Proof. Let $v_1, v_2 \in V(HS_n^{j_i})\backslash\{v\}$ with $v_1 = \langle x_1, y_1\rangle$, $v_2 = \langle x_2, y_2\rangle$. By the definition of HS_n, $x_1 \neq y_1$, $x_2 \neq y_2$, and $y_1 \neq y_2$, $v' = \langle x(1, n), y(1, n)\rangle$. Clearly, $v_1' = \langle y_1, x_1\rangle$ and $v_2' = \langle y_2, x_2\rangle$. Since $y_1 \neq y_2$, v_1' and v_2' belong to different copies of HS_n. Let $u \in V(HS_n^{j_i})\backslash\{v\}$, $x \neq y(1, n)$, and $u' = \langle y(1, n), x\rangle$. Since $x(1, n) = y(1, n)$, v', u' belong to the same copy of HS_n. □

3 Generalized 4-connectivity of HS_n

In this section, we will study the generalized 4-connectivity of hierarchical star networks. To prove the main result, the following results are useful.

Lemma 3.1. [4] *If there are two adjacent vertices of degree $\delta(G)$, then $\kappa_k(G) \leq \delta(G) - 1$ for $3 \leq k \leq |V(G)|$.*

Lemma 3.2. [1] *Let G be a k-connected graph, and let x and y be a pair of distinct vertices in G. Then there exist k internally disjoint paths $P_1, P_2, \dots, P_k$ in G connecting x and y.*

Lemma 3.3. [1] *Let $G = (V, E)$ be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V\backslash\{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y. That is, there exists a family of k internally vertex-disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

Lemma 3.4. [1] *Let $G = (V, E)$ be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X, Y)-paths in G.*

Theorem 3.1. [22] $\kappa_3(S_n) = n - 2$, *for $n \geq 3$.*

Theorem 3.2. [23] $\kappa_4(S_n) = n - 2$, *for $n \geq 3$.*

Lemma 3.5. *For any three vertices $x, y, z \in V(S_n)$, $n \geq 3$, there exist $n-2$ internally disjoint trees $\widehat{T}_1, \dots, \widehat{T}_{n-2}$ connecting x, y, z and each $\widehat{T}_i$ contains a vertex u_i such that u_i is distinct from x, y, z and $u_i \neq u_j$, for $1 \leq i \neq j \leq n-2$.*

Proof. By Theorem 3.2, $\kappa_4(S_n) = n - 2$. That is to say, for any four distinct vertices $x, y, z, u \in V(S_n)$, there are $n-2$ internally disjoint trees $T_1', \dots, T_{n-2}'$ connecting x, y, z, u in S_n. There exists some integer i such that $|V(T_i')| = 4$ and $|V(T_j')| \geq 5$, where $1 \leq i \neq j \leq n-2$.

Without loss of generality, suppose $|V(T_1')| = 4$. By Lemma 2.1, $d_{S_n}(u) = n - 1$.

Case 1. $d_{T_1'}(u) = 2$.

Obviously, for T_j', $2 \le j \le n-2$, $d_{T_j'}(u) = 1$. Let $\widehat{T}_1 = T_1'$ and $\widehat{T}_j = T_j' - u$ for $1 \le i \ne j \le n-2$. Then $\widehat{T}_1, \dots, \widehat{T}_{n-2}$ are the desired trees.

Case 2. $d_{T_1'}(u) = 1$.

Case 2.1. For $2 \le j \le n-2$, $d_{T_j'}(u) = 1$.

The proof is similar to that of Case 1.

Case 2.2. For $2 \le j \le n-2$, there exists some T_j' subject to $d_{T_j'}(u) = 2$.

Without loss of generality, suppose $d_{T_2'}(u) = 2$. Let $\widehat{T}_1 = T_1'$, $\widehat{T}_2 = T_2'$, $\widehat{T}_j = T_j' - u$ for $3 \le j \le n-2$. Then $\widehat{T}_1, \dots, \widehat{T}_{n-2}$ are the desired trees. □

Theorem 3.3. $\kappa_4(HS_n) = n-1$ *for* $n \ge 2$.

By Lemma 3.1, $\kappa_4(HS_n) \le \delta(HS_n) - 1 = n - 1$. To obtain Theorem 3.3, it suffices to establish the following claim.

Claim. For any given vertex set $S = \{x, y, z, w\} \subseteq V(HS_n)$, there exist $n-1$ internally disjoint S-trees in HS_n.

Proof. It proceeds by induction on n. Obviously, HS_2 is connected, there is a tree connecting x, y, z, w in HS_2. Suppose that $n \ge 3$ and the theorem holds for $n-1$. □

Recall that HS_n consists of $n!$ copies isomorphic to S_n, denoted by HS_n^{ℓ}, $\ell \in \Gamma_n$. We consider the following cases.

Case 1. The vertices of S are distributed in one copy of HS_n.

Without loss of generality, suppose $x, y, z, w \in V(HS_n^{\alpha})$, where $\alpha \in \Gamma_n$. By Theorem 3.2, there exist $n-2$ internally disjoint S-trees connecting x, y, z, w in HS_n^{α}, say $T_1, T_2, \dots, T_{n-2}$. Let $H = HS_n^{\Gamma_n - \{\alpha\}}$, by Lemma 2.3, H is connected. Clearly, $x', y', z', w' \in V(H)$, then there exists a tree $\widehat{T}_{n-1}$ connecting x', y', z', w' in H. Let $T_{n-1} = \widehat{T}_{n-1} \cup xx' \cup yy' \cup zz' \cup ww'$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 2. The vertices of S are distributed in two distinct copies of HS_n.

Without loss of generality, suppose $x, y, z \in V(HS_n^{\alpha})$ and $w \in V(HS_n^{\beta})$, where $\alpha, \beta \in \Gamma_n$, $\alpha \ne \beta$. By Lemma 3.5, there are $n-2$ internally disjoint trees connecting x, y, z, say $\widehat{T}_i$, and $u_i \in V(\widehat{T}_i)$ is distinct with x, y, z for $i \in \{1, \dots, n-2\}$.

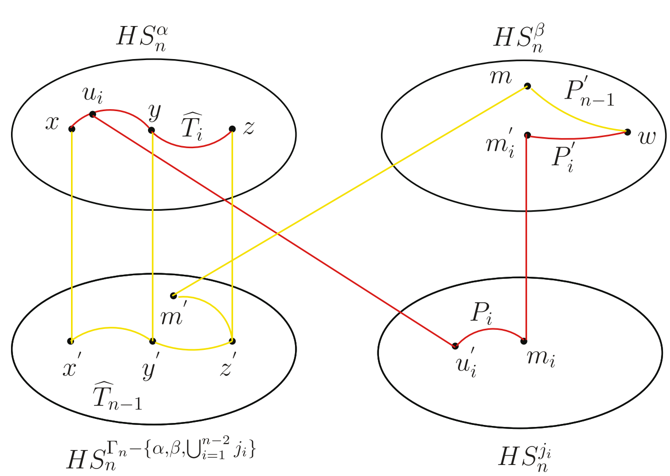

Figure 3: The illustration of Case 2.1.1.

Case 2.1. There is no cross edge between $\{\widehat{T}_1, \dots, \widehat{T}_{n-2}\}$ and HS_n^{β}.

Without loss of generality, suppose $u_i' \in V(HS_n^{j_i})$, where $j_i \in \Gamma_n - \{\alpha, \beta\}$, $i \in \{1, \dots, n-2\}$. First, we suppose that different vertices of $\{u_1', \dots, u_{n-2}'\}$ belong to different copies.

Case 2.1.1. $x', y', z' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$ for $i \in \{1, \dots, n-2\}$.

Clearly, there exists a path P_i joining u_i' and m_i in $HS_n^{j_i}$ such that $m_i' \in V(HS_n^{\beta})$. Let $W = \{m_1', \dots, m_{n-2}', m\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths joining w and W, say $P_1', \dots, P_{n-1}'$, $m_i' \in V(P_i')$, $m \in V(P_{n-1}')$, $m' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$, for $i \in \{1, \dots, n-2\}$. Clearly, $x', y', z', m' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$, and so there is a tree $\widehat{T}_{n-1}$ connecting x', y', z', m'. Let $T_i = \widehat{T}_i \cup u_iu_i' \cup P_i \cup m_im_i' \cup P_i'$ for $i \in \{1, \dots, n-2\}$ and $T_{n-1} = xx' \cup yy' \cup zz' \cup mm' \cup P_{n-1}' \cup \widehat{T}_{n-1}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 3).

Case 2.1.2. One of $\{x', y', z'\}$ belongs to some $HS_n^{j_i}$ for $i \in \{1, \dots, n-2\}$.

Suppose x' belongs to the copy $HS_n^{j_{n-2}}$, clearly u_{n-2}' also belongs to $HS_n^{j_{n-2}}$. Let $m_{n-2}, h \in V(HS_n^{j_{n-2}})$, $X = \{u_{n-2}', x'\}$, and $Y = \{m_{n-2}, h\}$ such that $m_{n-2}' \in V(HS_n^{\beta})$, $h' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$. By Lemma 3.4, there are two internally disjoint paths between X and Y, say P_{n-2} and P_{n-1}, where P_{n-2} is the path joining u_{n-2}' and m_{n-2} and P_{n-1} is the path joining x' and h. Let $W = \{m_1', \dots, m_{n-2}', m\}$, by Lemma 3.3, there are $n-1$ internally disjoint paths between w and W, say $P_1', \dots, P_{n-1}'$, and $m_i' \in V(P_i')$, $m \in V(P_{n-1}')$, such that $m' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$. In view of $y', z', m', h' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$, there exists a tree connecting y', z', m', h', say $\widehat{T}_{n-1}$. Let $T_i = \widehat{T}_i \cup u_iu_i' \cup P_i \cup m_im_i' \cup P_i'$ for $i \in \{1, \dots, n-2\}$ and $T_{n-1} = xx' \cup yy' \cup zz' \cup P_{n-1} \cup hh' \cup mm' \cup P_{n-1}' \cup \widehat{T}_{n-1}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 4).

Next, we suppose two of $\{u_1', \dots, u_{n-2}'\}$ belong to the same copy. Let $X' = (X \backslash \{x'\}) \cup \{u_{n-3}'\}$, $Y' = (Y \backslash \{h\}) \cup \{m_{n-3}\}$, and $m_{n-3}' \in V(HS_n^{\beta})$. We obtain $n-3$ internally disjoint S-tees $T_1, T_2, \dots, T_{n-4}, T_{n-2}$, similar to that of Case 2.1.2 by replacing X with X' and Y with Y'. In view of $x', y', z', m' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}})$, there is a tree connecting x', y', z', m' say $\widehat{T}_{n-1}$. Let $T_{n-3} = \widehat{T}_{n-3} \cup u_{n-3}u_{n-3}' \cup P_{n-3} \cup m_{n-3}m_{n-3}' \cup P_{n-3}'$, $T_{n-1} = xx' \cup yy' \cup zz' \cup mm' \cup P_{n-1}' \cup \widehat{T}_{n-1}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 2.2. There is one cross edge between $\{\widehat{T}_1, \dots, \widehat{T}_{n-2}\}$ and HS_n^{β}.

Consequently, we just consider the case different vertices of $\{u_1', \dots, u_{n-2}'\}$ belong to different copies and the case that two of $\{u_1', \dots, u_{n-2}'\}$ in the same copy is similar.

Case 2.2.1. For some $i \in \{1, \dots, n-2\}$, $u_i' \in V(HS_n^{\beta})$.

Without loss of generality, suppose $u_{n-2}' \in V(HS_n^{\beta})$, $u_i' \in V(HS_n^{j_i})$ for $i \in \{1, \dots, n-3\}$.

When $x', y', z' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}})$, we set $W' = (W \backslash \{m_{n-2}'\}) \cup \{u_{n-2}'\}$. We obtain $n-2$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-3}, T_{n-1}$, similar to that of Case 2.1.1 by replacing W with W'. And $T_{n-2} = \widehat{T}_{n-2} \cup P_{n-2}' \cup u_{n-2}u_{n-2}'$ (Figure 5[a]).

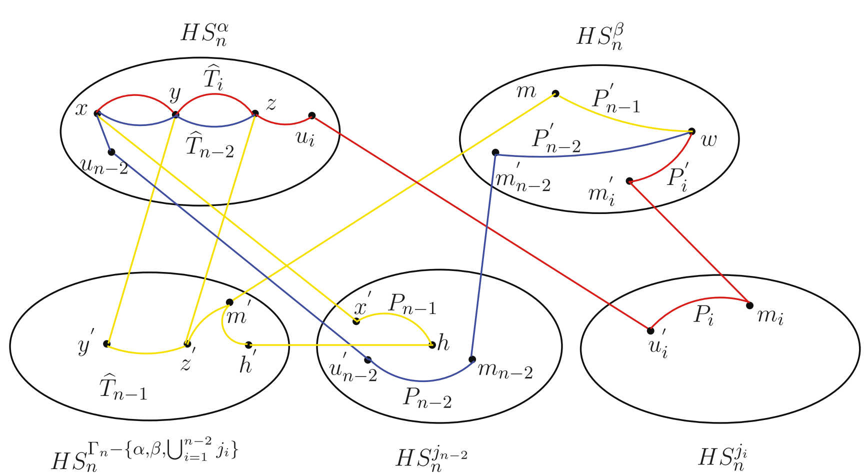

Figure 4: The illustration of Case 2.1.2.

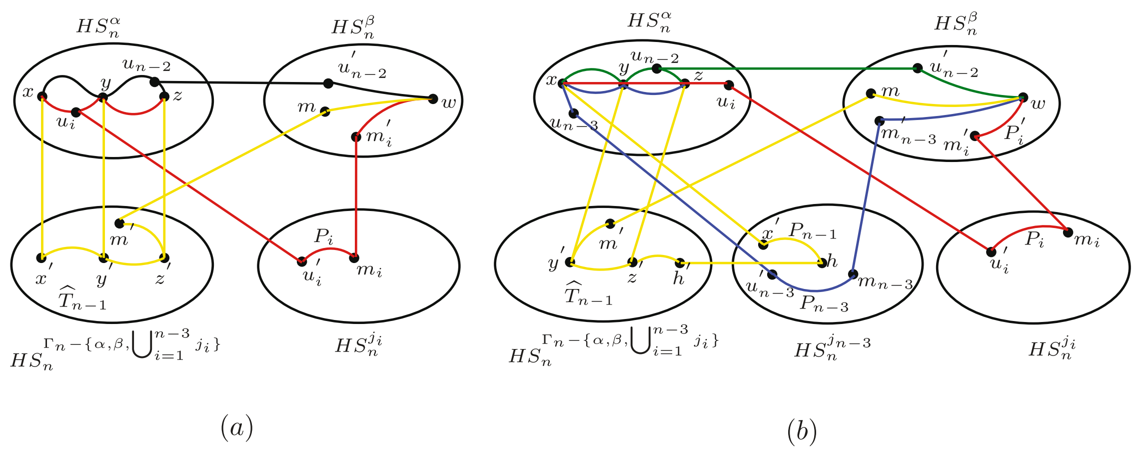

Figure 5: The illustrations of Case 2.2.1: (a) $x', y', z' \in V\left(HS_n^{\Gamma_n-\{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}}\right)$; and (b) $x', u'_{n-3} \in V(HS_n^{j_{n-3}})$.

When one of $\{x', y', z'\}$ and u_i' belong to the same copy, for some $i \in \{1, \ldots, n-3\}$, we suppose that x', u'_{n-3} belong to the same copy $HS_n^{j_{n-3}}$. Let $X' = (X \backslash \{u'_{n-2}\}) \cup \{u'_{n-3}\}$, $Y' = (Y \backslash \{m_{n-2}\}) \cup \{m_{n-3}\}$, and $W' = (W \backslash \{m'_{n-2}\}) \cup \{u'_{n-2}\}$. We obtain $n-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{n-3}, T_{n-1}$, similar to that of Case 2.1.2 by replacing X with X', Y with Y', W with W'. And $T_{n-2} = \widehat{T}_{n-2} \cup P'_{n-2} \cup u_{n-2}u'_{n-2}$ (Figure 5[b]).

Case 2.2.2. One of $\{x', y', z'\}$ belongs to HS_n^{β}.

Without loss of generality, suppose $z' \in V(HS_n^{\beta})$, $u_i' \in V(HS_n^{j_i})$, where $j_i \in \Gamma_n - \{\alpha, \beta\}$, $i \in \{1, \ldots, n-2\}$. There is a path joining u_i' and m_i in $HS_n^{j_i}$, say P_i, and $m_i' \in V(HS_n^{\beta})$, for $i \in \{1, \ldots, n-2\}$.

Case 2.2.2.1. $x', y' \in V(HS_n^{\Gamma_n-\{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$.

When $z' \in N_{HS_n^{\beta}}(w)$, we set $\widehat{T}_{n-1}$ be the tree connecting x', y', m'. We obtain $n-2$ internally disjoint S-trees $T_1, T_2, \ldots, T_{n-2}$, similar to that of Case 2.1.1. And $T_{n-1} = \widehat{T}_{n-1} \cup P'_{n-1} \cup xx' \cup yy' \cup zz' \cup mm'$.

When $z' \notin N_{HS_n^{\beta}}(w)$, we set P'_{n-1} be the path from w to z', then there exists a vertex a distinct from z' and w in P'_{n-1}. First, we suppose $a' \in V(HS_n^{\Gamma_n-\{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$. The proof is similar to the case of $z' \in N_{HS_n^{\beta}}(w)$. Next, we suppose $a' \in V(HS_n^{\alpha,j_i})$, for some $i \in \{1, \ldots, n-2\}$. Clearly, in this case, $w' \in V(HS_n^{\Gamma_n-\{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$, the proof is similar to the case of $a' \in V(HS_n^{\Gamma_n-\{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$ (Figure 6[a]).

Case 2.2.2.2. For some $i \in \{1, \ldots, n-2\}$, one of $\{x', y'\}$ belongs to $V(HS_n^{j_i})$.

Without loss of generality, suppose $x', u'_{n-2} \in V(HS_n^{j_{n-2}})$.

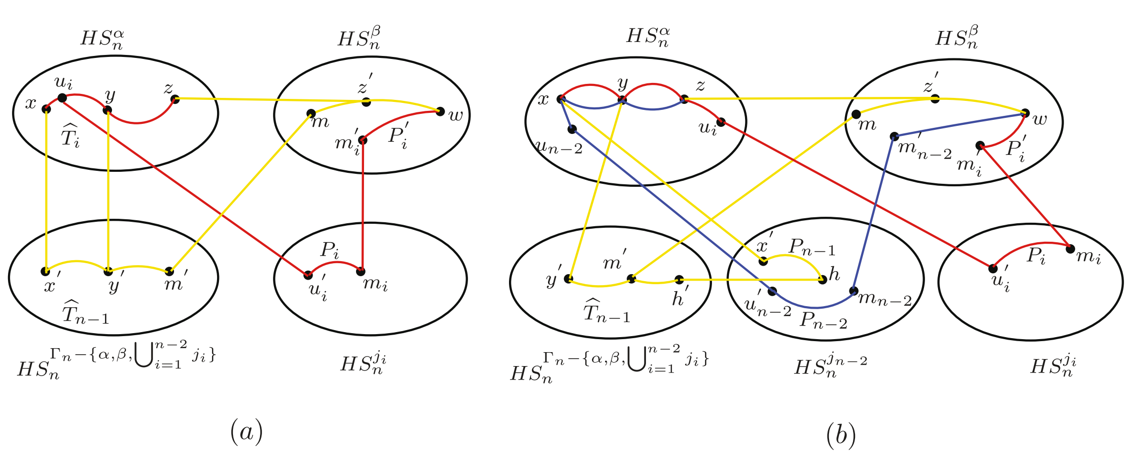

Figure 6: The illustrations of $z' \in N_{\{HS_n^{\beta}\}}(w)$: (a) Case 2.2.2.1 and (b) Case 2.2.2.2.

When $z' \in N_{HS_n^\beta}(w)$, we set $\widehat{T}_{n-1}$ be the tree connecting h', y', m'. We can obtain $n-2$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-2}$, similar to Case 2.1.2. And $T_{n-1} = \widehat{T}_{n-1} \cup P_{n-1} \cup P'_{n-1} \cup xx' \cup yy' \cup zz' \cup mm' \cup hh'$ (Figure 6[b]).

When $z' \notin N_{HS_n^\beta}(w)$, there exists a vertex a distinct from z' and w in P'_{n-1}. First, we suppose $a' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$. The proof is similar to the case of $z' \in N_{HS_n^\beta}(w)$. Next, we suppose $a' \in V(HS_n^{\alpha,j_i})$ for some $i \in \{1,\dots,n-2\}$. Clearly, in this case, $w' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$. The proof is similar to the case of $a' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i\}})$.

Case 2.3. There are two cross edges between $\{\widehat{T}_1, \dots, \widehat{T}_{n-2}\}$ and HS_n^β.

Case 2.3.1. One of $\{x', y', z'\}$ belongs to HS_n^β, and some $u'_i \in V(HS_n^\beta)$ for $i \in \{1,\dots,n-2\}$.

Without loss of generality, suppose $z', u'_{n-2} \in V(HS_n^\beta)$, and $u'_i \in V(HS_n^{j_i})$ for $i \in \{1,\dots,n-3\}$. There exists a path joining u'_i and m_i in $HS_n^{j_i}$, say P_i, and $m'_i \in V(HS_n^\beta)$, for $i \in \{1,\dots,n-3\}$.

Let $W = \{m'_1, \dots, m'_{n-3}, u'_{n-2}, z'\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between w and W, say $P'_1, \dots, P'_{n-1}$, and $m'_i \in V(P'_i)$ for $i \in \{1,\dots,n-3\}$, $u'_{n-2} \in V(P'_{n-2})$, $z' \in V(P'_{n-1})$. If $w \notin W$, clearly, $x', y', w' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}})$, there exists a tree connecting x', y', and w', say $\widehat{T}_{n-1}$. Let

$$\begin{cases} T_i = \widehat{T}_i \cup u_i u'_i \cup P_i \cup m_i m'_i \cup P'_i, & i \in \{1,\dots,n-3\} \\ T_{n-2} = \widehat{T}_{n-2} \cup u_{n-2}u'_{n-2} \cup P'_{n-2} \\ T_{n-1} = xx' \cup yy' \cup zz' \cup ww' \cup \widehat{T}_{n-1} \end{cases}$$

Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 7[a]). If $w \in W$, the proof is similar to the case of $w \notin W$.

Case 2.3.2. Two of $\{x', y', z'\}$ belong to HS_n^β.

Without loss of generality, suppose $x', y' \in V(HS_n^\beta)$, clearly, x' is adjacent to y'. Let $W' = (W \backslash \{u'_{n-2}, z'\}) \cup \{x', y'\}$, we obtain $n-3$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-3}$, similar to that of Case 2.3.1 by replacing W with W'. Let $x' \in V(P'_{n-2})$, $y' \in V(P'_{n-1})$. Since the girth of HS_n^β is 6, there is a vertex m in P'_{n-2} or P'_{n-1} such that m is distinct with x', y', and w. Without loss of generality, suppose $m \in V(P'_{n-2})$. Since there are two cross edges between HS_n^α and HS_n^β, m' and w' belong to different copies, z' and u'_{n-2} belong to different copies. We consider the case that m' belongs to different copies with z' or u'_{n-2}, and the case that m' belongs to the same copy with z' or u'_{n-2} is similar. Suppose $z' \in V(HS_n^{j_{n-2}})$, $m' \in V(HS_n^{j_{n-1}})$. By Lemma 2.3, $HS_n^{j_{n-2},j_{n-1}}$ is connected, then there is a path joining z' and m', say Q. Obviously, $u'_{n-2}, w' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}})$. By Lemma 2.3, $HS_n^{\Gamma_n - \{\alpha,\beta,\bigcup_{i=1}^{n-3} j_i\}}$ is connected, then there is a path joining u'_{n-2} and w', say R. Let $T_{n-2} = \widehat{T}_{n-2} \cup u_{n-2}u'_{n-2} \cup R \cup ww'$, $T_{n-1} = xx' \cup yy' \cup zz' \cup P'_{n-2} \cup P'_{n-1} \cup mm' \cup Q$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 7[b]).

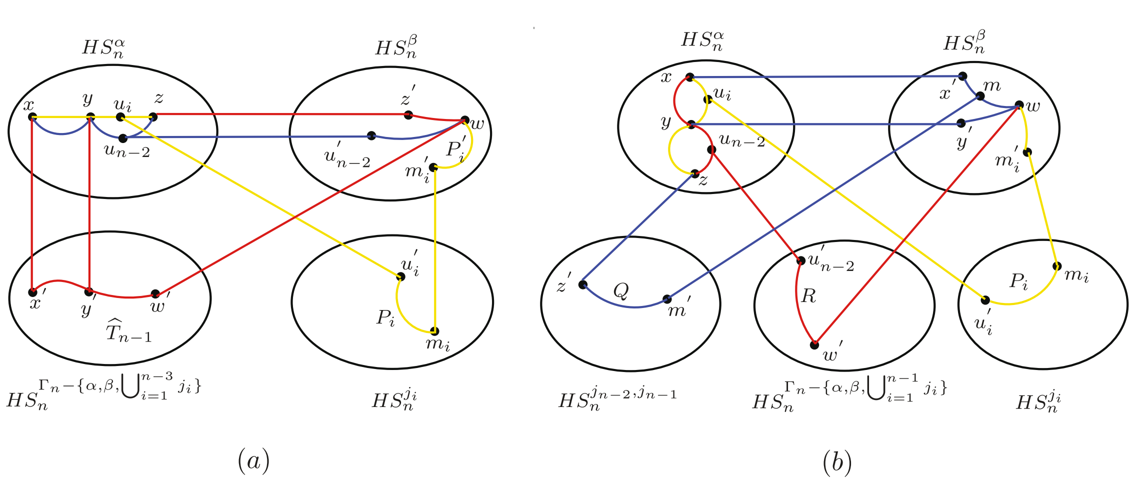

Figure 7: The illustrations of (a) Case 2.3.1 and (b) Case 2.3.2.

Case 2.3.3. Two of $\{u'_1, \ldots, u'_{n-2}\}$ belong to HS_n^{β}.

Let $W' = (W \backslash \{m'_{n-3}, z'\}) \cup \{u'_{n-3}, m\}$, and $m' \in V(HS_n^{\Gamma_n - \{\alpha, \beta, \bigcup_{i=1}^{n-4} j_i\}})$. We obtain $n-3$ internally disjoint S-trees $T_1, T_2, \ldots, T_{n-4}, T_{n-2}$, similar to that of Case 2.3.1 by replacing W with W'. Let $\widehat{T}_{n-1}$ be the tree connecting x', y', z', m', $T_{n-3} = \widehat{T}_{n-3} \cup u_{n-3}u'_{n-3} \cup P'_{n-3}$, $T_{n-1} = xx' \cup yy' \cup zz' \cup mm' \cup P'_{n-1} \cup \widehat{T}_{n-1}$. Then $T_1, T_2, \ldots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 3. The vertices of S are distributed equally in two distinct copies of HS_n.

Without loss of generality, suppose $x, y \in V(HS_n^{\alpha})$ and $z, w \in V(HS_n^{\beta})$, where $\alpha, \beta \in \Gamma_n$, $\alpha \neq \beta$. Since HS_n^{α} is isomorphic to S_n, by Lemma 3.2, there are $n-1$ internally disjoint paths joining x and y in HS_n^{α}, say $P_1, \ldots, P_{n-1}$. Similarly, there are $n-1$ internally disjoint paths joining z and w in HS_n^{β}, say $P'_1, \ldots, P'_{n-1}$. Let $x_i \in N(x) \cap V(P_i)$, $w_i \in N(w) \cap V(P'_i)$ for $i \in \{1, \ldots, n-1\}$. By Lemma 2.4, different vertices of $\{x'_1, \ldots, x'_{n-1}\}$ belong to distinct copies and different vertices of $\{w'_1, \ldots, w'_{n-1}\}$ belong to distinct copies. Obviously, at most one $x'_i \in V(HS_n^{\beta})$, and at most one $w'_i \in V(HS_n^{\alpha})$ for $i \in \{1, \ldots, n-1\}$. We suppose x'_i, w'_i belong to different copies, where $i \in \{1, \ldots, n-1\}$, and the case that for some $i \in \{1, \ldots, n-1\}$, x'_i, w'_i belong to the same copy is similar.

Case 3.1. None of $\{x'_1, \ldots, x'_{n-1}\}$ belongs to HS_n^{β}, and none of $\{w'_1, \ldots, w'_{n-1}\}$ belongs to HS_n^{α} for $i \in \{1, \ldots, n-1\}$.

As $n! - 2 \geq 2n - 2 (n \geq 3)$, we can suppose $x'_i \in V(HS_n^{\gamma_i})$, $w'_i \in V(HS_n^{\eta_i})$, and $\gamma_i, \eta_i \in \Gamma_n - \{\alpha, \beta\}$, $\gamma_i \neq \eta_i$ for $i \in \{1, \ldots, n-1\}$. By Lemma 2.3, $HS_n^{\gamma_i, \eta_i}$ is connected, there exists a path joining x'_i and w'_i, say $\widehat{P}_i$, for $i \in \{1, \ldots, n-1\}$. Let $T_i = P_i \cup P'_i \cup x_i x'_i \cup w_i w'_i \cup \widehat{P}_i$, where $i \in \{1, \ldots, n-1\}$. Then $T_1, T_2, \ldots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 3.2. One of $\{x'_1, \ldots, x'_{n-1}\}$ belongs to HS_n^{β}, and none of $\{w'_1, \ldots, w'_{n-1}\}$ belongs to HS_n^{α} for $i \in \{1, \ldots, n-1\}$.

Without loss of generality, suppose $x'_{n-1} \in V(HS_n^{\beta})$.

Case 3.2.1. $x'_{n-1} \in V(P'_i)$, where $i \in \{1, \ldots, n-1\}$.

Without loss of generality, suppose $x'_{n-1} \in V(P'_{n-1})$. We can obtain $n-2$ internally disjoint S-trees $T_1, \ldots, T_{n-2}$ similar to that of Case 3.1. Let $T_{n-1} = P_{n-1} \cup P'_{n-1} \cup x_{n-1}x'_{n-1}$. Then $T_1, T_2, \ldots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 3.2.2. $x'_{n-1} \notin V(P'_i)$, where $i \in \{1, \ldots, n-1\}$.

Let $W = \{w_1, \ldots, w_{n-1}\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths $Q_1, \ldots, Q_{n-1}$ between x'_{n-1} and W such that $w_i \in V(Q_i)$, where $i \in \{1, \ldots, n-1\}$. It is necessary to consider the following situations.

When $|V(Q_i) \cap V(P'_i)| = n-1$, that is, $V(Q_i) \cap V(P'_i) = W$, where $i \in \{1, \ldots, n-1\}$. We can obtain $n-2$ internally disjoint S-trees $T_1, \ldots, T_{n-2}$ similar to that of Case 3.1. Let $T_{n-1} = P_{n-1} \cup P'_{n-1} \cup Q_{n-1} \cup x_{n-1}x'_{n-1}$. Then $T_1, T_2, \ldots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

When $|V(Q_i) \cap V(P'_i)| \geq n$, where $i \in \{1, \ldots, n-1\}$. The proof is similar to that of Case 3.2.1.

Case 3.3. One of $\{x'_1, \ldots, x'_{n-1}\}$ belongs to HS_n^{β}, and one of $\{w'_1, \ldots, w'_{n-1}\}$ belongs to HS_n^{α}, where $i \in \{1, \ldots, n-1\}$.

The proof is similar to that of Case 3.2.

Case 4. The vertices of S are distributed in three distinct copies of HS_n.

Without loss of generality, suppose $x, y \in V(HS_n^{\alpha})$ and $z \in V(HS_n^{\beta})$, $w \in V(HS_n^{\gamma})$ where $\alpha, \beta, \gamma \in \Gamma_n$, $\alpha \neq \beta \neq \gamma$. Since HS_n^{α} is isomorphic to S_n, by Lemma 3.2, there exist $n-1$ internally disjoint paths joining x and y in HS_n^{α}, say $P_1, \ldots, P_{n-1}$. Let $N(x) = \{x_1, \ldots, x_{n-1}\}$ and $x_i \in N(x) \cap V(P_i)$, by Lemma 2.4, the external neighbors of $N(x)$ belong to different copies. Consequently, we just consider the case $y \notin N(x)$ and the proof of the case $y \in N(x)$ is similar.

Case 4.1. There are three cross edges between $N[x]$ and $HS_n^{\beta, \gamma}$.

Suppose $x'_i \in V(HS_n^{j_i})$, $x'_{n-2} \in V(HS_n^{\beta})$, $x'_{n-1} \in V(HS_n^{\gamma})$, where $j_i \in \Gamma_n - \{\alpha, \beta, \gamma\}$, $i \in \{1, \ldots, n-3\}$, and let $y' \in V(HS_n^{j_{n-2}})$. By Lemma 2.4, $x' \in V(HS_n^{\beta, \gamma})$.

When $x' \in V(HS_n^{\beta})$, there is a tree connecting x'_i, a_i, b_i in $HS_n^{j_i}$, say $\widehat{T}_i$, and $a'_i \in V(HS_n^{\beta})$, $b'_i \in V(HS_n^{\gamma})$ for $i \in \{1, \ldots, n-3\}$. There is a path joining y' and b_{n-2} in $HS_n^{j_{n-2}}$, say P, and $b'_{n-2} \in V(HS_n^{\gamma})$. Let $Z =$

$\{a'_1, \ldots, a'_{n-3}, x'_{n-2}, x'\}$, $W = \{b'_1, \ldots, b'_{n-2}, x'_{n-1}\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P'_1, \ldots, P'_{n-1}$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \ldots, \widehat{P}_{n-1}$, $a'_i \in V(P'_i)$ for $i \in \{1, \ldots, n-3\}$, $x'_{n-2} \in V(P'_{n-2})$, $x' \in V(P'_{n-1})$, $b'_i \in V(\widehat{P}_i)$ for $i \in \{1, \ldots, n-2\}$, $x'_{n-1} \in V(\widehat{P}_{n-1})$. Let

$$\begin{cases} T_i = P_i \cup P'_i \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x'_i \cup a_i a'_i \cup b_i b'_i,\ i \in \{1, \ldots, n-3\} \\ T_{n-2} = P_{n-2} \cup P'_{n-2} \cup \widehat{P}_{n-2} \cup x_{n-2}x'_{n-2} \cup yy' \cup P \cup b_{n-2}b'_{n-2} \\ T_{n-1} = P_{n-1} \cup P'_{n-1} \cup \widehat{P}_{n-1} \cup xx' \cup x_{n-1}x'_{n-1}. \end{cases}$$

Then $T_1, T_2, \ldots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 8).

When $x' \in V(HS_n^{\gamma})$, the proof is similar to the case of $x' \in V(HS_n^{\beta})$.

Case 4.2. There are two cross edges between $N[x]$ and $HS_n^{\beta,\gamma}$.

Case 4.2.1. For two integers $1 \le j \ne k \le n-1$, $x'_j \in V(HS_n^{\beta})$, $x' \in V(HS_n^{\gamma})$.

Without loss of generality, suppose $x'_{n-2} \in V(HS_n^{\beta})$, $x'_{n-1} \in V(HS_n^{\gamma})$, $x'_i \in V(HS_n^{j_i})$, for $j_i \in \Gamma_n - \{\alpha, \beta, \gamma\}$, $i \in \{1, \ldots, n-3\}$.

Case 4.2.1.1. For some $i \in \{1, \ldots, n-3\}$, $x' \in V(HS_n^{j_i})$.

Without loss of generality, suppose $x', x'_{n-3} \in V(HS_n^{j_{n-3}})$, and let $y' \in V(HS_n^{j_{n-2}})$. Let $u, v \in V(HS_n^{j_{n-3}})$, and $u' \in V(HS_n^{j_{n-1}})$, $v' \in V(HS_n^{\gamma})$. Let $A = \{x'_{n-3}, x'\}$, $B = \{u, v\}$. By Lemma 3.4, there are two disjoint paths

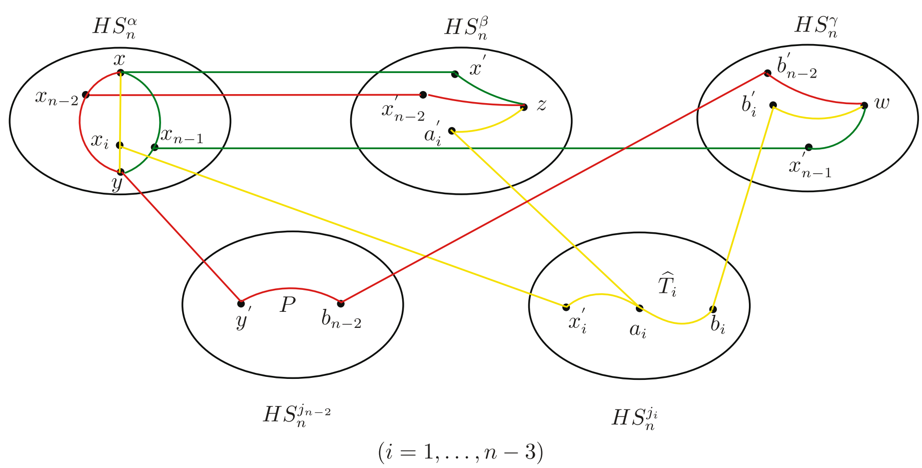

Figure 8: The illustration of $x' \in V(HS_n^{\beta})$.

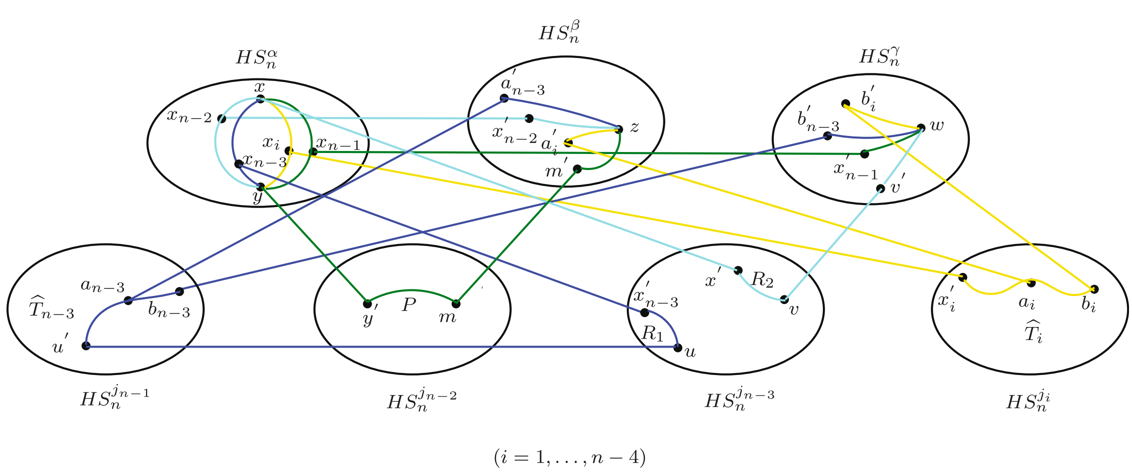

Figure 9: The illustration of Case 4.2.1.1.

between A and B, say R_1 and R_2, R_1 is the path joining x'_{n-3} and u, and R_2 is the path joining x' and v. There is a path joining y' and m in $HS_n^{j_{n-2}}$, say P and $m' \in V(HS_n^{\beta})$. There is a tree connecting x'_i, a_i and b_i in $HS_n^{j_i}$, say $\widehat{T}_i$ for $i \in \{1, \dots, n-4\}$, and $a'_i \in V(HS_n^{\beta})$, $b'_i \in V(HS_n^{\gamma})$. There is a tree connecting u', a_{n-3} and b_{n-3} in $HS_n^{j_{n-1}}$, say $\widehat{T}_{n-3}$ and $a'_{n-3} \in V(HS_n^{\beta})$, $b'_{n-3} \in V(HS_n^{\gamma})$. Let $Z = \{a'_1, \dots, a'_{n-3}, x'_{n-2}, m'\}$, $W = \{b'_1, \dots, b'_{n-3}, v', x'_{n-1}\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P'_1, \dots, P'_{n-1}$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and $a'_i \in V(P'_i)$, $x'_{n-2} \in V(P'_{n-2})$, $m' \in V(P'_{n-1})$, $b'_i \in V(\widehat{P}_i)$ for $i \in \{1, \dots, n-3\}$, $v' \in V(\widehat{P}_{n-2})$, $x'_{n-1} \in V(\widehat{P}_{n-1})$. Let

$$\begin{cases} T_i = P_i \cup P'_i \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x'_i \cup a_i a'_i \cup b_i b'_i,\ i \in \{1, \dots, n-4\} \\ T_{n-3} = P_{n-3} \cup P'_{n-3} \cup \widehat{P}_{n-3} \cup \widehat{T}_{n-3} \cup x_{n-3}x'_{n-3} \cup a_{n-3}a'_{n-3} \cup b_{n-3}b'_{n-3} \cup uu' \cup R_1 \\ T_{n-2} = P_{n-2} \cup P'_{n-2} \cup \widehat{P}_{n-2} \cup R_2 \cup x_{n-2}x'_{n-2} \cup xx' \cup vv' \\ T_{n-1} = P_{n-1} \cup P'_{n-1} \cup \widehat{P}_{n-1} \cup P \cup x_{n-1}x'_{n-1} \cup yy' \cup mm'. \end{cases}$$

Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 9).

Case 4.2.1.2. $x' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-3} j_i\}})$.

Without loss of generality, suppose $x' \in V(HS_n^{j_{n-2}})$. When $y' \in V(HS_n^{\beta})$ or $V(HS_n^{\gamma})$, the proof is similar to that of Case 4.1. When $y' \in V(HS_n^{j_i})$, for some $i \in \{1, \dots, n-3\}$, the proof is similar to that of Case 4.2.1.1. When $y' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$ (Figure 10).

Without loss of generality, suppose $y' \in V(HS_n^{j_{n-1}})$. There is a tree connecting x'_i, a_i, and b_i in $HS_n^{j_i}$, say $\widehat{T}_i$ and $a'_i \in V(HS_n^{\beta})$, $b'_i \in V(HS_n^{\gamma})$ for $i \in \{1, \dots, n-3\}$. There is a path joining x' and b_{n-2} in $HS_n^{j_{n-2}}$, say R and $b'_{n-2} \in V(HS_n^{\gamma})$. There is a path joining y' and m in $HS_n^{j_{n-1}}$, say Q and $m' \in V(HS_n^{\beta})$. Let $Z = \{a'_1, \dots, a'_{n-3}, x'_{n-2}, m'\}$, $W = \{b'_1, \dots, b'_{n-2}, x'_{n-1}\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P'_1, \dots, P'_{n-1}$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and $a'_i \in V(P'_i)$, for $i \in \{1, \dots, n-3\}$, $x'_{n-2} \in V(P'_{n-2})$, $m' \in V(P'_{n-1})$, $b'_i \in V(\widehat{P}_i)$ for $i \in \{1, \dots, n-2\}$, $x'_{n-1} \in V(\widehat{P}_{n-1})$. Let

$$\begin{cases} T_i = P_i \cup P'_i \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x'_i \cup a_i a'_i \cup b_i b'_i,\ i \in \{1, \dots, n-3\} \\ T_{n-2} = P_{n-2} \cup P'_{n-2} \cup \widehat{P}_{n-2} \cup x_{n-2}x'_{n-2} \cup R \cup xx' \cup b_{n-2}b'_{n-2} \\ T_{n-1} = P_{n-1} \cup P'_{n-1} \cup \widehat{P}_{n-1} \cup x_{n-1}x'_{n-1} \cup yy' \cup Q \cup mm'. \end{cases}$$

Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

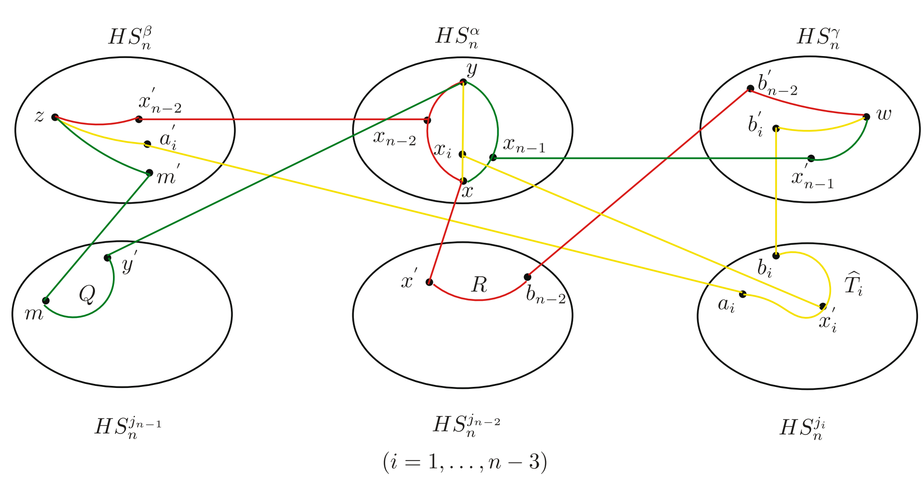

Figure 10: The illustration of $y' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$.

Case 4.2.2. For some $j \in \{1, \dots, n-1\}$, $x' \in V(HS_n^{\beta})$, $x'_j \in V(HS_n^{\gamma})$.

Without loss of generality, suppose $x' \in V(HS_n^{\beta})$, $x'_{n-1} \in V(HS_n^{\gamma})$.

Let $x'_i \in V(HS_n^{j_i})$, for $i \in \{1, \dots, n-2\}$ and $Z' = (Z \backslash \{x'_{n-2}\}) \cup \{a'_{n-2}\}$. We can obtain $n-2$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-3}, T_{n-1}$, similar to that of Case 4.1 by replacing Z with Z'. There is a tree connecting x'_{n-2}, a_{n-2}, b_{n-2} in $HS_n^{j_{n-2}}$, say $\widehat{T}_{n-2}$. Let $T_{n-2} = \widehat{T}_{n-2} \cup x_{n-2}x'_{n-2} \cup a_{n-2}a'_{n-2} \cup b_{n-2}b'_{n-2} \cup P_{n-2} \cup P'_{n-2} \cup \widehat{P}_{n-2}$ (Figure 11[a]). Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 4.2.3. For some $j \in \{1, \dots, n-1\}$, $x', x'_j \in V(HS_n^{\beta})$.

Without loss of generality, suppose $x', x'_{n-1} \in V(HS_n^{\beta})$, $x'_i \in V(HS_n^{j_i})$, where $j_i \in \Gamma_n - \{\alpha, \beta, \gamma\}$, for $i \in \{1, \dots, n-2\}$.

When $y' \in V(HS_n^{\gamma})$, there is a tree connecting x'_i, a_i, b_i in $HS_n^{j_i}$, say $\widehat{T}_i$ and $a'_i \in V(HS_n^{\beta})$, $b'_i \in V(HS_n^{\gamma})$ for $i \in \{1, \dots, n-3\}$. There is a path joining x'_{n-2} and b_{n-2} in $HS_n^{j_{n-2}}$, say P and $b'_{n-2} \in V(HS_n^{\gamma})$. Let $Z = \{a'_1, \dots, a'_{n-3}, x', x'_{n-1}\}$, $W = \{b'_1, \dots, b'_{n-2}, y'\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P'_1, \dots, P'_{n-1}$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and $a'_i \in V(P'_i)$ for $i \in \{1, \dots, n-3\}$, $x' \in V(P'_{n-2})$, $x'_{n-1} \in V(P'_{n-1})$, $b'_i \in V(\widehat{P}_i)$, for $i \in \{1, \dots, n-2\}$, $y' \in \widehat{P}_{n-1}$. Let

$$\begin{cases} T_i = P_i \cup P'_i \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x'_i \cup a_i a'_i \cup b_i b'_i,\ i \in \{1, \dots, n-3\} \\ T_{n-2} = P_{n-2} \cup P'_{n-2} \cup \widehat{P}_{n-2} \cup x_{n-2}x'_{n-2} \cup P \cup b_{n-2}b'_{n-2} \cup xx' \\ T_{n-1} = P_{n-1} \cup P'_{n-1} \cup \widehat{P}_{n-1} \cup x_{n-1}x'_{n-1} \cup yy'. \end{cases}$$

Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 11[b]).

When $y' \in V(HS_n^{\Gamma_n - \{\alpha, \beta, \gamma, \bigcup_{i=1}^{n-2} j_i\}})$, without loss of generality, suppose $y' \in V(HS_n^{j_{n-1}})$. There is a path joining y' and b_{n-1} in $HS_n^{j_{n-1}}$, say Q, and $b'_{n-1} \in V(HS_n^{\gamma})$. We obtain $n-2$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-2}$, similar to the case of $y' \in V(HS_n^{\gamma})$. Let $W' = (W \backslash \{y'\}) \cup \{b'_{n-1}\}$. $T_{n-1} = P_{n-1} \cup P'_{n-1} \cup \widehat{P}_{n-1} \cup x_{n-1}x'_{n-1} \cup Q \cup yy' \cup b_{n-1}b'_{n-1}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 4.3. There is one cross edge between $N[x]$ and $HS_n^{\beta,\gamma}$.

Case 4.3.1. $x' \in V(HS_n^{\beta})$.

Without loss of generality, suppose $x'_i \in V(HS_n^{j_i})$, where $j_i \in \Gamma_n - \{\alpha, \beta, \gamma\}$, $i \in \{1, \dots, n-1\}$. There is a tree connecting x'_i, a_i, b_i, say $\widehat{T}_i$, and $a'_i \in V(HS_n^{\beta})$, $b'_i \in V(HS_n^{\gamma})$ for $i \in \{1, \dots, n-1\}$. Let $Z = \{a'_1, \dots, a'_{n-1}\}$, $W = \{b'_1, \dots, b'_{n-1}\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P'_1, \dots, P'_{n-1}$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and $a'_i \in V(P'_i)$, $b'_i \in V(\widehat{P}_i)$ for $i \in \{1, \dots, n-1\}$. Let $T_i = P_i \cup P'_i \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x'_i \cup a_i a'_i \cup b_i b'_i$ for $i \in \{1, \dots, n-1\}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

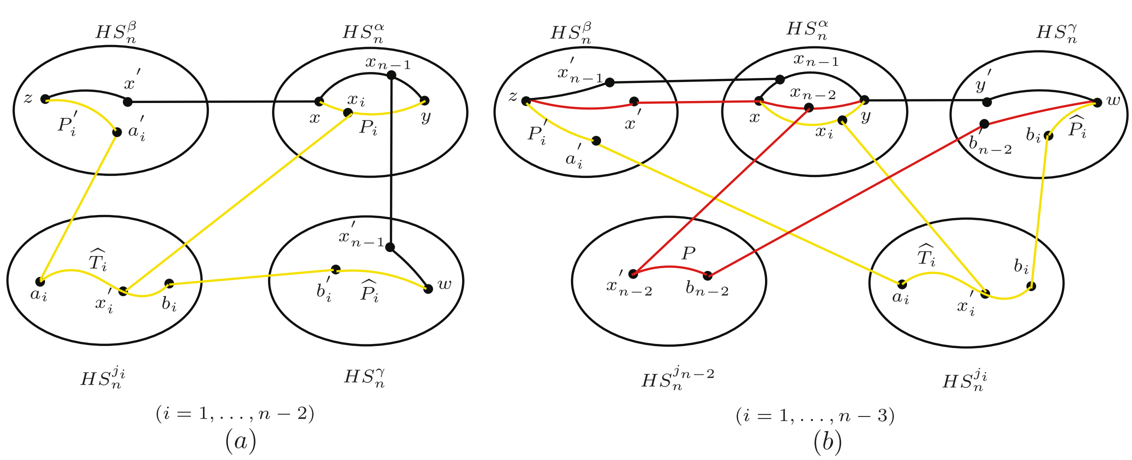

Figure 11: The illustrations of (a) Case 4.2.2 and (b) Case 4.2.3.

Case 4.3.2. For some $i \in \{1, \dots, n-2\}$, $x_i' \in V(HS_n^\beta)$.

Without loss of generality, suppose $x_{n-1}' \in V(HS_n^\beta)$, $x_i' \in V(HS_n^{j_i})$, where $j_i \in \Gamma_n - \{\alpha, \beta, \gamma\}$, for $i \in \{1, \dots, n-2\}$.

When $x' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$, without loss of generality, suppose $x' \in V(HS_n^{j_{n-1}})$. There is a path joining x' and b_{n-1} in $HS_n^{j_{n-1}}$, say R and $b_{n-1}' \in V(HS_n^\gamma)$. Let $Z = \{a_1', \dots, a_{n-2}', x_{n-1}'\}$, $W = \{b_1', \dots, b_{n-1}'\}$. By Lemma 3.3, there are $n-1$ internally disjoint paths between z and Z, say $P_1', \dots, P_{n-1}'$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and $a_i' \in V(P_i')$, for $i \in \{1, \dots, n-2\}$, $x_{n-1}' \in V(P_{n-1}')$, $b_i' \in V(\widehat{P}_i)$ for $i \in \{1, \dots, n-1\}$. Let $T_i = P_i \cup P_i' \cup \widehat{P}_i \cup \widehat{T}_i \cup x_i x_i' \cup a_i a_i' \cup b_i b_i'$ for $i \in \{1, \dots, n-2\}$, $T_{n-1} = P_{n-1} \cup P_{n-1}' \cup \widehat{P}_{n-1} \cup x_{n-1}x_{n-1}' \cup R \cup xx' \cup b_{n-1}b_{n-1}'$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 12).

When $x' \in V(HS_n^{j_i})$, for some $i \in \{1, \dots, n-2\}$, without loss of generality, suppose $x' \in V(HS_n^{j_{n-2}})$. We obtain $n-2$ internally disjoint S-trees $T_1, T_2, \dots, T_{n-2}$, similar to the case of $x' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$. Since there are two cross edges between HS_n^α and $HS_n^{j_{n-2}}$, $y' \notin V(HS_n^{\alpha,\beta,\bigcup_{i=1}^{n-2} j_i})$. We suppose $y' \in V(HS_n^\gamma)$ and the proof of the case $y' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$ is similar. Let $\widehat{P}_{n-1}$ be the path joining w and y', $T_{n-1} = P_{n-1} \cup P_{n-1}' \cup \widehat{P}_{n-1} \cup yy'$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees.

Case 4.4. There are no cross edges between $N[x]$ and $HS_n^{\beta,\gamma}$.

The proof is similar to that of Case 4.3.1.

Case 5. The vertices of S are distributed in four distinct copies of HS_n.

Without loss of generality, suppose $x \in V(HS_n^\alpha)$, $y \in V(HS_n^\beta)$, $z \in V(HS_n^\gamma)$, and $w \in V(HS_n^\eta)$, where $\alpha, \beta, \gamma, \eta \in \Gamma_n$, $\alpha \neq \beta \neq \gamma \neq \eta$. Let $X = \{x_1, \dots, x_{n-1}\}$, $Y = \{y_1, \dots, y_{n-1}\}$, $Z = \{z_1, \dots, z_{n-1}\}$, $W = \{w_1, \dots, w_{n-1}\}$, and $x_i', y_i', z_i', w_i' \in V(HS_n^{j_i})$ for $i \in \{1, \dots, n-1\}$. We suppose $x \notin X$, $y \notin Y$, $z \notin Z$, $w \notin W$, and the proof of the case $x \in X$ or $y \in Y$ or $z \in Z$ or $w \in W$ is similar. By Lemma 3.3, there are $n-1$ internally disjoint paths between x and X, say $P_1, \dots, P_{n-1}$, $n-1$ internally disjoint paths between y and Y, say $P_1', \dots, P_{n-1}'$, $n-1$ internally disjoint paths between z and Z, say $P_1'', \dots, P_{n-1}''$, $n-1$ internally disjoint paths between w and W, say $\widehat{P}_1, \dots, \widehat{P}_{n-1}$ and such that $x_i \in V(P_i)$, $y_i \in V(P_i')$, $z_i \in V(P_i'')$, $w_i \in V(\widehat{P}_i)$ for $i \in \{1, \dots, n-1\}$. Clearly, there exists a tree connecting x_i', y_i', z_i', w_i' in $HS_n^{j_i}$, say $\widehat{T}_i$ for $i \in \{1, \dots, n-1\}$. Let $T_i = P_i \cup P_i' \cup P_i'' \cup \widehat{P}_i \cup x_i x_i' \cup y_i y_i' \cup z_i z_i' \cup w_i w_i' \cup \widehat{T}_i$ for $i \in \{1, \dots, n-1\}$. Then $T_1, T_2, \dots, T_{n-1}$ are $n-1$ internally disjoint S-trees (Figure 13).

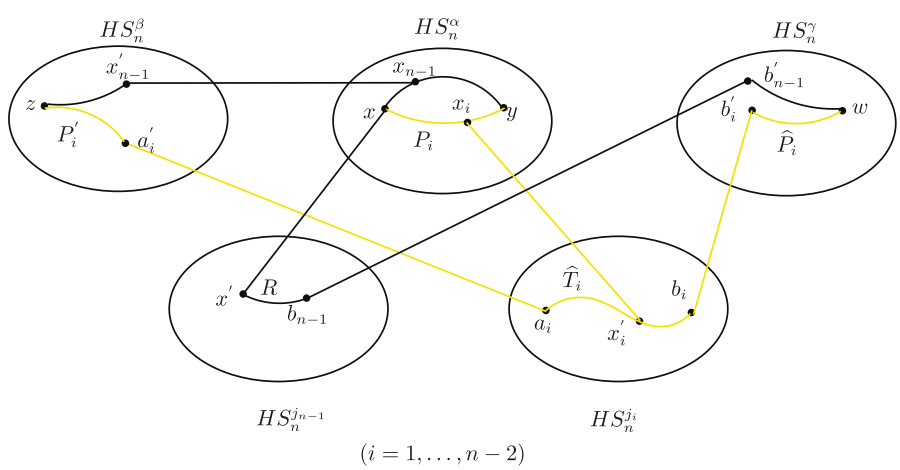

Figure 12: The illustration of $x' \in V(HS_n^{\Gamma_n - \{\alpha,\beta,\gamma,\bigcup_{i=1}^{n-2} j_i\}})$.

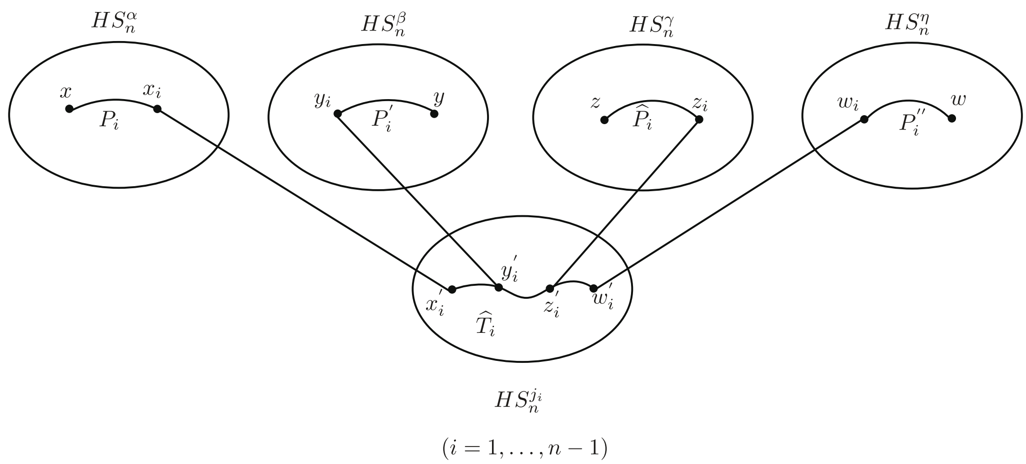

Figure 13: The illustration of Case 5.

4 Concluding remarks

The hierarchical star networks have some attractive properties to design interconnection networks. In this article, we focus on the hierarchical star graphs, which is an invariant of the star network and denoted by HS_n. We show that $\kappa_4(HS_n) = n - 1$ for $n \geq 2$. So far, the results about generalized k-connectivity of networks are almost about $k = 3$ and there are few results about larger k. In the future work, the generalized k-connectivity of the networks for $k \geq 5$ would be an interesting problem.

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