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Abstract

We give a general approach to infinite dimensional non-Gaussian analysis which generalizes the work [KSWY95]. For given measure we construct a family of biorthogonal systems. We study their properties and their Gel'fand triples that they generate. As an example we consider the measures of Poisson type.

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Contents

1 Introduction

Non-Gaussian analysis was already introduced in [AKS93] for smooth probability measure on infinite dimensional linear spaces, using biorthogonal decomposition as a natural extension of the chaos decomposition that is well known in Gaussian analysis. This biorthogonal "Appell" system has been constructed for smooth measures by Yu. L. Daletskii [Dal91]. For a detailed description of its use in infinite dimensional analysis and for the proof of the results which were announced in [AKS93] we refer to [ADKS96] which was based on quasi-invariance of the measures and smoothness of the logarithmic derivatives.

Kondratiev et al. [KSWY95] considered the case of non-degenerate measures on the dual of a nuclear space with analytic characteristic functionals and no further conditions such as quasi-invariance of the measure or smoothness of the logarithmic derivative was required. In this case the important example of Poisson noise is now accessible. Again for a given measure μ with analytic Laplace transform [KSWY95] construct an Appell biorthogonal system $\mathbf{A}^{\mu}$ as a pair $(\mathbf{P}^{\mu},\mathbf{Q}^{\mu})$ of Appell polynomials $\mathbf{P}^{\mu}$ and a canonical system of generalized functions $\mathbf{Q}^{\mu}$, properly associated to the measure μ. Hence within this framework they obtained:

- explicit description of the test function space introduced in [ADKS96];
- the test functions space is identical for all measures that they consider;
- characterization theorems for generalized as well as test functions was obtained analogously as in Gaussian analysis, see [KLP+96] for more references;
- extension of the Wick product and the corresponding Wick calculus [KLS96] as well as full description of positive distributions (as measures).

Aim of the present work. As in [KSWY95] we consider the case of non-degenerate measures on the dual of a nuclear space with analytic Laplace transform but instead of the **μ-exponential** $e_{\mu}(\cdot,\cdot)$ we use the **generalized μ-exponential** $e^{\alpha}_{\mu}(\cdot,\cdot)$ where α is a holomorphic function α on $\mathcal{N}_{\mathbb{C}}$ which is invertible in a neighborhood of zero, i.e., $\alpha\in\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}},\mathcal{N}_{\mathbb{C}})$. Hence using $e^{\alpha}_{\mu}(\cdot,\cdot)$ we construct an generalized Appell orthogonal system $\mathbf{A}^{\mu,\alpha}$ as a pair

$(\mathbf{P}^{\mu,\alpha},\mathbf{Q}^{\mu,\alpha})$ of generalized Appell polynomials $\mathbf{P}^{\mu,\alpha}$ and a system of generalized functions $\mathbf{Q}^{\mu,\alpha}$.
Central results. In the above framework

- we obtain an explicit description of the test function space introduced in [ADKS96];
- the spaces of test functions turns out to be the same for all $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}},\mathcal{N}_{\mathbb{C}})$ and for all measures that we consider;
- characterization theorems for generalized as well as test functions are obtained analogously as in the Gaussian case;
- the spaces of distributions for a fixed measure μ are again identical for all function α in the above conditions;
- the well known Wick product and the corresponding Wick calculus [KLS96] extends rather directly;
- in the important case of Poisson white noise a special choice of α produces the orthogonal system of Charlier polynomials, see Example 5.2.

2 General theory

2.1 Some facts on nuclear triples

We start with a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. For a given separable nuclear space $\mathcal{N}$ densely topologically embedded in $\mathcal{H}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

The dual pairing $\langle \cdot,\cdot \rangle$ of $\mathcal{N}'$ and $\mathcal{N}$ then is realized as an extension of the inner product in $\mathcal{H}$

$$\langle f, \xi \rangle = (f, \xi) \quad f \in \mathcal{H},\ \xi \in \mathcal{N}.$$

Instead of reproducing the abstract definition of nuclear spaces (see e.g., [Sch71]) we give a complete (and convenient) characterization in terms of projective limits of decreasing chains of Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$.

Theorem 2.1 *The nuclear Fréchet space $\mathcal{N}$ can be represented as*

$$\mathcal{N} = \bigcap_{p\in\mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p,\ p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$, $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The topology of $\mathcal{N}$ is given by the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

The Hilbert norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$. Without loss of generality we always suppose that $\forall p \in \mathbb{N}$, $\forall \xi \in \mathcal{N} : |\xi| \leq |\xi|_p$ and that the system of norms is ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ if $p < q$. By general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p}.$$

with inductive limit topology τ_{ind} by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}'_p,\ p \in \mathbb{N}\}$. The inductive limit topology (w.r.t. this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$. Let us mention that in our setting the topology τ_{ind} coincides with the Mackey topology $\tau(\mathcal{N}', \mathcal{N})$ and the strong topology $\beta(\mathcal{N}', \mathcal{N})$, see e.g., [HKPS93, Appendix 5].

Further we want to introduce the notion of tensor power of a nuclear space. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^{\otimes n}$, $n \in \mathbb{N}$ of Hilbert spaces. Since there is no danger of confusion we will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_{-p}^{\otimes n}$ respectively. Using the definition

$$\mathcal{N}^{\otimes n} := \operatorname{pr}\lim_{p \in \mathbb{N}} \mathcal{H}_p^{\otimes n},$$

one can prove [Sch71] that $\mathcal{N}^{\otimes n}$ is a nuclear space which is called the n-th tensor power of $\mathcal{N}$.

The dual space of $\mathcal{N}^{\otimes n}$ can be written

$$\mathcal{N}'^{\otimes n} = \operatorname{ind}\lim_{p \in \mathbb{N}} \mathcal{H}_{-p}^{\otimes n}.$$

We also want to introduce the (Boson or symmetric) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$$

with the convention $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$ and the Hilbert norm

$$\|\vec{\varphi}\|^2_{\Gamma(\mathcal{H})} = \sum_{n=0}^{\infty} n! \left|\varphi^{(n)}\right|^2, \ \vec{\varphi} = \left\{\varphi^{(n)} \,|\, n \in \mathbb{N}_0\right\} \in \Gamma(\mathcal{H}).$$

2.2 Holomorphy on locally convex spaces

We shall collect some facts from the theory of holomorphic functions in locally convex topological vector spaces $\mathcal{E}$ (over the complex field $\mathbb{C}$), see e.g., [Din81]. Let $\mathcal{L}(\mathcal{E}^n)$ be the space of n-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric n-linear forms. Also let $P^n(\mathcal{E})$ denote the n-homogeneous polynomials on $\mathcal{E}$. There is a linear bijection $\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \widehat{A} \in P^n(\mathcal{E})$. Now let $\mathcal{U} \subset \mathcal{E}$ be open and consider a function $G : \mathcal{U} \to \mathbb{C}$. G is said to be **G-holomorphic** if for all $\theta_0 \in \mathcal{U}$ and for all

$\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C}$: $\lambda \mapsto G(\theta_0 + \lambda\theta)$ is holomorphic in some neighborhood of zero in $\mathbb{C}$. If G is G-holomorphic then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{d^n G(\eta)}$ such that

$$G(\theta + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!}\widehat{d^n G(\eta)}(\theta)$$

for all θ from some open neighborhood $\mathcal{V}$ of zero. G is said to be **holomorphic**, if for all η in $\mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that

$$\sum_{n=0}^{\infty} \frac{1}{n!}\widehat{d^n G(\eta)}(\theta)$$

converges uniformly on $\mathcal{V}$ to a continuous function. Of course, $\widehat{d^n G(\eta)}(\theta)$ is the n-th partial derivative of G at η in direction θ. We say that G is holomorphic at θ_0 if there is an open set $\mathcal{U}$ containing θ_0 such that G is holomorphic on $\mathcal{U}$. The following Proposition can be found e.g., in [Din81].

Proposition 2.2 *G is holomorphic if and only if it is G-holomorphic and locally bounded.*

Let us explicitly consider a function holomorphic at the point $0 \in \mathcal{E} = \mathcal{N}_{\mathbb{C}}$, then

1) there exist p and $\varepsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \leq \varepsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function of one complex variable $\lambda \mapsto G(\xi_0 + \lambda\xi)$ is holomorphic at $0 \in \mathbb{C}$, and

2) there exists $c > 0$ such that for all $\xi \in \mathcal{N}_{\mathbb{C}}$ with $|\xi|_p \leq \varepsilon : |G(\xi)| \leq c$.
As we do not want to discern between different restrictions of one function, we consider germs of holomorphic functions, i.e., we identify F and G if there exists an open neighborhood $\mathcal{U} : 0 \in \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ such that $F(\xi) = G(\xi)$ for all $\xi \in \mathcal{U}$. Thus we define $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ as the algebra of germs of functions holomorphic at zero equipped with the inductive topology given by the following family of norms

$$\mathrm{n}_{p,l,\infty}(G) = \sup_{|\theta|_p \leq 2^{-l}} |G(\theta)|, \quad p, l \in \mathbb{N}.$$

For later use we need the space $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ of holomorphic functions from $\mathcal{N}_{\mathbb{C}}$ to $\mathcal{N}_{\mathbb{C}}$. Let $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ be open and consider a function $\alpha : \mathcal{U} \to \mathcal{N}_{\mathbb{C}}$. α is said to be holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$ iff

1. it is G-holomorphic; i.e., there exist p and $\epsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_{\mathbb{C}}$ with $|\xi_0|_p \leq \epsilon$ and for all $\xi \in \mathcal{N}_{\mathbb{C}}$ the function of one complex variable $\lambda \mapsto \alpha(\xi_0 + \lambda\xi)$ is holomorphic at $0 \in \mathbb{C}$;

2. α is locally bounded, i.e., for all $p \in \mathbb{N}$ there exist $C_p > 0$ such that $\forall \eta \in A$ with $|\eta|_p \leq C_p$ then $\forall p' \in \mathbb{N}$ there exist $C_{p'}$ such that $\forall \eta \in A$ $|\alpha(\eta)|_{p'} \leq C_{p'}$, where A is an bounded set in $\mathcal{N}_{\mathbb{C}}$.

If α is holomorphic at $0 \in \mathcal{N}_{\mathbb{C}}$, then for every $\eta \in \mathcal{U}$ there exists a sequence of homogeneous polynomials $\frac{1}{n!}\widehat{d^n\alpha(\eta)}$ such that

$$\theta \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n\alpha(\eta)}\,(\theta)$$

converges and define a continuous function on some neighborhood of zero.

Let use now introduce spaces of entire functions which will be useful later. Let $\mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}})$ denote the set of all entire functions on $\mathcal{H}_{-p,\mathbb{C}}$ of growth $k \in [1,2]$ and type 2^{-l}, $p, l \in \mathbb{Z}$. This is a linear space with norm

$$\mathrm{n}_{p,l,k}(\varphi) = \sup_{z \in \mathcal{H}_{-p,\mathbb{C}}} |\varphi(z)| \exp\left(-2^{-l}|z|^k_{-p}\right),\ \varphi \in \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}).$$

The space of entire functions on $\mathcal{N}'_{\mathbb{C}}$ of growth k and minimal type is naturally introduced by

$$\mathcal{E}^k_{\min}(\mathcal{N}'_{\mathbb{C}}) := \operatorname*{pr\,lim}_{p,l \in \mathbb{N}} \mathcal{E}^k_{2^{-l}}(\mathcal{H}_{-p,\mathbb{C}}),$$

see e.g., [Kon91], [BK95], [Oue91]. We will also need the space of entire functions on $\mathcal{N}_{\mathbb{C}}$ of growth k and finite type:

$$\mathcal{E}^k_{\max}(\mathcal{N}_{\mathbb{C}}) := \operatorname*{ind\,lim}_{p,l \in \mathbb{N}} \mathcal{E}^k_{2^l}(\mathcal{H}_{p,\mathbb{C}}).$$

2.3 Measures on linear topological spaces

To introduce probability measures on the vector space $\mathcal{N}'$, we consider $\mathcal{C}_\sigma(\mathcal{N}')$ the σ-algebra generated by cylinder sets on $\mathcal{N}'$, which coincides with the Borel σ-algebras $\mathcal{B}_\sigma(\mathcal{N}')$ and $\mathcal{B}_\beta(\mathcal{N}')$ generated by the weak and strong topology on $\mathcal{N}'$, respectively. Thus we will consider this σ-algebra as the **natural** σ-algebra on $\mathcal{N}'$. Detailed definitions of the above notions and proofs of the mentioned relations can be found in e.g., [BK95].

We will restrict our investigations to a special class of measures μ on $\mathcal{C}_\sigma(\mathcal{N}')$ which satisfy two additional assumptions. The first one concerns some analyticity of the Laplace transformation

$$l_\mu(\theta) := L_\mu 1(\theta) = \int_{\mathcal{N}'} \exp\langle x, \theta\rangle \, d\mu(x) =: \mathbb{E}_\mu(\exp\langle \cdot, \theta\rangle), \quad \theta \in \mathcal{N}_{\mathbb{C}}.$$

Here we also have introduced the convenient notion of expectation $\mathbb{E}_\mu$ of a μ-integrable function.

Assumption 1 The measure μ has an analytic Laplace transform in a neighborhood of zero. That means there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$ of zero, such that l_μ is holomorphic on $\mathcal{U}$, i.e., $l_\mu \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. This class of **analytic measures** is denoted by $\mathcal{M}_a(\mathcal{N}')$.

An equivalent description of analytic measures is given by the following lemma and the proof can be founded in [KSW95].

Lemma 2.3 *The following statements are equivalent*

1) $\mu \in \mathcal{M}_a(\mathcal{N}')$;

2) $\exists p_\mu \in \mathbb{N}, \quad \exists C > 0 : \left| \int_{\mathcal{N}'} \langle x, \theta\rangle^n d\mu(x) \right| \leq n!\, C^n \, |\theta|^n_{p_\mu}, \ \theta \in \mathcal{H}_{p_\mu,\mathbb{C}}$;

3) $\exists p'_\mu \in \mathbb{N}, \quad \exists \varepsilon_\mu > 0 : \int_{\mathcal{N}'} \exp(\varepsilon_\mu |x|_{-p'_\mu}) d\mu(x) < \infty.$

For $\mu \in \mathcal{M}_a(\mathcal{N}')$ the estimate in statement 2 of the above lemma allows to define the moment kernels $M^\mu_n \in \mathcal{N}'^{\widehat{\otimes} n}$. This is done by extending the above estimate by a simple polarization argument and applying the kernel theorem. The kernels are determined by

$$l_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle M^\mu_n, \theta^{\otimes n} \right\rangle \tag{2.1}$$

or equivalently

$$\left\langle M^\mu_n, \theta_1 \widehat{\otimes} \ldots \widehat{\otimes} \theta_n \right\rangle = \left. \frac{\partial^n}{\partial t_1 \ldots \partial t_n} l_\mu(t_1\theta_1 + \ldots + t_n\theta_n) \right|_{t_1 = \ldots = t_n = 0}.$$

Moreover, if $p > p_\mu$ is such that the embedding $i_{p,p_\mu} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_\mu}$ is Hilbert-Schmidt then

$$|M^\mu_n|_{-p} \leq \left(nC \left\| i_{p,p_\mu} \right\|_{HS}\right)^n \leq n! \left(eC \left\| i_{p,p_\mu} \right\|_{HS}\right)^n. \tag{2.2}$$

Definition 2.4 *A function* $\varphi : \mathcal{N}' \to \mathbb{C}$ *of the form*

$$\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle,\ x \in \mathcal{N}',\ N \in \mathbb{N},$$

is called a **continuous polynomial** (*short* $\varphi \in \mathcal{P}(\mathcal{N}')$) *iff* $\varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$, $\forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Now we are ready to formulate the second assumption on μ:

Assumption 2 For all $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\varphi = 0$ μ-almost everywhere we have $\varphi \equiv 0$. In the following a measure with this property will be called **non-degenerate**.

Note: Assumption 2 is equivalent to:
Let $\varphi \in \mathcal{P}(\mathcal{N}')$ with $\int_A \varphi d\mu = 0$ for all $A \in \mathcal{C}_\sigma(\mathcal{N}')$ then $\varphi \equiv 0$.
A sufficient condition can be obtained by regarding admissible shifts of the measure μ. If $\mu(\cdot + \xi)$ is absolutely continuous with respect to μ for all $\xi \in \mathcal{N}$, i.e., there exists the Radon-Nikodym derivative

$$\rho_\mu(\xi, x) = \frac{d\mu(x+\xi)}{d\mu(x)} \in L^1(\mathcal{N}', \mu),\ x \in \mathcal{N}',$$

then we say that μ is **$\mathcal{N}$-quasi-invariant** see e.g., [GV68], [Sko74]. This is sufficient to ensure Assumption 2, see e.g., [KV91], [BK95].

3 The Appell system

The space $\mathcal{P}(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ see e.g., [Sch71, Chap. II 6.1, Chap. II 7.4]

$$\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$$

via

$$\varphi\left(x\right) = \sum_{n=0}^{\infty} \left\langle x^{\otimes n}, \varphi^{(n)} \right\rangle \longleftrightarrow \vec{\varphi} = \left\{ \varphi^{(n)} \,|\, n \in \mathbb{N}_0 \right\}.$$

Note that only a finite number of $\varphi^{(n)}$ is a non-zero. The notion of convergence of sequences in this topology on $\mathcal{P}(\mathcal{N}')$ is the following: for $\varphi \in \mathcal{P}(\mathcal{N}')$, such that

$$\varphi\left(x\right) = \sum_{n=0}^{N(\varphi)} \left\langle x^{\widehat{\otimes}n}, \varphi^{(n)} \right\rangle$$

let $p_n : \mathcal{P}(\mathcal{N}') \to \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ denote the mapping p_n defined by $p_n(\varphi) := \varphi^{(n)}$. A sequence $\{\varphi_j,\, j \in \mathbb{N}\}$ of smooth polynomials converge to $\varphi \in \mathcal{P}(\mathcal{N}')$ iff the $N(\varphi_j)$ are bounded and $p_n\varphi_j \underset{j\to\infty}{\longrightarrow} p_n\varphi$ in $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ for all $n \in \mathbb{N}$.

Now we can introduce the dual space $\mathcal{P}'_\mu(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to $L^2(\mu)$. As a result we have constructed the triple

$$\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}')$$

The (bilinear) dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\mu$ between $\mathcal{P}'_\mu(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_\mu = (\varphi, \overline{\psi})_{L^2(\mu)}, \quad \varphi \in L^2\left(\mu\right),\ \psi \in \mathcal{P}\left(\mathcal{N}'\right).$$

3.1 $\mathbf{P}^\mu$-system

Because of the holomorphy of l_μ and since that $l_\mu(0) = 1$, there exists a neighborhood of zero

$$\mathcal{U}_0 = \left\{ \theta \in \mathcal{N}_{\mathbb{C}} \mid 2^{q_0} \left|\theta\right|_{p_0} < 1 \right\}$$

$p_0, q_0 \in \mathbb{N}$, $p_0 \geq p'_\mu$, $2^{-q_0} \leq \varepsilon_\mu$ (p'_μ, ε_μ from Lemma 2.3) such that $l_\mu(\theta) \neq 0$ for $\theta \in \mathcal{U}_0$ and the normalized μ-exponential

$$e_\mu(\theta; z) := \frac{\exp\langle z, \theta\rangle}{l_\mu(\theta)} \quad \text{for } \theta \in \mathcal{U}_0, \quad z \in \mathcal{N}'_{\mathbb{C}}, \tag{3.1}$$

is well defined. We use the holomorphy of $\theta \mapsto e_\mu(\theta; z)$ to expand it in a power series in θ similar to the case corresponding to the construction of one dimensional Appell polynomials [Bou76]. We have in analogy to [AKS93], [ADKS96]

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n e_\mu(0, z)}(\theta),$$

where $\widehat{d^n e_\mu(0, z)}$ is an n-homogeneous form polynomial. But since $e_\mu(\theta; z)$ is not only G-holomorphic but holomorphic we know that $\theta \mapsto e_\mu(\theta; z)$ is also locally bounded. Thus Cauchy's inequality for Taylor series [Din81] may be applied, $\rho \leq 2^{-q_0}$, $p \geq p_0$

$$\begin{aligned} \left| \frac{1}{n!} \widehat{d^n e_\mu(0, z)}(\theta) \right| &\leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} |e_\mu(\theta; z)| \, |\theta|_p^n \\ &\leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} \frac{1}{l_\mu(\theta)} \exp\left(\rho \, |z|_{-p}\right) |\theta|_p^n \end{aligned} \tag{3.2}$$

if $z \in \mathcal{H}_{-p,\mathbb{C}}$. This inequality extends by polarization [Din81] to an estimate sufficient for the kernel theorem. Thus we have a representation

$$\widehat{d^n e_\mu(0, z)}(\theta) = \left\langle P_n^\mu(z), \theta^{\otimes n} \right\rangle,$$

where $P_n^\mu(z) \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$. The kernel theorem really gives a little more: $P_n^\mu(z) \in \mathcal{H}^{\widehat{\otimes} n}_{-p',\mathbb{C}}$ for any p' ($> p \geq p_0$) such that the embedding operator

$$i_{p',p} : \mathcal{H}_{p',\mathbb{C}} \hookrightarrow \mathcal{H}_{p,\mathbb{C}}$$

is Hilbert-Schmidt. Thus we have

$$e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^\mu(z), \theta^{\otimes n} \right\rangle \quad \text{for } \theta \in \mathcal{U}_0, \ z \in \mathcal{N}'_{\mathbb{C}}. \tag{3.3}$$

We will also use the notation

$$P_n^\mu(\varphi^{(n)})(\cdot) := \left\langle P_n^\mu(\cdot), \varphi^{(n)} \right\rangle, \quad \varphi^{(n)} \in \mathcal{N}^{\widehat{\otimes} n}_{\mathbb{C}}, \quad n \in \mathbb{N},$$

which is called **Appell polynomial**. Thus for any measure satisfying Assumption 1 we have defined the $\mathbf{P}^\mu$-system

$$\mathbf{P}^\mu = \left\{ \left\langle P_n^\mu(\cdot), \varphi^{(n)} \right\rangle \mid \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n},\ n \in \mathbb{N}_0 \right\}.$$

The following proposition collects some properties of the polynomials $P_n^\mu(z)$, (for the proof we refer to [KSWY95]).

Proposition 3.1 *For $x, y \in \mathcal{N}'$, $n \in \mathbb{N}$ the following holds*

(P1)
$$P_n^\mu(x) = \sum_{k=0}^{n} \binom{n}{k} x^{\otimes k} \widehat{\otimes} P_{n-k}^\mu(0). \tag{3.4}$$

(P2)
$$x^{\otimes n} = \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(x) \widehat{\otimes} M_{n-k}^\mu. \tag{3.5}$$

(P3)
$$\begin{aligned} P_n^\mu(x+y) &= \sum_{k+l+m=n} \frac{n!}{k!\, l!\, m!} P_k^\mu(x) \widehat{\otimes} P_l^\mu(y) \widehat{\otimes} M_m^\mu \\ &= \sum_{k=0}^{n} \binom{n}{k} P_k^\mu(x) \widehat{\otimes} y^{\otimes(n-k)}. \end{aligned} \tag{3.6}$$

(P4) Further we observe

$$\mathbb{E}_\mu(\langle P_m^\mu(\cdot), \varphi^{(m)} \rangle) = 0 \quad \text{for } m \neq 0\ , \varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} m}. \tag{3.7}$$

(P5) For all $p > p_0$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0}$ is Hilbert–Schmidt and for all $\varepsilon > 0$ small enough ($\varepsilon \leq (2^{q_0} e \left\| i_{p,p_0} \right\|_{HS})^{-1}$) there exists a constant $C_{p,\varepsilon} > 0$ with

$$|P_n^\mu(z)|_{-p} \leq C_{p,\varepsilon}\, n!\, \varepsilon^{-n}\, e^{(\varepsilon |z|_{-p})}, \quad z \in \mathcal{H}_{-p,\mathbb{C}}. \tag{3.8}$$

The following lemma describes the set of polynomials $\mathcal{P}(\mathcal{N}')$.

Lemma 3.2 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$\varphi(x) = \sum_{n=0}^{N} \left\langle P_n^\mu(x), \varphi^{(n)} \right\rangle, \quad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n} \tag{3.9}$$

and vice versa, any functional of the form (3.9) is a smooth polynomial.

3.2 $\mathbf{Q}^\mu$-system

To give an internal description of the type (3.9) for $\mathcal{P}'_\mu(\mathcal{N}')$ we have to construct an appropriate system of generalized functions, the $\mathbf{Q}^\mu$-system. We propose to construct the $\mathbf{Q}^\mu$-system using differential operators.

For $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ define a differential operator, $D(\Phi^{(n)})$, of order n and constant coefficients $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$, such that, applied to monomials $\langle x^{\otimes m}, \varphi^{(m)} \rangle$, $\varphi^{(m)} \in \mathcal{N}^{\widehat{\otimes} m}_{\mathbb{C}}$, $m \in \mathbb{N}$

$$D\left(\Phi^{(n)}\right)\left\langle x^{\otimes m}, \varphi^{(m)}\right\rangle = \begin{cases} \dfrac{m!}{(m-n)!}\left\langle x^{\otimes(m-n)}\widehat{\otimes}\Phi^{(n)}, \varphi^{(m)}\right\rangle & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} \tag{3.10}$$

and extend by linearity from the monomials to $\mathcal{P}(\mathcal{N}')$.

Lemma 3.3 *$D(\Phi^{(n)})$ is a continuous linear operator from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$.*

Remark For $\Phi^{(1)} \in \mathcal{N}'$ we have the usual Gâteaux derivative as e.g., in white noise analysis [HKPS93]

$$D\left(\Phi^{(1)}\right)\varphi = D_{\Phi^{(1)}}\varphi := \frac{d}{dt}\,\varphi\left(\cdot + t\Phi^{(1)}\right)\Big|_{t=0}$$

for $\varphi \in \mathcal{P}(\mathcal{N}')$. Moreover we have $D((\Phi^{(1)})^{\otimes n}) = (D_{\Phi^{(1)}})^n$, thus $D((\Phi^{(1)})^{\otimes n})$ is a differential operator of order n.

In view of Lemma 3.3 it is possible to define the adjoint operator

$$D(\Phi^{(n)})^* : \mathcal{P}'_\mu(\mathcal{N}') \longrightarrow \mathcal{P}'_\mu(\mathcal{N}'), \quad \Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}.$$

Further we introduce the constant function $1 \in L^2(\mu) \subset \mathcal{P}'_\mu(\mathcal{N}')$ such that $1\,(x) \equiv 1$ for all $x \in \mathcal{N}'$, so

$$\langle\!\langle 1, \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi\,(x)\, d\mu\,(x) = \mathbb{E}_\mu\,(\varphi)\,.$$

Now we are ready to define the $\mathbf{Q}^\mu$-system.

Definition 3.4 *For any $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ we define a generalized function $Q^\mu_n(\Phi^{(n)}) \in \mathcal{P}'_\mu(\mathcal{N}')$ by*

$$Q^\mu_n\left(\Phi^{(n)}\right) = D\left(\Phi^{(n)}\right)^* 1.$$

We want to introduce an additional formal notation which stresses the linearity of $\Phi^{(n)} \mapsto Q_n^\mu(\Phi^{(n)}) \in \mathcal{P}'_\mu(\mathcal{N}')$:

$$\left\langle Q_n^\mu, \Phi^{(n)} \right\rangle := Q_n^\mu \left(\Phi^{(n)} \right)$$

Example 3.5 *The simplest non trivial case can be studied using finite dimensional real analysis. We consider the nuclear "triple"*

$$\mathbb{R} \subseteq \mathbb{R} \subseteq \mathbb{R}$$

where the dual pairing between a "test function" and a "distribution" degenerates to multiplication. On $\mathbb{R}$ *we consider a measure* $d\mu(x) = \rho(x)\, dx$ *where* ρ *is a positive* C^∞*-function on* $\mathbb{R}$ *such that assumptions 1 and 2 are fulfilled. In this setting the adjoint of the differentiation operator is given by*

$$\left(\frac{d}{dx} \right)^* f(x) = -\left(\left(\frac{d}{dx} \right) + \beta(x) \right) f(x), \quad f \in C^\infty(\mathbb{R}),$$

where β *is the logarithmic derivative of the measure* μ *and given by*

$$\beta = \frac{\rho'}{\rho}.$$

This enables us to calculate the $\mathbf{Q}^\mu$*-system. One has*

$$\begin{aligned} Q_n^\mu(x) &= \left(\left(\frac{d}{dx} \right)^* \right)^n 1 \\ &= (-1)^n \left(\frac{d}{dx} + \beta(x) \right)^n 1 \\ &= (-1)^n \frac{\rho^{(n)}(x)}{\rho(x)}, \end{aligned}$$

where the last equality can be seen by simple induction (for ρ *non smooth this construction produce* **generalized** *functions* Q_n^μ *even in this one dimensional case).*

If $\rho(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$ *is the Gaussian density, then* Q_n^μ *is related to the n-th Hermite polynomial:*

$$Q_n^\mu(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}} \right).$$

Definition 3.6 *We define the* $\mathbf{Q}^\mu$*-system in* $\mathcal{P}'_\mu(\mathcal{N}')$ *by*

$$\mathbf{Q}^\mu = \left\{ Q_n^\mu \left(\Phi^{(n)} \right) \mid \Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}},\ n \in \mathbb{N}_0 \right\},$$

and the pair $(\mathbf{P}^\mu, \mathbf{Q}^\mu)$ *will be called the* **Appell system** $\mathbf{A}^\mu$ *generated by the measure* μ.

We have the following central property of the Appell system $\mathbf{A}^\mu$.

Theorem 3.7 (Biorthogonality w.r.t. μ**)**

$$\left\langle\!\left\langle Q_n^\mu(\Phi^{(n)}),\ P_m^\mu \left(\varphi^{(m)} \right) \right\rangle\!\right\rangle_\mu = \delta_{m,n}\, n!\, \langle \Phi^{(n)}, \varphi^{(n)} \rangle \tag{3.11}$$

for $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ *and* $\varphi^{(m)} \in \mathcal{N}^{\widehat{\otimes} m}_{\mathbb{C}}$.

Now we are going to characterize the space $\mathcal{P}'_\mu(\mathcal{N}')$.

Theorem 3.8 *For all* $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ *there exists a unique sequence* $\{\Phi^{(n)} \mid n \in \mathbb{N}_0\}$, $\Phi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ *such that*

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu \left(\Phi^{(n)} \right) \equiv \sum_{n=0}^{\infty} \left\langle Q_n^\mu, \Phi^{(n)} \right\rangle \tag{3.12}$$

and vice versa, every series of the form (3.12) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

The proofs of this result can be found in [KSWY95].

4 The triple $(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_\mu$

4.1 Test functions

On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the Appell decomposition from Lemma 3.2. Let

$$\varphi(x) = \sum_{n=0}^{N} \left\langle P_n^\mu(x), \varphi^{(n)} \right\rangle \in \mathcal{P}(\mathcal{N}')$$

be given, then $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$ for each $p \geq 0$ $(n \in \mathbb{N}_0)$. Thus we may define for any $p, q \in \mathbb{N}$ a Hilbert norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|_{p,q,\mu}^2 = \sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left|\varphi^{(n)}\right|_p^2$$

The completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|_{p,q,\mu}$ is denoted by $(\mathcal{H}_p)^1_{q,\mu}$.

Definition 4.1 *We define*

$$(\mathcal{N})^1_\mu := \operatorname*{pr\,lim}_{p,q\in\mathbb{N}} (\mathcal{H}_p)^1_{q,\mu}$$

This space have the following properties (for the proofs see [KSWY95] and references therein).

Theorem 4.2 $(\mathcal{N})^1_\mu$ *is a nuclear space. The topology* $(\mathcal{N})^1_\mu$ *is uniquely defined by the topology on* $\mathcal{N}$*: It does not depend on the choice of the family of norms* $\{|\cdot|_p\}$.

Theorem 4.3 *There exists* $p', q' > 0$ *such that for all* $p \geq p'$, $q \geq q'$ *the topological embedding* $(\mathcal{H}_p)^1_{q,\mu} \subset L^2(\mu)$ *holds.*

Corollary 4.4 $(\mathcal{N})^1_\mu$ *is continuously and densely embedded in* $L^2(\mu)$.

Theorem 4.5 *Any test function* φ *in* $(\mathcal{N})^1_\mu$ *has a uniquely defined extension to* $\mathcal{N}'_\mathbb{C}$ *as an element of* $\mathcal{E}^1_{\min}(\mathcal{N}'_\mathbb{C})$.

In this construction one unexpected moment was the following:

Theorem 4.6 *For all measures* $\mu \in \mathcal{M}_a(\mathcal{N}')$ *we have the topological identity*

$$(\mathcal{N})^1_\mu = \mathcal{E}^1_{\min}(\mathcal{N}').$$

Since this last theorem states that the space of test functions $(\mathcal{N})^1_\mu$ is isomorphic to $\mathcal{E}^1_{\min}(\mathcal{N}')$ for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$, we will drop the subscript μ. The test function space $(\mathcal{N})^1$ is the same for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$.

4.2 Distributions

The space $(\mathcal{N})_\mu^{-1}$ of distributions corresponding to the space of test functions $(\mathcal{N})^1$ can be viewed as a subspace of $\mathcal{P}'_\mu(\mathcal{N}')$, since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1$ topologically, i.e.,

$$(\mathcal{N})_\mu^{-1} \subset \mathcal{P}'_\mu(\mathcal{N}')$$

Let us now introduce the Hilbert subspace $(\mathcal{H}_{-p})_{-q,\mu}^{-1}$ of $\mathcal{P}'_\mu(\mathcal{N}')$ for which the norm

$$\|\Phi\|_{-p,-q,\mu}^2 := \sum_{n=0}^{\infty} 2^{-qn} \left|\Phi^{(n)}\right|_{-p}^2$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu \left(\Phi^{(n)}\right) \in \mathcal{P}'_\mu(\mathcal{N}')$$

from Theorem 3.8. The space $(\mathcal{H}_{-p})_{-q,\mu}^{-1}$ is the dual space of $(\mathcal{H}_p)_{q,\mu}^1$ with respect to $L^2(\mu)$ (because of the biorthogonality of $\mathbf{P}^\mu$- and $\mathbf{Q}^\mu$-systems). By the general duality theory

$$(\mathcal{N})_\mu^{-1} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})_{-q,\mu}^{-1}$$

is the dual space of $(\mathcal{N})^1$ with respect to $L^2(\mu)$. So, we have the topological nuclear triple

$$(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})_\mu^{-1}.$$

The action of

$$\Phi = \sum_{n=0}^{\infty} Q_n^\mu \left(\Phi^{(n)}\right) \in (\mathcal{N})_\mu^{-1}$$

on a test function

$$\varphi = \sum_{n=0}^{\infty} \left\langle P_n^\mu, \varphi^{(n)}\right\rangle \in (\mathcal{N})^1$$

is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle.$$

Example 4.7 (Generalized Radon-Nikodym derivative) *We want to define a generalized function $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property*

$$\langle\!\langle \rho_\mu(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x), \ \varphi \in (\mathcal{N})^1 .$$

That means we have to establish the continuity of $\rho_\mu(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$. If $p' \geq p$ is sufficiently large and $\epsilon > 0$ is small enough, there exists $q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x) \right| &\leq C \|\varphi\|_{p',q,\mu} \int_{\mathcal{N}'} \exp\left(\epsilon |x-z|_{-p'}\right) d\mu(x) \\ &\leq C \|\varphi\|_{p',q,\mu} \exp\left(\epsilon |z|_{-p'}\right) \int_{\mathcal{N}'} \exp\left(\epsilon |x|_{-p'}\right) d\mu(x) . \end{aligned}$$

If ϵ is chosen sufficiently small the last integral exists. Thus we have in fact $\rho_\mu(z,\cdot) \in (\mathcal{N})^{-1}_\mu$. It is clear that whenever the Radon-Nikodym derivative $\frac{d\mu(x+\xi)}{d\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu(z,\cdot)$ defined above. We will show that in $(\mathcal{N})^{-1}_\mu$ we have the canonical expansion

$$\rho_\mu(z,\cdot) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q^\mu_n\left(z^{\otimes n}\right) .$$

Since both sides are in $(\mathcal{N})^{-1}_\mu$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi^{(n)} \in \mathcal{N}^{\widehat{\otimes} n}_{\mathbb{C}}$ we have

$$\begin{aligned} &\left\langle\!\left\langle \rho_\mu(z,\cdot), \left\langle P^\mu_n, \varphi^{(n)} \right\rangle \right\rangle\!\right\rangle_\mu \\ =\ & \int_{\mathcal{N}'} \left\langle P^\mu_n(x-z), \varphi^{(n)} \right\rangle d\mu(x) \\ =\ & \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{\mathcal{N}'} \left\langle P^\mu_k(x) \widehat{\otimes} z^{\otimes(n-k)}, \varphi^{(n)} \right\rangle d\mu(x) \\ =\ & (-1)^n \left\langle z^{\otimes n}, \varphi^{(n)} \right\rangle \\ =\ & \left\langle\!\!\left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k Q^\mu_k\left(z^{\otimes k}\right), \left\langle P^\mu_n, \varphi^{(n)} \right\rangle \right\rangle\!\!\right\rangle_\mu , \end{aligned}$$

where we have used (3.6), (3.7) and the biorthogonality of $\mathbf{P}^\mu$- and $\mathbf{Q}^\mu$-systems. In other words, we have proven that $\rho_\mu(-z,\cdot)$ is the generating *function of the* $\mathbf{Q}^{\mu}$*-system*

$$\rho_\mu(-z,\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{\mu}\left(z^{\otimes n}\right).$$

4.3 Integral transformations

4.3.1 Normalized Laplace transform S_μ

We first introduce the Laplace transform of a function $\varphi \in L^2(\mu)$. The global assumption $\mu \in \mathcal{M}_a(\mathcal{N}')$ guarantees the existence of $p'_\mu \in \mathbb{N}, \epsilon_\mu > 0$ such that

$$\int_{\mathcal{N}'} \exp\left(-\epsilon_\mu |x|_{-p_\mu}\right) d\mu(x) < \infty$$

by Lemma 2.3. Thus $\exp(\langle \cdot, \theta\rangle) \in L^2(\mu)$ if $2\,|\theta|_{p'_\mu} < \epsilon_\mu$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$. Then by Cauchy-Schwarz inequality the Laplace transform defined by

$$L_\mu\varphi(\theta) := \int_{\mathcal{N}'} \varphi(x) \exp\langle x, \theta\rangle \, d\mu(x)$$

is well defined for $\varphi \in L^2(\mu)$, $\theta \in \mathcal{H}_{p'_\mu,\mathbb{C}}$. Now we are interested to extend this integral transform from $L^2(\mu)$ to the space of distributions $(\mathcal{N})_\mu^{-1}$.

Since our construction of test functions and distributions spaces is closely related to $\mathbf{P}^{\mu}$- and $\mathbf{Q}^{\mu}$-systems it is useful to introduce the so called S_μ-transform

$$S_\mu\varphi(\theta) := \frac{L_\mu\varphi(\theta)}{l_\mu(\theta)} = \int_{\mathcal{N}'} \varphi(x)\, e_\mu(\theta; x)\, d\mu(x).$$

The μ-exponential $e_\mu(\theta;\cdot)$ is not a test function in $(\mathcal{N})^1$, see [KSWY95, Example 6], so the definition of the S_μ-transform of a distribution $\Phi \in (\mathcal{N})_\mu^{-1}$ must be more careful. Every such Φ is of finite order, i.e., $\exists p, q \in \mathbb{N}$ such that $\Phi \in (\mathcal{H}_{-p})_{-q,\mu}^{-1}$ and $e_\mu(\theta;\cdot)$ is in the corresponding dual space $(\mathcal{H}_p)_{q,\mu}^{1}$ if $\theta \in \mathcal{H}_{p,\mathbb{C}}$ is such that $2^q\,|\theta|_p^2 < 1$. Then we can define a consistent extension of S_μ-transform.

$$S_\mu\Phi(\theta) := \langle\!\langle \Phi, e_\mu(\theta,\cdot)\rangle\!\rangle_\mu$$

if θ is chosen in the above way. The biorthogonality of $\mathbf{P}^{\mu}$- and $\mathbf{Q}^{\mu}$-system implies

$$S_\mu\Phi(\theta) = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, \theta^{\otimes n}\right\rangle,$$

moreover $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$, see [KSWY95, Theorem 35].

4.3.2 Convolution C_μ

We define the convolution of a function $\varphi \in (\mathcal{N})^1$ with the measure μ by

$$C_\mu \varphi (y) := \int_{\mathcal{N}'} \varphi (x + y) \, d\mu (x) , \quad y \in \mathcal{N}'.$$

For any $\varphi \in (\mathcal{N})^1$, $z \in \mathcal{N}'_{\mathbb{C}}$, the convolution has the representation

$$C_\mu \varphi (z) = \langle\!\langle \rho_\mu (-z, \cdot) , \varphi \rangle\!\rangle_\mu .$$

If $\varphi \in (\mathcal{N})^1$ has the canonical $\mathbf{P}^\mu$-decomposition

$$\varphi = \sum_{n=0}^{\infty} \left\langle P_n^\mu , \varphi^{(n)} \right\rangle ,$$

then

$$C_\mu \varphi (z) = \sum_{n=0}^{\infty} \left\langle z^{\otimes n} , \varphi^{(n)} \right\rangle .$$

In Gaussian analysis C_μ- and S_μ-transform coincide. It is a typical non-Gaussian effect that these two transformations differ from each other.

4.4 Characterization theorems

Now we will characterize the spaces of test and generalized functions by the integral transforms introduced in the previous section.

We will start to characterize the space $(\mathcal{N})^1$ in terms of the convolution C_μ.

Theorem 4.8 *The convolution C_μ is a topological isomorphism from $(\mathcal{N})^1$ on $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.*

Remark. Since we have identified $(\mathcal{N})^1$ and $\mathcal{E}^1_{\min}(\mathcal{N}')$ by Theorem 4.6, the above assertion can be restated as follows. We have

$$C_\mu : \mathcal{E}^1_{\min}(\mathcal{N}') \to \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}}),$$

as a topological isomorphism.

The next Theorem characterizes distributions from $(\mathcal{N})^{-1}_\mu$ in terms of S_μ-transform.

Theorem 4.9 *The S_μ-transform is a topological isomorphism from $(\mathcal{N})_\mu^{-1}$ on* $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$.

Detailed proofs of the above theorems can be founded in [KSWY95, Theorems 33, 35].

5 Generalized Appell Systems

5.1 Description of the $\mathbf{P}^{\mu,\alpha}$-system

Remember that the μ-exponential is the generating function of the $\mathbf{P}^{\mu}$-system, i.e., if $\theta \in \mathcal{U}_0 \subset \mathcal{N}_{\mathbb{C}}$ and $z \in \mathcal{N}'_{\mathbb{C}}$, then

$$e_{\mu}(\theta, z) := \frac{\exp\langle z, \theta\rangle}{l_{\mu}(\theta)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left\langle P_n^{\mu}(z), \theta^{\otimes n}\right\rangle, \; P_n^{\mu}(z) \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}.$$

In view to generalize the Appell system we consider $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ an invertible function such that $\alpha(0) = 0$; moreover we have the following decomposition

$$\alpha(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!}\left\langle \alpha^{(n)}(0), \theta^{\otimes n}\right\rangle, \quad \theta \in \mathcal{U}_{\alpha} \subset \mathcal{N}_{\mathbb{C}} \tag{5.1}$$

where $\alpha^{(n)}(0) \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}}$ since α is vector valued. Analogously for the inverse function $\alpha^{-1} =: g_{\alpha}$, we have

$$g_{\alpha}(\theta) = \sum_{n=1}^{\infty} \frac{1}{n!}\left\langle g_{\alpha}^{(n)}(0), \theta^{\otimes n}\right\rangle, \; \theta \in \mathcal{V}_{\alpha} \subset \mathcal{N}_{\mathbb{C}}, \tag{5.2}$$

where $g_{\alpha}^{(n)}(0) \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}} \otimes \mathcal{N}_{\mathbb{C}}$. Now we introduce a new normalized exponential using the function α, i.e.,

$$e_{\mu}^{\alpha}(\theta; z) := e_{\mu}(\alpha(\theta); z) = \frac{\exp\langle z, \alpha(\theta)\rangle}{l_{\mu}(\alpha(\theta))}, \; \theta \in \mathcal{U}'_{\alpha} \subset \mathcal{U}_{\alpha}, \; z \in \mathcal{N}'_{\mathbb{C}}.$$

Using the same procedure as in Section 3 there exist $P_n^{\mu,\alpha}(z) \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ called **generalized Appell polynomial** or α-**polynomial** such that

$$e_{\mu}^{\alpha}(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle P_n^{\mu,\alpha}(z), \theta^{\otimes n}\rangle, \; \theta \in \mathcal{U}'_{\alpha}, \; z \in \mathcal{N}'_{\mathbb{C}}, \tag{5.3}$$

which for fixed $z \in \mathcal{N}'_{\mathbb{C}}$ converges uniformly on some neighborhood of zero on $\mathcal{N}_{\mathbb{C}}$. Hence we have constructed the $\mathbf{P}^{\mu,\alpha}$-system

$$\mathbf{P}^{\mu,\alpha} = \left\{\left\langle P_n^{\mu,\alpha}(\cdot), \varphi_{\alpha}^{(n)}\right\rangle \mid \varphi_{\alpha}^{(n)} \in \mathcal{N}^{\widehat{\otimes} n}_{\mathbb{C}}, \; n \in \mathbb{N}\right\}.$$

In this case the related moments kernels of the measure μ are determined by

$$l_\mu^\alpha(\theta) := l_\mu(\alpha(\theta)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle M_n^{\mu,\alpha}, \theta^{\otimes n} \right\rangle, \ \theta \in \mathcal{N}_{\mathbb{C}}, \ M_n^{\mu,\alpha} \in \mathcal{N}'^{\widehat{\otimes} n}.$$

Let us collect some properties of the polynomials $P_n^{\mu,\alpha}(z)$.

Proposition 5.1 *For $z, w \in \mathcal{N}'$, $n \in \mathbb{N}$ the following holds*

(P_α1)

$$P_n^{\mu,\alpha}(z) = \sum_{m=1}^{n} \frac{1}{m!} \left\langle P_m^\mu(z), A_n^m \right\rangle, \tag{5.4}$$

where A_n^m are related to the kernels of α and are given in the proof, see (5.12) below;

(P_α2)

$$z^{\otimes n} = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle P_m^{\mu,\alpha}(z), B_k^m \rangle \widehat{\otimes} M_{n-k}^\mu, \tag{5.5}$$

where B_k^m are related with the kernels of g_α and are given in the proof, see (5.13) below;

(P_α3)

$$P_n^{\mu,\alpha}(z+w) = \sum_{k+l+m=n} \frac{n!}{k!l!m!} P_k^{\mu,\alpha}(z) \widehat{\otimes} P_l^{\mu,\alpha}(w) \widehat{\otimes} M_m^{\mu,\alpha}. \tag{5.6}$$

(P_α4)

$$P_n^{\mu,\alpha}(z+w) = \sum_{k=0}^{n} \binom{n}{k} P_k^{\mu,\alpha}(z) \widehat{\otimes} P_{n-k}^{\delta_0,\alpha}(w). \tag{5.7}$$

(P_α5) Further, we observe

$$\mathbb{E}_\mu(\langle P_m^{\mu,\alpha}(\cdot), \varphi_\alpha^{(m)} \rangle) = 0 \quad \text{for} \quad m \neq 0, \ \varphi_\alpha^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} m}. \tag{5.8}$$

(P_α6) For all $p' > p$ such that the embedding $\mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$ is of Hilbert-Schmidt class and for all $\epsilon > 0$ there exist $\sigma_\epsilon > 0$ such that

$$|P_n^{\mu,\alpha}(z)|_{-p'} \leq 2\, n! \sigma_\epsilon^{-n} \ \exp(\varepsilon |z|_{-p}), \quad z \in \mathcal{H}_{-p',\mathbb{C}}, n \in \mathbb{N}_0, \tag{5.9}$$

where σ_ϵ is chosen in such a way that $|\alpha(\theta)| \leq \epsilon$ and $|l_\mu(\alpha(\theta))| \geq 1/2$ for $|\theta|_p = \sigma_\epsilon$.

Proof. ($P_\alpha 1$) Analogously with (3.3) we have

$$e_\mu^\alpha(\theta;z) := \frac{\exp\langle z,\alpha(\theta)\rangle}{l_\mu(\alpha(\theta))} = \sum_{m=0}^{\infty}\frac{1}{m!}\left\langle P_m^\mu(z),\alpha(\theta)^{\otimes m}\right\rangle. \tag{5.10}$$

Using the representation from (5.1) we compute $\alpha(\theta)^{\otimes m}$:

$$\begin{aligned}\alpha(\theta)^{\otimes m} &= \sum_{l=1}^{\infty}\frac{1}{l!}\left\langle\alpha^{(l)}(0),\theta^{\otimes l}\right\rangle\otimes\cdots\otimes\sum_{l=1}^{\infty}\frac{1}{l!}\left\langle\alpha^{(l)}(0),\theta^{\otimes l}\right\rangle \\ &= \sum_{l_1,\ldots,l_m=1}^{\infty}\frac{1}{l_1!\cdots l_m!}\left\langle\alpha^{(l_1)}(0)\otimes\cdots\otimes\alpha^{(l_m)}(0),\theta^{\otimes(l_1+\ldots+l_m)}\right\rangle \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\left\langle A_n^m,\theta^{\otimes n}\right\rangle, \end{aligned} \tag{5.11}$$

where

$$A_n^m = \begin{cases}\displaystyle\sum_{l_1+\ldots+l_m=n}\frac{n!}{l_1!\cdots l_m!}\alpha^{(l_1)}(0)\otimes\cdots\otimes\alpha^{(l_m)}(0) & \text{for } n\geq m \\ 0 & \text{for } n<m\end{cases}. \tag{5.12}$$

Now we introduce (5.11) in (5.10) to obtain

$$\begin{aligned}e_\mu^\alpha(\theta;z) &= \sum_{m=0}^{\infty}\frac{1}{m!}\left\langle P_m^\mu(z),\sum_{n=1}^{\infty}\frac{1}{n!}\left\langle A_n^m,\theta^{\otimes n}\right\rangle\right\rangle \\ &= \sum_{n=1}^{\infty}\frac{1}{n!}\left\langle\sum_{m=0}^{n}\frac{1}{m!}\left\langle P_m^\mu(z),A_n^m\right\rangle,\theta^{\otimes n}\right\rangle.\end{aligned}$$

By definition

$$e_\mu^\alpha(\theta;z) = \sum_{n=0}^{\infty}\frac{1}{n!}\left\langle P_n^{\mu,\alpha}(z),\theta^{\otimes n}\right\rangle,$$

so we conclude that

$$P_n^{\mu,\alpha}(z) = \sum_{m=1}^{n}\frac{1}{m!}\left\langle P_m^\mu(z),A_n^m\right\rangle.$$

(P$_\alpha$2) Since $\theta = \alpha(g_\alpha(\theta))$ we have

$$e_\mu(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\mu,\alpha}(z), g_\alpha(\theta)^{\otimes n} \right\rangle .$$

Having in mind (5.2) we first compute $g_\alpha(\theta)^{\otimes n}$:

$$\begin{aligned} g_\alpha(\theta)^{\otimes n} &= \sum_{l=1}^{\infty} \frac{1}{l!} \left\langle g_\alpha^{(l)}(0), \theta^{\otimes l} \right\rangle \otimes \cdots \otimes \sum_{l=1}^{\infty} \frac{1}{l!} \left\langle g_\alpha^{(l)}(0), \theta^{\otimes l} \right\rangle \\ &= \sum_{l_1,\ldots,l_n=1}^{\infty} \frac{1}{l_1! \cdots l_n!} \left\langle g_\alpha^{(l_1)}(0) \otimes \cdots \otimes g_\alpha^{(l_n)}(0), \theta^{\otimes(l_1+\ldots+l_n)} \right\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle B_m^n, \theta^{\otimes m} \right\rangle , \end{aligned}$$

where

$$B_m^n = \begin{cases} \displaystyle\sum_{l_1+\ldots+l_n=m} \frac{m!}{l_1! \cdots l_n!} g_\alpha^{(l_1)}(0) \otimes \cdots \otimes g_\alpha^{(l_n)}(0) & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} . \tag{5.13}$$

Hence

$$\begin{aligned} e_\mu(\theta, z) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\mu,\alpha}(z), \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle B_m^n, \theta^{\otimes m} \right\rangle \right\rangle \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle \sum_{n=0}^{m} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), B_m^n \rangle, \theta^{\otimes m} \right\rangle . \end{aligned}$$

On the other hand

$$e_\mu(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^\mu(z), \theta^{\otimes n} \right\rangle ,$$

so we conclude that

$$P_m^\mu(z) = \sum_{n=1}^{m} \frac{1}{n!} \langle P_n^{\mu,\alpha}(z), B_m^n \rangle . \tag{5.14}$$

The result follows using property (P2) of the polynomials $P_n^\mu(z)$.

($\mathrm{P}_\alpha 3$) Let us start from the equation of the generating functions

$$e_\mu^\alpha(\theta, z+w) = e_\mu^\alpha(\theta, z)\, e_\mu^\alpha(\theta, w)\, l_\mu^\alpha(\theta).$$

This implies

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\mu,\alpha}(z+w), \theta^{\otimes n} \right\rangle \\ = & \sum_{k,l,m=0}^{\infty} \frac{1}{k!l!m!} \left\langle P_k^{\mu,\alpha}(z) \widehat{\otimes} P_l^{\mu,\alpha}(w) \widehat{\otimes} M_m^{\mu,\alpha}, \theta^{\otimes(k+l+m)} \right\rangle, \end{aligned}$$

from this ($\mathrm{P}_\alpha 3$) follows immediately.
($\mathrm{P}_\alpha 4$) We note that

$$e_\mu^\alpha(\theta; z+w) = e_\mu^\alpha(\theta; z) \exp\langle w, \alpha(\theta)\rangle, \quad \theta \in \mathcal{U}_0 \subset \mathcal{N}_{\mathbb{C}}.$$

Now, since $l_{\delta_0}(\theta) = 1$, we have the following decomposition

$$\exp\langle w, \alpha(\theta)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_n^{\delta_0,\alpha}(w), \theta^{\otimes n} \right\rangle, \tag{5.15}$$

where for $\alpha \equiv \mathrm{id}$, $P_n^{\delta_0,\alpha}(w) = w^{\otimes n}$. The result follows as done in ($\mathrm{P}_\alpha 3$).
($\mathrm{P}_\alpha 5$) To see this we use, $\theta \in \mathcal{N}_{\mathbb{C}}$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_\mu \left(\left\langle P_m^{\mu,\alpha}(\cdot), \theta^{\otimes n} \right\rangle \right) = \mathbb{E}_\mu \left(e_\mu^\alpha(\theta; \cdot) \right) = \frac{\mathbb{E}_\mu(\exp\langle \cdot, \alpha(\theta)\rangle)}{l_\mu(\alpha(\theta))} = 1.$$

Then the polarization identity and a comparison of coefficients give the result.
($\mathrm{P}_\alpha 6$) Using the definition of $P_n^{\mu,\alpha}$ and Cauchy's inequality for Taylor series we have

$$\begin{aligned} \left| \left\langle P_n^{\mu,\alpha}(z), \theta^{\otimes n} \right\rangle \right| &= n! \left| d^n \widehat{e_\mu^\alpha(0; z)}(\theta) \right|_{-p} \\ &\leq n! \frac{1}{\sigma_\epsilon^n} \sup_{|\theta|_p = \sigma_\epsilon} \frac{\exp\left(|\alpha(\theta)|_p |z|_{-p} \right)}{|l_\mu(\alpha(\theta))|} |\theta|_p^n \\ &\leq 2n! \sigma_\epsilon^{-n} \exp\left(\epsilon |z|_{-p} \right) |\theta|_p^n. \end{aligned}$$

The result follows by polarization and kernel theorem. ■

Let us give a concrete example which furnish good arguments to use the $\mathbf{P}^{\mu,\alpha}$-system.

Example 5.2 (Poisson noise) *Let us consider the classical (real) Schwartz triple*

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}).$$

The **Poisson white noise measure** π *is defined as a probability measure on* $\mathcal{C}_\sigma(S'(\mathbb{R}))$ *with Laplace transform*

$$l_\pi(\theta) = \exp\left[\int_{\mathbb{R}} (\exp\theta(t) - 1)\,dt\right] = \exp\left[\langle \exp\theta(\cdot) - 1, 1\rangle\right], \quad \theta \in S_{\mathbb{C}}(\mathbb{R}),$$

see e.g., [GV68]. It is not hard to see that l_π *is a holomorphic function on* $S_{\mathbb{C}}(\mathbb{R})$*, so assumption 1 is satisfied. But to check Assumption 2, we need additional considerations.*

First of all we remark that for any $\xi \in S(\mathbb{R})$*,* $\xi \neq 0$ *the measure* π *and* $\pi(\cdot + \xi)$ *are orthogonal (see [GGV75] for a detailed analysis). It means that* π *is not* $S(\mathbb{R})$*-quasi-invariant and the note after Assumption 2 is not applicable now.*

Let some $\varphi \in \mathcal{P}(S'(\mathbb{R}))$*,* $\varphi = 0$ π*-a.s. be given. We need to show that then* $\varphi \equiv 0$*. To this end we will introduce a system of orthogonal polynomials in the space* $L^2(S'(\mathbb{R}), \pi)$ *which can be constructed in the following way. The mapping*

$$\theta(\cdot) \mapsto \alpha(\theta)(\cdot) = \log(1 + \theta(\cdot)) \in S_{\mathbb{C}}(\mathbb{R}), \quad \theta \in S_{\mathbb{C}}(\mathbb{R})$$

is holomorphic on a neighborhood $\mathcal{U} \subset S_{\mathbb{C}}(\mathbb{R})$*,* $0 \in \mathcal{U}$*. Then*

$$e_\pi^\alpha(\theta; x) = \frac{\exp\langle x, \alpha(\theta)\rangle}{l_\pi(\alpha(\theta))} = \exp\left[\langle x, \alpha(\theta)\rangle - \langle \theta, 1\rangle\right], \quad \theta \in \mathcal{U},\ x \in S'(\mathbb{R})$$

is a holomorphic function on $\mathcal{U}$ *for any* $x \in S'(\mathbb{R})$*. The Taylor decomposition and the kernel theorem give*

$$e_\pi^\alpha(\theta; x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle C_n(x), \theta^{\otimes n}\right\rangle,$$

where $C_n : S'(\mathbb{R}) \to S'(\mathbb{R})^{\widehat{\otimes} n}$ *are polynomial mappings. For* $\varphi^{(n)} \in S_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} n}$*,* $n \in \mathbb{N}_0$*, we define Charlier polynomials*

$$x \mapsto C_n\left(\varphi^{(n)}; x\right) := \left\langle C_n(x), \varphi^{(n)}\right\rangle \in \mathbb{C},\ x \in S'(\mathbb{R}).$$

Due to [Ito88], [IK88] we have the following orthogonality property:

$$\forall \varphi^{(n)} \in S_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} n}, \ \forall \psi^{(m)} \in S_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} m}$$

$$\int C_n\left(\varphi^{(n)}\right) C_m\left(\psi^{(m)}\right) d\pi = \delta_{nm} n! \left\langle \varphi^{(n)}, \psi^{(n)} \right\rangle .$$

Now the rest is simple. Any continuous polynomial φ has a uniquely defined decomposition

$$\varphi(x) = \sum_{n=0}^{N} \left\langle C_n(x), \varphi^{(n)} \right\rangle, \quad x \in S'(\mathbb{R}),$$

where $\varphi^{(n)} \in S_{\mathbb{C}}(\mathbb{R})^{\widehat{\otimes} n}$. If $\varphi = 0$ π-a.e., then

$$\|\varphi\|^2_{L^2(\pi)} = \sum_{n=0}^{N} n! \left\langle \varphi^{(n)}, \overline{\varphi^{(n)}} \right\rangle = 0.$$

Hence $\varphi^{(n)} = 0$, $n = 0, \ldots, N$, i.e., $\varphi \equiv 0$. So Assumption 2 is satisfied.

Lemma 5.3 *For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation*

$$\varphi(x) = \sum_{n=0}^{N} \left\langle P_n^{\mu,\alpha}(x), \varphi_\alpha^{(n)} \right\rangle, \quad \varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n} \tag{5.16}$$

and vice versa, any functional of the form (5.16) is a smooth polynomial.

Proof. The representation from Definition 2.4 and equation (5.16) can be transformed into one another using (5.4) and (5.5). ■

5.2 Description of the $\mathbf{Q}^{\mu,\alpha}$-system

5.2.1 Using S_μ-transform

By assumption we know that α is invertible with inverse given by g_α and $\alpha(\theta) \in \mathcal{V}_\alpha \subset \mathcal{N}_{\mathbb{C}}$, $\forall \theta \in \mathcal{U}_\alpha$. For given $\Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n}$ we define a generalized function $Q_n^{\mu,\alpha}(\Phi_\alpha^{(n)})$ via the S_μ-transform

$$S_\mu\left(Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right)\right)(\theta) := \left\langle \Phi_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \right\rangle, \quad \theta \in \mathcal{V}_\alpha. \tag{5.17}$$

5.2.2 Using differential operators

Using the kernels $g_\alpha^{(n)}(0)$ of g_α, see (5.2), we define a *differential operator (of infinite order)* from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')\otimes\mathcal{N}_{\mathbb{C}}$ as follows

$$G_\alpha=\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle g_\alpha^{(n)}(0),\triangledown^{\otimes n}\right\rangle,$$

such that, if $\varphi\in\mathcal{P}(\mathcal{N}')$ and $\xi\in\mathcal{N}'_{\mathbb{C}}$ we have

$$G_\alpha^\xi(\varphi)(x):=\langle\xi,G_\alpha(\varphi)(x)\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle\xi,\left\langle g_\alpha^{(n)}(0),\triangledown^{\otimes n}\varphi(x)\right\rangle\right\rangle,\quad x\in\mathcal{N}',$$

i.e., $G_\alpha^\xi:\mathcal{P}(\mathcal{N}')\to\mathcal{P}(\mathcal{N}')$ and formally $G_\alpha:=g_\alpha(\triangledown)$.

Let we state the following useful lemma.

Lemma 5.4 *For all $\xi\in\mathcal{N}'_{\mathbb{C}}$, $x\in\mathcal{N}'$ and $\theta\in\mathcal{N}_{\mathbb{C}}$ we have*

$$\langle\xi,g_\alpha(\triangledown)\rangle(\exp\langle x,\theta\rangle)=\langle\xi,g_\alpha(\theta)\rangle\exp\langle x,\theta\rangle.$$

Proof. Using the representation given in (5.2) we have

$$\langle\xi,g_\alpha(\triangledown)\rangle=\sum_{n=0}^{\infty}\frac{1}{n!}\left\langle g_{\alpha,\xi}^{(n)}(0),\nabla^{\otimes n}\right\rangle,\quad g_{\alpha,\xi}^{(n)}(0)=\left\langle g_\alpha^{(n)}(0),\xi\right\rangle\in\mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}.$$

For simplicity we put $g_{\alpha,\xi}^{(n)}(0)\equiv\Psi^{(n)}$. At first we apply the operator to some monomial. For given $\theta\in\mathcal{N}_{\mathbb{C}}$, $m\geq n$

$$\begin{aligned}\left\langle\Psi^{(n)},\nabla^{\otimes n}\right\rangle\langle x,\theta\rangle^m&=\left\langle\Psi^{(n)},\nabla^{\otimes n}\right\rangle\left\langle x^{\otimes m},\theta^{\otimes m}\right\rangle\\&=m(m-1)\cdots(m-n+1)\left\langle\Psi^{(n)}\widehat{\otimes}x^{\otimes(m-n)},\theta^{\otimes m}\right\rangle\\&=m(m-1)\cdots(m-n+1)\langle x,\theta\rangle^{m-n}\left\langle\Psi^{(n)},\theta^{\otimes n}\right\rangle,\end{aligned}$$

where we used (3.10) in the second equality. Now expand the given function, $\exp\langle x,\theta\rangle$, in the Taylor series and applying the above result we get

$$\begin{aligned}&\left\langle\Psi^{(n)},\nabla^{\otimes n}\right\rangle\exp\langle x,\theta\rangle\\=&\left\langle\Psi^{(n)},\nabla^{\otimes n}\right\rangle\sum_{m=0}^{\infty}\frac{\langle x,\theta\rangle^m}{m!}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=n}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{m!} \left\langle \Psi^{(n)} \widehat{\otimes} x^{\otimes(m-n)}, \theta^{\otimes m} \right\rangle \\
&= \left\langle \Psi^{(n)}, \theta^{\otimes n} \right\rangle \sum_{m=n}^{\infty} \frac{1}{(m-n)!} \langle x, \theta \rangle^{m-n} \\
&= \left\langle \Psi^{(n)}, \theta^{\otimes n} \right\rangle \exp \langle x, \theta \rangle .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\langle \xi, g_\alpha(\bigtriangledown) \rangle (\exp \langle x, \theta \rangle) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \Psi^{(n)}, \nabla^{\otimes n} \right\rangle \exp \langle x, \theta \rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \Psi^{(n)}, \theta^{\otimes n} \right\rangle \exp \langle x, \theta \rangle \\
&= \langle \xi, g_\alpha(\theta) \rangle (\exp \langle x, \theta \rangle) .
\end{aligned}
$$

■

Theorem 5.5 *Under the above conditions the* $Q_n^{\mu,\alpha}(\xi^{\otimes n})$ *are given by*

$$
Q_n^{\mu,\alpha}\left(\xi^{\otimes n}\right)(\cdot) = \left(\langle \xi, g_\alpha(\bigtriangledown) \rangle^{*n} 1\right)(\cdot) . \tag{5.18}
$$

Proof. Applying the S_μ-transform to the r.h.s of (5.18) we have

$$
\begin{aligned}
S_\mu\left(\langle \xi, g_\alpha(\bigtriangledown) \rangle^{*n} 1\right)(\theta) &= \left\langle\!\left\langle \langle \xi, g_\alpha(\bigtriangledown) \rangle^{*n} 1, e_\mu(\theta, \cdot) \right\rangle\!\right\rangle_\mu \\
&= \left\langle\!\left\langle 1, \langle \xi, g_\alpha(\bigtriangledown) \rangle^{n} e_\mu(\theta, \cdot) \right\rangle\!\right\rangle_\mu \\
&= \frac{1}{l_\mu(\theta)} \int_{\mathcal{N}'} \langle \xi, g_\alpha(\bigtriangledown) \rangle^n \exp \langle x, \theta \rangle \, \mathrm{d}\mu(x) \\
&= \frac{\langle \xi, g_\alpha(\theta) \rangle^n}{l_\mu(\theta)} \int_{\mathcal{N}'} \exp \langle x, \theta \rangle \, \mathrm{d}\mu(x) \\
&= \langle \xi, g_\alpha(\theta) \rangle^n .
\end{aligned} \tag{5.19}
$$

On the other hand the S_μ-transform of the l.h.s. (5.18), by (5.17), is the same as (5.19) which prove the result. ■

Example 5.6 *As an illustration of* G_α *we use again the Poisson measure* π *(see Example 5.2) and* $\alpha(\theta)(\cdot) = \log(1 + \theta(\cdot))$, $\theta \in S(\mathbb{R})$. *For this choice we have*

$$
g_\alpha(\theta)(\cdot) = \exp \theta(\cdot) - 1 = \sum_{n=1}^{\infty} \frac{\theta^n(\cdot)}{n!}.
$$

On the other hand, from (5.2) we have

$$g_\alpha(\theta)(\cdot) = \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle g_\alpha^{(n)}(0), \theta^{\otimes n} \right\rangle (\cdot),$$

so we conclude that

$$g_\alpha^{(n)}(0) = \delta(t_1 - t) \cdots \delta(t_n - t).$$

We introduce the notation of functional derivative (see [IK88]),

$$\nabla_{\delta_t}(\theta) = \frac{\delta}{\delta\theta(t)}, \quad \theta \in S(\mathbb{R}), \ t \in \mathbb{R}.$$

With this, we easily see that for $\nabla_h = \langle \nabla, h \rangle$ we have

$$(\exp(\nabla_h) f)(\cdot) = f(\cdot + h), \quad f \in \mathcal{P}(S'(\mathbb{R})), \ h \in S(\mathbb{R}).$$

Hence

$$(g_\alpha(\nabla_{\delta_t})(\theta))(f(\cdot)) = \left(\exp\left(\frac{\delta}{\delta\theta(t)}\right) - 1\right) f(\cdot) = f(\cdot + \delta_t) - f(\cdot)$$

and if $\xi \in S_{\mathbb{C}}(\mathbb{R})$ we have

$$\langle g_\alpha(\nabla_{\delta_t}), \xi \rangle f(\cdot) = \int_{\mathbb{R}} [f(\cdot + \delta_t) - f(\cdot)] \xi(t) \, dt.$$

Therefore if $f \in \mathcal{P}(S'(\mathbb{R}))$ then

$$G_\alpha : f(\cdot) \longmapsto f(\cdot + \delta_t) - f(\cdot).$$

This mapping can be considered as a **"gradient"** *operator on the Poisson space $(S'(\mathbb{R}),\mathcal{B}(S'(\mathbb{R})),\pi)$.*

Definition 5.7 *We define the $\mathbf{Q}^{\mu,\alpha}$-system in $\mathcal{P}'_\mu(\mathcal{N}')$ by*

$$\mathbf{Q}^{\mu,\alpha} = \left\{ Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right) \mid \Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n}, \ n \in \mathbb{N}_0 \right\},$$

and the pair $(\mathbf{P}^{\mu,\alpha}, \mathbf{Q}^{\mu,\alpha})$ will be called the **generalized Appell system** $\mathbf{A}^{\mu,\alpha}$ *generated by the measure μ and given mapping $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$.*

Now we are going to discuss the central property of the generalized Appell system $\mathbf{A}^{\mu,\alpha}$.

Theorem 5.8 (Biorthogonality of $\mathbf{Q}^{\mu,\alpha}$ and $\mathbf{P}^{\mu,\alpha}$ w.r.t. μ)

$$\left\langle\!\left\langle Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right), P_m^{\mu,\alpha}\left(\varphi_\alpha^{(m)}\right)\right\rangle\!\right\rangle_\mu = \delta_{nm} n! \left\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)}\right\rangle, \tag{5.20}$$

for $\Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes} n}$ *and* $\varphi_\alpha^{(m)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} m}$.

Proof. By definition of S_μ we have

$$S_\mu\left(Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right)\right)(\theta) := \left\langle\!\left\langle Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right), e_\mu(\theta,\cdot)\right\rangle\!\right\rangle_\mu$$

if we substitute $\theta \mapsto \alpha(\eta)$, then we obtain

$$\begin{aligned} S_\mu\left(Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right)\right)(\alpha(\eta)) &= \left\langle\!\left\langle Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right), e_\mu(\alpha(\eta),\cdot)\right\rangle\!\right\rangle_\mu \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \left\langle\!\left\langle Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right), \left\langle P_m^{\mu,\alpha}(\cdot), \eta^{\otimes m}\right\rangle\right\rangle\!\right\rangle_\mu . \end{aligned}$$

Substituting of θ by $\alpha(\eta)$ in (5.17) give us

$$S_\mu\left(Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right)\right)(\alpha(\eta)) = \left\langle \Phi_\alpha^{(n)}, \eta^{\otimes n}\right\rangle .$$

Then a comparison of coefficients and the polarization identity give the desired result. ■

Now we characterize the space $\mathcal{P}'_\mu(\mathcal{N}')$.

Theorem 5.9 *For all* $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ *there exists a unique sequence* $\{\Phi_\alpha^{(n)} \mid n \in \mathbb{N}_0\}$, $\Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes} n}$ *such that*

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right) \equiv \sum_{n=0}^{\infty} \left\langle Q_n^{\mu,\alpha}, \Phi_\alpha^{(n)}\right\rangle \tag{5.21}$$

and vice versa, every series of the form (5.21) generates a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$.

Proof. For $\Phi \in \mathcal{P}'_\mu(\mathcal{N}')$ we can uniquely define $\Phi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\prime\widehat{\otimes}n}$ by

$$\left\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \right\rangle := \frac{1}{n!} \left\langle\!\left\langle \Phi, \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle \right\rangle\!\right\rangle_\mu , \quad \varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n},$$

which is well defined since $\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$. The continuity of $\varphi_\alpha^{(n)} \mapsto \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle$ follows from the continuity of $\varphi \mapsto \langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu$, $\varphi \in \mathcal{P}(\mathcal{N}')$. This implies that

$$\varphi \longmapsto \sum_{n=0}^{\infty} n! \left\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \right\rangle$$

is continuous on $\mathcal{P}(\mathcal{N}')$. This defines a generalized function in $\mathcal{P}'_\mu(\mathcal{N}')$, which we denote by

$$\sum_{n=0}^{\infty} Q_n^{\mu,\alpha} \left(\Phi_\alpha^{(n)} \right).$$

In view of Theorem 5.8 it is easy to see that

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} \left(\Phi_\alpha^{(n)} \right).$$

To see the converse consider a series of the form (5.21) and $\varphi \in \mathcal{P}(\mathcal{N}')$. Then there exists $\varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that we have the representation

$$\varphi = \sum_{n=0}^{N} P_n^{\mu,\alpha} \left(\varphi_\alpha^{(n)} \right).$$

So we have

$$\left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} \left(\Phi_\alpha^{(n)} \right), \varphi \right\rangle\!\!\right\rangle_\mu = \sum_{n=0}^{N} n! \left\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \right\rangle,$$

because of Theorem 5.8. The continuity of

$$\varphi \longmapsto \left\langle\!\!\left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} \left(\Phi_\alpha^{(n)} \right), \varphi \right\rangle\!\!\right\rangle_\mu$$

follows because $\varphi_\alpha^{(n)} \mapsto \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle$ is continuous for all $n \in \mathbb{N}$. ■

6 Test functions on a linear space with measure

6.1 Test functions spaces

We will construct the test function space $(\mathcal{N})^1_{\mu,\alpha}$ using $\mathbf{P}^{\mu,\alpha}$-system and study some properties. On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the representation from (5.16)

$$\varphi(\cdot) = \sum_{n=0}^{N} \left\langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)} \right\rangle,$$

with $\varphi_\alpha^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$ for each $p > 0$ ($n \in \mathbb{N}$). Thus we may define for any $p, q \in \mathbb{N}$ a Hilbert norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|^2_{p,q,\mu,\alpha} = \sum_{n=0}^{N} (n!)^2 \, 2^{nq} \left|\varphi_\alpha^{(n)}\right|_p^2 < \infty$$

The completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|^2_{p,q,\mu,\alpha}$ is called $(\mathcal{H}_p)^1_{q,\mu,\alpha}$.

Definition 6.1 *We define*

$$(\mathcal{N})^1_{\mu,\alpha} := \operatorname{pr}\lim_{p,q \in \mathbb{N}} (\mathcal{H}_p)^1_{q,\mu,\alpha}$$

Theorem 6.2 $(\mathcal{N})^1_{\mu,\alpha}$ *is a nuclear space. The topology in* $(\mathcal{N})^1_{\mu,\alpha}$ *is uniquely defined by the topology on* $\mathcal{N}$*. It does not depend on the choice of the family of norms* $\{|\cdot|_p\}$*.*

Proof. Nuclearity of $(\mathcal{N})^1_{\mu,\alpha}$ follows essentially from that of $\mathcal{N}$. For fixed p, q choose p' such that the embedding

$$i_{p',p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p$$

is Hilbert-Schmidt and consider the embedding

$$I_{p',q',p,q,\alpha} : (\mathcal{H}_{p'})^1_{q',\mu,\alpha} \hookrightarrow (\mathcal{H}_p)^1_{q,\mu,\alpha}.$$

Then $I_{p',q',p,q,\alpha}$ is induced by

$$I_{p',q',p,q,\alpha}(\varphi) = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha}, i_{p',p}^{\otimes n} \varphi_\alpha^{(n)} \right\rangle \quad \text{for} \quad \varphi = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle \in (\mathcal{H}_{p'})^1_{q',\mu,\alpha}.$$

Its Hilbert-Schmidt norm, for a given orthonormal basis of $(\mathcal{H}_{p'})^1_{q',\mu,\alpha}$, can be estimate by

$$\|I_{p',q',p,q,\alpha}\|_{HS}^2 = \sum_{n=0}^{\infty} 2^{n(q-q')} \|i_{p',p}\|_{HS}^{2n}$$

which is finite for a suitably chosen q'.

To prove the independence of the family of norms, let us assume that we are given two different systems of Hilbert norms $|\cdot|_p$ and $|\cdot|'_k$, such that they induce the same topology on $\mathcal{N}$. For fixed k and l we have to estimate $\|\cdot\|'_{k,l,\mu,\alpha}$ by $\|\cdot\|_{p,q,\mu,\alpha}$ for some p, q (and vice versa which is completely analogous). But for all $f \in \mathcal{N}$ we have $|f|'_k \leq C\,|f|_p$ for some constant C and some p, since $|\cdot|'_k$ has to be continuous with respect to the projective limit topology on $\mathcal{N}$. That means that the injection i from $\mathcal{H}_p$ into the completion $\mathcal{K}_k$ of $\mathcal{N}$ with respect to $|\cdot|'_k$ is a mapping bounded by C. We denote by i also its linear extension from $\mathcal{H}_{p,\mathbb{C}}$ into $\mathcal{K}_{k,\mathbb{C}}$. It follows that $i^{\otimes n}$ is bounded by C^n from $\mathcal{H}_{p,\mathbb{C}}^{\otimes n}$ into $\mathcal{K}_{k,\mathbb{C}}^{\otimes n}$. Now we choose q such that $2^{\frac{q-l}{2}} \geq C$. Then

$$\begin{aligned}
\|\cdot\|'_{k,l,\mu,\alpha} &= \sum_{n=0}^{\infty} (n!)^2\, 2^{nl}\, |\cdot|'^2_k \\
&\leq \sum_{n=0}^{\infty} (n!)^2\, 2^{nl} C^{2n}\, |\cdot|_p^2 \\
&\leq \|\cdot\|_{p,q,\mu,\alpha}
\end{aligned}$$

which is exactly what we need. ■

Lemma 6.3 *There exist $p, C, K > 0$ such that for all $n \in \mathbb{N}_0$*

$$\int |P_n^{\mu,\alpha}(z)|_{-p}^2\, d\mu(z) \leq 4\,(n!)^2\, C^n K. \tag{6.1}$$

Proof. We can use the estimate (5.9) and Lemma 2.3 to conclude the result. ■

Theorem 6.4 *There exists $p', q' > 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu,\alpha} \subset L^2(\mu)$ holds.*

Proof. Elements of the space $(\mathcal{N})^1_{\mu,\alpha}$ are defined as series convergent in the given topology. Now we need the convergence of the series in $L^2(\mu)$. Choose q' such that $C > 2^{q'}$ (C from estimate (6.1)). Let us take an arbitrary

$$\varphi = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle \in \mathcal{P}\left(\mathcal{N}'\right).$$

For $p > p'$ (p' from the Lemma 6.3) and $q > q'$ the following estimates hold

$$\begin{aligned}
\|\varphi\|_{L^2(\mu)} &\leq \sum_{n=0}^{\infty} \left\| \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle \right\|_{L^2(\mu)} \\
&\leq \sum_{n=0}^{\infty} \left| \varphi_\alpha^{(n)} \right|_p \left\| \left| P_n^{\mu,\alpha} \right|_{-p} \right\|_{L^2(\mu)} \\
&\leq 2K^{1/2} \sum_{n=0}^{\infty} n! 2^{nq/2} \left| \varphi_\alpha^{(n)} \right|_p \left(C2^{-q} \right)^{n/2} \\
&\leq 2K^{1/2} \left(\sum_{n=0}^{\infty} \left(C2^{-q} \right)^n \right)^{1/2} \left(\sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left| \varphi_\alpha^{(n)} \right|_p^2 \right)^{1/2} \\
&= 2K^{1/2} \left(1 - C2^{-q} \right)^{-1/2} \|\varphi\|_{p,q,\mu,\alpha}.
\end{aligned}$$

Taking the closure the inequality extends to the whole space $(\mathcal{H}_p)^1_{q,\mu,\alpha}$. ■

Corollary 6.5 $(\mathcal{N})^1_{\mu,\alpha}$ *is continuously and densely embedded in $L^2(\mu)$.*

6.2 Description of test functions

Proposition 6.6 *Any test function φ in $(\mathcal{N})^1_{\mu,\alpha}$ has a uniquely defined extension to $\mathcal{N}'_{\mathbb{C}}$ as an element of $\mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})$.*

Proof. Any element φ in $(\mathcal{N})^1_{\mu,\alpha}$ is defined as a series of the following type

$$\varphi = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle, \quad \varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n},$$

such that

$$\|\varphi\|^2_{p,q,\mu,\alpha} = \sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left|\varphi^{(n)}_{\alpha}\right|^2_p < \infty$$

for each $p, q \in \mathbb{N}$. So we need to show the convergence of the series

$$\sum_{n=0}^{\infty} \left\langle P^{\mu,\alpha}_n (z) , \varphi^{(n)}_{\alpha} \right\rangle, \quad z \in \mathcal{H}_{-p,\mathbb{C}}$$

to an entire function in z. Let $\epsilon > 0$ and $\sigma_{\epsilon} > 0$ as in ($\mathrm{P}_{\alpha}6$) of Proposition 5.1. We use (5.9) and estimate as follows

$$\begin{aligned}
& \sum_{n=0}^{\infty} \left|\left\langle P^{\mu,\alpha}_n (z) , \varphi^{(n)}_{\alpha} \right\rangle\right| \\
\leq \; & \sum_{n=0}^{\infty} \left|P^{\mu,\alpha}_n (z)\right|_{-p} \left|\varphi^{(n)}_{\alpha}\right|_p \\
\leq \; & 2 \sum_{n=0}^{\infty} n! \left|\varphi^{(n)}_{\alpha}\right|_p \sigma_{\epsilon}^{-n} \\
\leq \; & 2 \exp\left(\epsilon \left|z\right|_{-p'}\right) \left(\sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left|\varphi^{(n)}_{\alpha}\right|^2_p\right)^{1/2} \left(\sum_{n=0}^{\infty} 2^{-nq} \sigma_{\epsilon}^{-2n}\right)^{1/2} \\
\leq \; & 2 \, \|\varphi\|_{p,q,\mu,\alpha} \left(1 - 2^{-q} \sigma_{\epsilon}^{-2}\right)^{-1/2} \exp\left(\epsilon \left|z\right|_{-p'}\right),
\end{aligned}$$

if $2^q > \sigma_{\epsilon}^{-2}$ and p' is such that $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is Hilbert-Schmidt. That means the series

$$\sum_{n=0}^{\infty} \left\langle P^{\mu,\alpha}_n (z) , \varphi^{(n)}_{\alpha} \right\rangle$$

converges uniformly and absolutely in any neighborhood of zero of any space $\mathcal{H}_{-p,\mathbb{C}}$. Since each term $\langle P^{\mu,\alpha}_n (z) , \varphi^{(n)}_{\alpha} \rangle$ is entire in z the uniform convergence implies that

$$z \longmapsto \sum_{n=0}^{\infty} \left\langle P^{\mu,\alpha}_n (z) , \varphi^{(n)}_{\alpha} \right\rangle$$

is entire on each $\mathcal{H}_{-p,\mathbb{C}}$ and hence on $\mathcal{N}'_{\mathbb{C}}$. This complete the proof. ■

The following corollary gives an explicit estimate on the growth of test functions and is a consequence of the above Proposition.

Corollary 6.7 *For all $p > p'$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is of the Hilbert-Schmidt class and for all $\epsilon > 0$ there exists σ_ϵ (σ_ϵ from Proposition 5.1), such that for $p \in \mathbb{N}$ we obtain the following bound*

$$|\varphi(z)| \leq C \|\varphi\|_{p,q,\mu,\alpha} \exp\left(\epsilon |z|_{-p'}\right), \ \varphi \in (\mathcal{N})^1_{\mu,\alpha}, \ z \in \mathcal{H}_{-p,\mathbb{C}},$$

where $2^q > \sigma_\epsilon^{-2}$ and

$$C = 2\left(1 - 2^{-q}\sigma_\epsilon^{-2}\right)^{-1/2}.$$

Remark 6.8 *Proposition 6.6 states*

$$(\mathcal{N})^1_{\mu,\alpha} \subseteq \mathcal{E}^1_{\min}(\mathcal{N}')$$

as sets, where

$$\mathcal{E}^1_{\min}(\mathcal{N}') = \left\{\varphi|_{\mathcal{N}'} \,|\, \varphi \in \mathcal{E}^1_{\min}(\mathcal{N}'_{\mathbb{C}})\right\}.$$

Now we are going to show that the converse also holds.

Theorem 6.9 *For all functions $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$, as in Subsection 5.1, and for all measure $\mu \in \mathcal{M}_a(\mathcal{N}')$, we have the topological identity*

$$(\mathcal{N})^1_{\mu,\alpha} = \mathcal{E}^1_{\min}(\mathcal{N}').$$

Proof. Let $\varphi(z) \in \mathcal{E}^1_{\min}(\mathcal{N}')$ be given such that

$$\varphi(z) = \sum_{n=0}^{\infty} \left\langle z^{\otimes n}, \psi^{(n)} \right\rangle,$$

with

$$|||\varphi|||^2_{p,q,1} = \sum_{n=0}^{\infty} (n!)^2 \, 2^{nq} \left|\psi^{(n)}\right|^2_p < \infty$$

for each $p, q \in \mathbb{N}$. So we have

$$\left|\psi^{(n)}\right|_p \leq (n!)^{-1} \, 2^{-nq/2} \, |||\varphi|||_{p,q,1}.$$

On the other hand, we can use (5.5) to evaluate $\varphi(z)$ as

$$\varphi(z) \;=\; \sum_{n=0}^{\infty} \left\langle z^{\otimes n}, \psi^{(n)} \right\rangle$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\left\langle \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\frac{1}{m!}\left\langle P_m^{\mu,\alpha}(z), B_k^m\right\rangle \widehat{\otimes} M_{n-k}^{\mu}, \psi^{(n)}\right\rangle \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\frac{1}{m!}\left\langle \left\langle P_m^{\mu,\alpha}(z), B_k^m\right\rangle, \left(M_{n-k}^{\mu}, \psi^{(n)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right\rangle \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\frac{1}{m!}\left\langle P_m^{\mu,\alpha}(z), \left\langle B_k^m, \left(M_{n-k}^{\mu}, \psi^{(n)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right\rangle\right\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}\left\langle P_m^{\mu,\alpha}(z), \left\langle B_{k+m}^m, \left(M_{n-k}^{\mu}, \psi^{(n+m)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right\rangle\right\rangle \\
&= \sum_{m=0}^{\infty}\left\langle P_m^{\mu,\alpha}(z), \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}\left\langle B_{k+m}^m, \left(M_{n-k}^{\mu}, \psi^{(n+m)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right\rangle\right\rangle,
\end{aligned}
$$

such that, if

$$
\varphi(z) = \sum_{m=0}^{\infty}\left\langle P_m^{\mu,\alpha}(z), \varphi_\alpha^{(m)}\right\rangle,
$$

then we conclude that

$$
\varphi_\alpha^{(m)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}\left\langle B_{k+m}^m, \left(M_{n-k}^{\mu}, \psi^{(n+m)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right\rangle.
$$

Now for $p \in \mathbb{N}$ we need estimate $|\varphi_\alpha^{(n)}|_p$ by $\|\!|\cdot|\!\|_{p,q,1}$ since the nuclear topology given by the norms $\|\!|\cdot|\!\|_{p,q,1}$, is equivalent to the projective topology induced by the norms $\mathrm{n}_{p,l,k}$ (see [KSWY95]). Now we estimate $\varphi_\alpha^{(m)}$ as follows

$$
\begin{aligned}
\left|\varphi_\alpha^{(m)}\right|_p &\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}\left|B_{k+m}^m\right|_{\mathcal{H}_{-p}^{\widehat{\otimes}(k+m)}\otimes\mathcal{H}_p^{\widehat{\otimes}m}}\left|\left(M_{n-k}^{\mu}, \psi^{(n+m)}\right)_{\mathcal{H}^{\widehat{\otimes}(n-k)}}\right|_p \\
&\leq \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+m}{k+m}\frac{1}{m!}\left|B_{k+m}^m\right|_{\mathcal{H}_{-p}^{\widehat{\otimes}(k+m)}\otimes\mathcal{H}_p^{\widehat{\otimes}m}}\left|M_{n-k}^{\mu}\right|_{-p}\left|\psi^{(n+m)}\right|_p.
\end{aligned}
$$

Let us, at first, estimate the norm

$$
\left|B_{k+m}^m\right|_{-p,p} := \left|B_{k+m}^m\right|_{\mathcal{H}_{-p}^{\widehat{\otimes}(k+m)}\otimes\mathcal{H}_p^{\widehat{\otimes}m}}.
$$

To do this we choose $p > p_\mu$ such that $\left\|i_{p,p_\mu}\right\|_{HS}$ is finite and define

$$
D_{\alpha,\epsilon} := \sup_{|\theta|_p=\epsilon}\left|g_\alpha(\theta)\right|_p \quad \text{and} \quad \tilde{\epsilon} := \frac{\epsilon}{e\left\|i_{p,p_\mu}\right\|_{HS}}.
$$

So, with this

$$
\begin{aligned}
|B_m^n|_{-p,p} &\le \sum_{l_1,\dots,l_n=m} \frac{m!}{l_1!\cdots l_n!} \left|g_\alpha^{(l_1)}(0)\right|_{-p,p} \cdots \left|g_\alpha^{(l_n)}(0)\right|_{-p,p} \\
&\le \sum_{l_1,\dots,l_n=m} \frac{m! l_1!\cdots l_n!}{l_1!\cdots l_n!} D_{\alpha,\epsilon}^n \tilde{\epsilon}^{-m} \\
&\le m! D_{\alpha,\epsilon}^n 2^m \tilde{\epsilon}^{-m},
\end{aligned}
$$

that means

$$
\left|B_{k+m}^m\right|_{-p,p} \le (k+m)! D_{\alpha,\epsilon}^m 2^{k+m} \tilde{\epsilon}^{-(k+m)}.
$$

Now let $q \in \mathbb{N}$ such that $2^{q/2} > K_p$ ($K_p := eC \left\|i_{p,p_\mu}\right\|_{HS}$ as in (2.2)) and such that $2/(\tilde{\epsilon} K_p) < 1$, then we obtain

$$
\begin{aligned}
& \left|\varphi_\alpha^{(m)}\right|_p \\
\le\ & \sum_{n=0}^\infty \sum_{k=0}^n \binom{n+m}{k+m} \frac{1}{m!} (m+k)! D_{\alpha,\epsilon}^m \frac{2^{k+m}}{\tilde{\epsilon}^{k+m}} (n-k)! (K_p)^{n-k} \frac{2^{-(n+m)q/2}}{(n+m)!} \|\varphi\|_{p,q,1} \\
\le\ & \|\varphi\|_{p,q,1} \frac{2^{-mq/2}}{m!} D_{\alpha,\epsilon}^m \sum_{n=0}^\infty \left(2^{-q/2} K_p\right)^n \sum_{k=0}^n \left(\frac{2}{\tilde{\epsilon} K_p}\right)^k \\
\le\ & \|\varphi\|_{p,q,1} \frac{2^{-mq/2} 2^m}{m! \tilde{\epsilon}^m} D_{\alpha,\epsilon}^m \left(1 - 2^{-q/2} K_p\right)^{-1} \frac{\tilde{\epsilon} K_p}{\tilde{\epsilon} K_p - 2} \\
\equiv\ & L_{p,q,\alpha,\tilde{\epsilon}} \frac{2^{-mq/2} 2^m}{m! \tilde{\epsilon}^m} D_{\alpha,\epsilon}^m \|\varphi\|_{p,q,1}.
\end{aligned}
$$

For $q' < q$ such that $2^2 \tilde{\epsilon}^{-2} 2^{(q'-q)} D_{\alpha,\epsilon} < 1$ this follows the following estimate

$$
\begin{aligned}
\|\varphi\|_{p,q',\mu,\alpha}^2 &\le \sum_{m=0}^\infty (m!)^2 2^{mq'} \left|\varphi^{(m)}\right|_p^2 \\
&\le \|\varphi\|_{p,q,1}^2 L_{p,q,\alpha,\tilde{\epsilon}}^2 \sum_{m=0}^\infty \left(2^2 \tilde{\epsilon}^{-2} 2^{(q'-q)} D_{\alpha,\epsilon}\right)^m < \infty.
\end{aligned}
$$

This complete the proof. ■

Since we now have proved that the space of test functions $(\mathcal{N})_{\mu,\alpha}^1$ is isomorphic to $\mathcal{E}_{\min}^1(\mathcal{N}')$, for all measures $\mu \in \mathcal{M}_a(\mathcal{N}')$ and for all holomorphic invertible function $\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$, such that $\alpha(0) = 0$, we will now drop the subscript μ, α. The test function space $(\mathcal{N})^1$ is the same for all measures and functions α in the above conditions.

Corollary 6.10 *$(\mathcal{N})^1$ is an algebra under pointwise multiplication.*

Corollary 6.11 *$(\mathcal{N})^1$ admits 'scaling', i.e., for $\lambda \in \mathbb{C}$ the scaling operator $\sigma_\lambda : (\mathcal{N})^1 \to (\mathcal{N})^1$ defined by $\sigma_\lambda \varphi(x) := \varphi(\lambda x)$, $\varphi \in (\mathcal{N})^1$, $x \in \mathcal{N}'$ is well-defined.*

Corollary 6.12 *For all $z \in \mathcal{N}'_{\mathbb{C}}$ the space $(\mathcal{N})^1$ is invariant under the shift operator $\tau_z : \varphi \mapsto \varphi(\cdot + z)$.*

7 Distributions

In this section we will introduce and study the space $(\mathcal{N})^{-1}_{\mu,\alpha}$ of distributions corresponding to the space of test functions $(\mathcal{N})^{1}$ ($\equiv (\mathcal{N})^{1}_{\mu,\alpha}$). The goal is to prove that, for a fixed measure μ and for all function α, as in the subsection 5.1, the space $(\mathcal{N})^{-1}_{\mu,\alpha} = (\mathcal{N})^{-1}_{\mu}$, see Theorem 7.3 below.

Since $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^{1}$ the space $(\mathcal{N})^{-1}_{\mu,\alpha}$ can be viewed as a subspace of $\mathcal{P}'_{\mu}(\mathcal{N}')$, i.e.,

$$(\mathcal{N})^{-1}_{\mu,\alpha} \subset \mathcal{P}'_{\mu}(\mathcal{N}') .$$

Let us now introduce the Hilbert subspace $(\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha}$ of $\mathcal{P}'_{\mu}(\mathcal{N}')$ for which the norm

$$\|\Phi\|^{2}_{-p,-q,\mu,\alpha} := \sum_{n=0}^{\infty} 2^{-qn} \left|\Phi^{(n)}_{\alpha}\right|^{2}_{-p}$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu,\alpha}_{n}\left(\Phi^{(n)}_{\alpha}\right) \in \mathcal{P}'_{\mu}(\mathcal{N}')$$

from Theorem 5.9. The space $(\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha}$ is the dual space of $(\mathcal{H}_{p})^{1}_{q,\mu,\alpha}$ with respect to $L^{2}(\mu)$ (because of the biorthogonality of $\mathbf{P}^{\mu,\alpha}$- and $\mathbf{Q}^{\mu,\alpha}$-systems). By general duality theory

$$(\mathcal{N})^{-1}_{\mu,\alpha} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\mu,\alpha}$$

is the dual space of $(\mathcal{N})^{1}$ with respect to $L^{2}(\mu)$. As noted in Section 2 there exists a natural topology on co-nuclear spaces (which coincide with the inductive limit topology). We will consider $(\mathcal{N})^{-1}_{\mu,\alpha}$ as a topological vector space with this topology. So we have the nuclear triple

$$(\mathcal{N})^{1} \subset L^{2}(\mu) \subset (\mathcal{N})^{-1}_{\mu,\alpha} .$$

The action of a distribution

$$\Phi = \sum_{n=0}^{\infty} Q^{\mu,\alpha}_{n}(\Phi^{(n)}_{\alpha}) \in (\mathcal{N})^{-1}_{\mu,\alpha}$$

on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle P^{\mu,\alpha}_{n}, \varphi^{(n)}_{\alpha}\rangle \in (\mathcal{N})^{1}$$

is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} n! \left\langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \right\rangle .$$

For a more detailed characterization of the singularity of distributions in $(\mathcal{N})_{\mu,\alpha}^{-1}$ we will introduce some subspaces in this distribution space. For $\beta \in [0,1]$ we define

$$(\mathcal{H}_{-p})_{-q,\mu,\alpha}^{-\beta} := \left\{ \Phi \in \mathcal{P}'_\mu(\mathcal{N}') \mid \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-nq} \left|\Phi_\alpha^{(n)}\right|_{-p}^2 < \infty \right. \\ \left. \text{for } \Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right) \right\}$$

and

$$(\mathcal{N})_{\mu,\alpha}^{-\beta} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})_{-q,\mu,\alpha}^{-\beta} .$$

It is clear that the singularity increases with increasing β :

$$(\mathcal{N})_{\mu,\alpha}^{-0} \subset (\mathcal{N})_{\mu,\alpha}^{-\beta_1} \subset (\mathcal{N})_{\mu,\alpha}^{-\beta_2} \subset (\mathcal{N})_{\mu,\alpha}^{-1}$$

if $\beta_1 \le \beta_2$. We will also consider $(\mathcal{N})_{\mu,\alpha}^{-\beta}$ as equipped with the natural topology.

Example 7.1 (Generalized Radon-Nikodym derivative) *We want to define a generalized function $\rho_\mu^\alpha(z,\cdot) \in (\mathcal{N})_{\mu,\alpha}^{-1}$, $z \in \mathcal{N}'_{\mathbb{C}}$ with the following property*

$$\langle\!\langle \rho_\mu^\alpha(z,\cdot), \varphi \rangle\!\rangle_\mu = \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x), \quad \varphi \in (\mathcal{N})^1 .$$

That means we have to establish the continuity of $\rho_\mu^\alpha(z,\cdot)$. Let $z \in \mathcal{H}_{-p,\mathbb{C}}$. If $p \ge p'$ is sufficiently large and $\epsilon > 0$ small enough, Corollary 6.7 applies, i.e., $\exists q \in \mathbb{N}$ and $C > 0$ such that

$$\begin{aligned} \left| \int_{\mathcal{N}'} \varphi(x-z)\, d\mu(x) \right| &\le C \|\varphi\|_{p,q,\mu,\alpha} \int_{\mathcal{N}'} \exp\left(\epsilon |x-z|_{-p'}\right) d\mu(x) \\ &\le C \|\varphi\|_{p,q,\mu,\alpha} \exp\left(\epsilon |z|_{-p'}\right) \int_{\mathcal{N}'} \exp\left(\epsilon |x|_{-p'}\right) d\mu(x) . \end{aligned}$$

If ϵ is chosen sufficiently small the last integral exists (Lemma 2.3-3). Thus we have in fact $\rho_\mu^\alpha(z,\cdot) \in (\mathcal{N})^{-1}_{\mu,\alpha}$. It is clear that whenever the Radon-Nikodym derivative $\frac{d\mu(x+\xi)}{d\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case μ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu^\alpha(\xi,\cdot)$ defined above. We will show that in $(\mathcal{N})^{-1}_{\mu,\alpha}$ we have the canonical expansion

$$\rho_\mu^\alpha(z,\cdot) = \sum_{n=0}^\infty \frac{1}{n!}(-1)^n \left\langle Q_n^{\mu,\alpha}(\cdot), P_n^{\delta_0,\alpha}(-z)\right\rangle$$

where $P_n^{\delta_0,\alpha}(-z)$ is defined in (5.15). It is easy to see that the r.h.s. defines an element in $(\mathcal{N})^{-1}_{\mu,\alpha}$. Since both sides are in $(\mathcal{N})^{-1}_{\mu,\alpha}$ it is sufficient to compare their action on a total set from $(\mathcal{N})^1$. For $\varphi_\alpha^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ we have

$$\begin{aligned}
&\left\langle\!\left\langle \rho_\mu^\alpha(z,\cdot), \left\langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\right\rangle\right\rangle\!\right\rangle_\mu \\
= &\left\langle\!\!\left\langle \sum_{k=0}^\infty \frac{1}{k!}(-1)^k \left\langle Q_k^{\mu,\alpha}(\cdot), P_k^{\delta_0,\alpha}(-z)\right\rangle, \left\langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\right\rangle \right\rangle\!\!\right\rangle_\mu \\
= &\left\langle P_n^{\delta_0,\alpha}(-z), \varphi_\alpha^{(n)}\right\rangle,
\end{aligned}$$

where we have used the biorthogonality property of the $\mathbf{Q}^{\mu,\alpha}$- and $\mathbf{P}^{\mu,\alpha}$- systems. On the other hand

$$\begin{aligned}
&\left\langle\!\left\langle \rho_\mu^\alpha(z,\cdot), \left\langle P_n^{\mu,\alpha}(\cdot), \varphi_\alpha^{(n)}\right\rangle\right\rangle\!\right\rangle_\mu \\
= &\int_{\mathcal{N}'} \left\langle P_n^{\mu,\alpha}(x-z), \varphi_\alpha^{(n)}\right\rangle d\mu(x) \\
= &\sum_{k=0}^n \binom{n}{k} \int_{\mathcal{N}'} \left\langle P_k^{\mu,\alpha}(x)\,\widehat{\otimes}\, P_{n-k}^{\delta_0,,\alpha}(-z), \varphi_\alpha^{(n)}\right\rangle d\mu(x) \\
= &\sum_{k=0}^n \binom{n}{k} \mathbb{E}_\mu\left(\left\langle P_k^{\mu,\alpha}(\cdot)\,\widehat{\otimes}\, P_{n-k}^{\delta_0,,\alpha}(-z), \varphi_\alpha^{(n)}\right\rangle\right) \\
= &\left\langle P_n^{\delta_0,\alpha}(-z), \varphi_\alpha^{(n)}\right\rangle,
\end{aligned}$$

where we made use of the relation (5.8). This had to be shown. In other words, we have proven that $\rho_\mu^\alpha(z,\cdot)$ is the generating function of the $\mathbf{Q}^{\mu,\alpha}$-system.

$$\rho_\mu^\alpha(-z,\cdot) = \sum_{n=0}^\infty \frac{1}{n!} \left\langle Q_n^{\mu,\alpha}(\cdot), P_n^{\delta_0,\alpha}(z)\right\rangle$$

Example 7.2 (Delta function) *For $z \in \mathcal{N}'_{\mathbb{C}}$ we define a distribution by the following $\mathbf{Q}^{\mu,\alpha}$-decomposition:*

$$\delta_z = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n^{\mu,\alpha} \left(P_n^{\mu,\alpha} (z) \right).$$

If $p \in \mathbb{N}$ is large enough and $\epsilon > 0$ sufficiently small there exists $\sigma_\epsilon > 0$ according to (5.9) such that

$$\begin{aligned} \|\delta_z\|^2_{-p,-q,\mu,\alpha} &= \sum_{n=0}^{\infty} (n!)^{-2} \, 2^{-nq} \left| P_n^{\mu,\alpha} (z) \right|^2_{-p} \\ &\leq 4 \exp \left(2\epsilon \, |z|_{-p} \right) \sum_{n=0}^{\infty} \sigma_\epsilon^{-2n} 2^{-nq}, \qquad z \in \mathcal{H}_{-p,\mathbb{C}}, \end{aligned}$$

which is finite for sufficiently large $q \in \mathbb{N}$. Thus $\delta_z \in (\mathcal{N})^{-1}_{\mu,\alpha}$.

For

$$\varphi = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \right\rangle \in (\mathcal{N})^1$$

the action of δ_z is given by

$$\langle\!\langle \delta_z, \varphi \rangle\!\rangle_\mu = \sum_{n=0}^{\infty} \left\langle P_n^{\mu,\alpha} (z) \, , \varphi_\alpha^{(n)} \right\rangle = \varphi(z)$$

because of the biorthogonality property, see Theorem 5.8 pag. 32. This means that δ_z (in particular for z real) plays the role of a "δ-function" (evaluation map) in the calculus we discuss.

Theorem 7.3 *For a fixed measure μ and for all function α, as in subsection 5.1, we have*

$$(\mathcal{N})^{-1}_{\mu,\alpha} = (\mathcal{N})^{-1}_{\mu},$$

i.e., the space of distributions is the same for all functions α in the above conditions.

Proof. Let $\Phi \in (\mathcal{N})^{-1}_{\mu,\alpha}$ be given, then by Theorem 5.9 there exists generalized kernels $\Phi_\alpha^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ such that Φ has the following representation

$$\Phi = \sum_{n=0}^{\infty} \left\langle Q_n^{\mu,\alpha}, \Phi_\alpha^{(n)} \right\rangle.$$

Now we use the definition of $Q_n^{\mu,\alpha}$ given in (5.17) to obtain

$$\begin{aligned} S_\mu \Phi(\theta) &= \sum_{n=0}^{\infty} \left\langle \Phi_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \right\rangle \\ &= S_\mu \widehat{\Phi}(g_\alpha(\theta)), \qquad \theta \in \mathcal{N}_{\mathbb{C}}, \end{aligned} \tag{7.1}$$

where

$$\widehat{\Phi} = \sum_{n=0}^{\infty} \left\langle Q_n^\mu, \Phi_\alpha^{(n)} \right\rangle \in (\mathcal{N})_\mu^{-1}.$$

Hence by characterization Theorem 4.9 $S_\mu \widehat{\Phi} \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. But from (7.1) we see that

$$S_\mu \Phi = \left(S_\mu \widehat{\Phi} \right) \circ g_\alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}}),$$

since this is the composition of two holomorphic functions (see [Din81]), again by the characterization Theorem 4.9 we conclude that $\Phi \in (\mathcal{N})_\mu^{-1}$. Hence $(\mathcal{N})_{\mu,\alpha}^{-1} \subseteq (\mathcal{N})_\mu^{-1}$.

Conversely, let $\Psi \in (\mathcal{N})_\mu^{-1}$ be given, i.e.,

$$\Psi = \sum_{n=0}^{\infty} \left\langle Q_n^\mu, \Psi^{(n)} \right\rangle, \quad \Psi^{(n)} \in \mathcal{N}_{\mathbb{C}}'^{\widehat{\otimes} n}.$$

We want to prove that $\Psi \in (\mathcal{N})_{\mu,\alpha}^{-1}$. Due to (5.17) and the definition of $(\mathcal{N})_\mu^{-1}$ it is sufficient to show that

$$S_\mu \Psi(\theta) = \sum_{n=0}^{\infty} \left\langle \widehat{\Psi}_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_{\mathbb{C}},$$

where $\widehat{\Psi}_\alpha^{(n)}$ satisfy, for $p, q \in \mathbb{N}$

$$\sum_{n=0}^{\infty} 2^{-nq} \left| \widehat{\Psi}_\alpha^{(n)} \right|_{-p}^2 < \infty.$$

On the other hand, for a given $\theta \in \mathcal{N}_{\mathbb{C}}$

$$S_\mu \Psi(\theta) = \sum_{n=0}^{\infty} \left\langle \Psi^{(n)}, \theta^{\otimes n} \right\rangle =: G(\theta)$$

and, consequently $G \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. But we can write

$$G(\theta) = G(\alpha \circ g_\alpha(\theta)) = \widehat{G}(g_\alpha(\theta)),$$

where $\widehat{G} = G \circ \alpha$, with $G \circ \alpha \in \mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. Therefore

$$\widehat{G}(g_\alpha(\theta)) = \sum_{n=0}^{\infty} \left\langle \widehat{G}_\alpha^{(n)}, g_\alpha(\theta)^{\otimes n} \right\rangle,$$

where the coefficients $\widehat{G}_\alpha^{(n)}$ verify

$$\sum_{n=0}^{\infty} 2^{-nq} \left| \widehat{G}_\alpha^{(n)} \right|_{-p}^2 < \infty.$$

Therefore with $\widehat{\Psi}_\alpha^{(n)} = \widehat{G}_\alpha^{(n)}$ follows the result, i.e., $\Psi \in (\mathcal{N})_{\mu,\alpha}^{-1}$. ■

8 The Wick product

Here we give the natural generalization of the **Wick multiplication** in the present setting.

Definition 8.1 *Let $\Phi, \Psi \in (\mathcal{N})_\mu^{-1}$. Then we define the* **Wick product** $\Phi \diamond \Psi$ *by*

$$S_\mu (\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi \,.$$

This is well defined because $\mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ is an algebra and thus by characterization theorem there exists an element in $(\mathcal{N})_\mu^{-1}$ $\Phi \diamond \Psi$ such that $S_\mu(\Phi \diamond \Psi) = S_\mu\Phi \cdot S_\mu\Psi$.

From this it follows

$$Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right) \diamond Q_m^{\mu,\alpha}\left(\Psi_\alpha^{(m)}\right) = Q_{n+m}^{\mu,\alpha}\left(\Phi_\alpha^{(n)} \widehat{\otimes} \Psi_\alpha^{(m)}\right),$$

$\Phi_\alpha^{(n)} \in \mathcal{N}_\mathbb{C}^{\prime\widehat{\otimes} n}$ and $\Psi_\alpha^{(m)} \in \mathcal{N}_\mathbb{C}^{\prime\widehat{\otimes} m}$. So in terms of $\mathbf{Q}^{\mu,\alpha}$-decomposition

$$\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}\left(\Phi_\alpha^{(n)}\right) \quad \text{and} \quad \Psi = \sum_{m=0}^{\infty} Q_m^{\mu,\alpha}\left(\Psi_\alpha^{(m)}\right)$$

the Wick product is given by

$$\Phi \diamond \Psi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha}\left(\Xi_\alpha^{(n)}\right),$$

where

$$\Xi_\alpha^{(n)} = \sum_{k=0}^{n} \Phi_\alpha^{(k)} \widehat{\otimes} \Psi_\alpha^{(n-k)}.$$

This allows for a concrete norm estimate.

Proposition 8.2 *The Wick product is continuous on $(\mathcal{N})_\mu^{-1}$. In particular the following estimate holds for $\Phi \in (\mathcal{H}_{-p_1})^{-1}_{-q_1,\mu,\alpha}$, $\Psi \in (\mathcal{H}_{-p_2})^{-1}_{-q_2,\mu,\alpha}$ and $p = \max(p_1, p_2)$, $q = q_1 + q_2 + 1$*

$$\|\Phi \diamond \Psi\|_{-p,-q,\mu,\alpha} \le \|\Phi\|_{-p_1,-q_1,\mu,\alpha} \|\Psi\|_{-p_2,-q_2,\mu,\alpha} \,.$$

Proof. We can estimate as follows

$$\begin{aligned}
\|\Phi\diamond\Psi\|^2_{-p,-q,\mu,\alpha} &= \sum_{n=0}^{\infty} 2^{-nq}\left|\Xi_\alpha^{(n)}\right|^2_{-p}\\
&= \sum_{n=0}^{\infty} 2^{-nq}\left(\sum_{k=0}^{n}\left|\Phi_\alpha^{(k)}\right|_{-p}\left|\Psi_\alpha^{(n-k)}\right|_{-p}\right)^2\\
&\le \sum_{n=0}^{\infty} 2^{-nq}(n+1)\sum_{k=0}^{n}\left|\Phi_\alpha^{(k)}\right|^2_{-p}\left|\Psi_\alpha^{(n-k)}\right|^2_{-p}\\
&\le \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^{-nq_1}\left|\Phi_\alpha^{(k)}\right|^2_{-p} 2^{-nq_2}\left|\Psi_\alpha^{(n-k)}\right|^2_{-p}\\
&\le \left(\sum_{n=0}^{\infty} 2^{-nq_1}\left|\Phi_\alpha^{(n)}\right|^2_{-p}\right)\left(\sum_{n=0}^{\infty} 2^{-nq_2}\left|\Psi_\alpha^{(n)}\right|^2_{-p}\right)\\
&= \|\Phi\|^2_{-p_1,-q_1,\mu,\alpha}\|\Psi\|^2_{-p_2,-q_2,\mu,\alpha}.
\end{aligned}$$

■

Similar to the Gaussian case the special properties of the space $(\mathcal{N})^{-1}_\mu$ allow the definition of **Wick analytic functions** under very general assumptions. This has proven to be of some relevance to solve equations e.g., of the type $\Phi\diamond X=\Psi$ for $X\in(\mathcal{N})^{-1}_\mu$. See [KLS96] for the Gaussian case.

Proposition 8.3 *For any $n\in\mathbb{N}$ and any α as in Subsection 5.1 we have $Q_n^{\mu,\alpha}=(Q_1^{\mu,\alpha})^{\diamond n}$.*

Proof. Let $\Phi^{(1)}\in\mathcal{N}'_{\mathbb{C}}$ be given. Thus, if $\theta\in\mathcal{N}_{\mathbb{C}}$, follows

$$\begin{aligned}
S_\mu\left[\left(Q_1^{\mu,\alpha}\left(\Phi^{(1)}\right)\right)^{\diamond n}\right](\theta) &= \left\langle\Phi^{(1)},g_\alpha(\theta)\right\rangle^n\\
&= \left\langle\left(\Phi^{(1)}\right)^{\widehat{\otimes}n},(g_\alpha(\theta))^{\otimes n}\right\rangle\\
&= S_\mu\left[Q_n^{\mu,\alpha}\left(\left(\Phi^{(1)}\right)^{\widehat{\otimes}n}\right)\right](\theta).
\end{aligned}$$

■

Theorem 8.4 *Let $F:\mathbb{C}\to\mathbb{C}$ be analytic in a neighborhood of the point $z_0=\mathbb{E}(\Phi)$, $\Phi\in(\mathcal{N})^{-1}_\mu$. Then $F^\diamond(\Phi)$ defined by $S_\mu(F^\diamond(\Phi))=F(S_\mu\Phi)$ exists in $(\mathcal{N})^{-1}_\mu$.*

Proof. By Theorems 7.3 and 4.9 we have $S_\mu\Phi \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. Then $F(S_\mu\Phi) \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$ since the composition of two analytic functions is also analytic. Again by the above mentioned theorems we find that $F^\diamond(\Phi)$ exists in $(\mathcal{N})^{-1}_\mu$. ■

Remark 8.5 *If $F(z)$ have the following representation*

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then the Wick series

$$\sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\diamond n}$$

(where $\Psi^{\diamond n} = \Psi \diamond \cdots \diamond \Psi$ n-times) converges in $(\mathcal{N})^{-1}_\mu$ and

$$F^\diamond(\Phi) = \sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\diamond n}$$

holds.

Example 8.6 *The above mentioned equation $\Phi \diamond X = \Psi$ can be solved if $\mathbb{E}_\mu(\Phi) = S_\mu\Phi(0) \neq 0$. That implies $(S_\mu\Phi)^{-1} \in \mathrm{Hol}_0(\mathcal{N}_\mathbb{C})$. Thus*

$$\Phi^{\diamond(-1)} = S^{-1}_\mu((S_\mu\Phi)^{-1}) \in (\mathcal{N})^{-1}_\mu.$$

Then $X = \Phi^{\diamond(-1)} \diamond \Psi$ is the solution in $(\mathcal{N})^{-1}_\mu$. For more instructive examples we refer the reader to Section 5 of [KLS96].

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