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GENERALIZED ASYMPTOTIC EXPANSIONS OF CORNISH-FISHER TYPE

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1. Introduction and summary. Let $\{F_n(x)\}$ be a sequence of distribution functions depending on a parameter n, and converging to a limiting distribution $\Phi(x)$ as n increases. Then a generalized expansion of Cornish-Fisher type is an asymptotic relation between the quantiles of F_n and Φ. The original Cornish-Fisher formulae [3], [5] provided leading terms of these expansions in the case of normal Φ, expressing a normal deviate in terms of the corresponding quantile of F_n and its cumulants (the "normalizing" expansion) and, conversely, the quantiles of F_n in terms of its cumulants and the corresponding quantiles of Φ (the "inverse" expansion). The value of both these asymptotic formulae has been well illustrated by their use in approximating the quantiles of complicated distributions (Johnson and Welch [9], Fisher [4], Goldberg and Levine [6]), and for obtaining random quantiles for distribution sampling applications (Teichroew [13], Bol'shev [2]). For a survey of the literature on Cornish-Fisher expansions, and some discussion of their validity, see Wallace ([14], Section 4).

In Sections 2, 3 of the present paper, formal expansions are obtained which generalize the Cornish-Fisher relations to arbitrary analytic Φ. Essentially, these expansions provide algorithms for transforming an asymptotic expansion of F_n in terms of the "standard" distribution Φ into asymptotic relations between the quantiles of these distributions. The "standardizing" expansion of the quantile u of Φ in terms of the corresponding quantile x of F_n is expressed (Section 2) in terms of a sequence of functions defined by a differential recurrence operator. A similar differential operator appears in the generalized "inverse" expansion for x in terms of u (Section 3), which arises from the application of Lagrange's inversion formula to the equation of quantiles. An asymptotic expansion for quantiles of the Wilks likelihood ratio criterion is given as an example.

Formal expansions in terms of the cumulants of F_n and Φ are obtained in Section 4 by developing F_n about Φ as a Charlier differential series and collecting terms of like degree in the resulting exponential series. For known cumulants and for normal Φ these formal expressions reduce, as shown in Section 5, to a general form of the Cornish-Fisher expansions, in which the polynomial terms are represented as sums of products of Hermite polynomials. This representation is shown in Section 6 to account for some properties of the Cornish-Fisher polynomials.

2. The general standardizing expansion. If x and u are corresponding quantiles of F_n and Φ respectively, then

$$F_n(x) = \Phi(u) \tag{1}$$

and it is required to solve this equation for u in terms of x.

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The density function $\phi(u)$ of the distribution Φ will be assumed to be arbitrarily differentiable. Then writing

$$Z_n(x) = F_n(x) - \Phi(x), \tag{2}$$

it follows from (1) that

$$Z_n(x) = \int_x^u \phi(t)\, dt. \tag{3}$$

If the equation

$$\zeta = \int_x^u \phi(t)\, dt \tag{4}$$

is regarded as defining a function $u(\zeta)$ with $u(0) = x$, then $u(\zeta)$ may be developed in a formal Taylor series about $\zeta = 0$.

Differentiation of (4) yields

$$du/d\zeta = [\phi(u)]^{-1}. \tag{5}$$

Now writing

$$D_u \equiv d/du, \qquad \psi(u) = -\phi'(u)/\phi(u) = D_u \log (1/\phi(u)); \tag{6}$$

it is found by induction that

$$d^r u/d\zeta^r = c_r(u)[\phi(u)]^{-r}, \tag{7}$$

where the c_r are defined recursively by

$$c_1(u) \equiv 1, \qquad c_{r+1}(u) = (r\psi(u) + D_u)c_r(u), \qquad (r = 1, 2, \cdots). \tag{8}$$

Since $u = x$ when $\zeta = 0$, the Taylor series is seen to be

$$u(\zeta) = x + \textstyle\sum_{r=1}^{\infty} c_r(x)(\zeta/\phi(x))^r/r!; \tag{9}$$

and applying this result to (3), the general standardizing expansion is obtained in the form

$$u = x + \textstyle\sum_{r=1}^{\infty} c_r(x)(Z_n(x)/\phi(x))^r/r!. \tag{10}$$

In many applications $F_n(x)$ is known to have an asymptotic expansion of the form

$$\begin{aligned} F_n(x) &= \Phi(x) + \phi(x)[n^{-1}p_1(x) + n^{-2}p_2(x) + \cdots] \\ &= \Phi(x) + \phi(x)z_n(x), \end{aligned} \tag{11}$$

say, where the $p_r(x)$ may be polynomials in x. In terms of $z_n(x)$, (10) becomes

$$u = x + \textstyle\sum_{r=1}^{\infty} c_r(x)(z_n(x))^r/r!, \tag{12}$$

which expresses the quantile u directly as a series in terms of x whose rth term is $O(n^{-r})$.

When the limiting distribution, $\Phi(x)$, is the unit normal distribution,

$$\psi(x) = D_x \log ((2\pi)^{\frac{1}{2}}e^{x^2/2}) = x, \tag{13}$$

and $c_r(x)$ is an $(r - 1)$th degree polynomial in x:

$$c_1(x) \equiv 1, \quad c_2(x) = x, \quad c_3(x) = 2x^2 + 1, \quad c_4(x) = 6x^3 + 7x, \cdots. \tag{14}$$

In this case, (12) is essentially the Cornish-Fisher normalizing expansion, which will be considered in more detail in Sections 5, 6.

For other applications, however, the appropriate limiting function Φ may be the distribution of χ^2 with ν degrees of freedom. Then

$$\psi(x) = \tfrac{1}{2} - (\tfrac{1}{2}\nu - 1)x^{-1}, \tag{15}$$

and $c_r(x)$ is an $(r - 1)$th degree polynomial in x^{-1}.

3. The general inverse expansion. The solution of (1) for x in terms of u could be obtained from (10) or (12) by inverting the series, which suggests the application of Lagrange's inversion formula. This formula provides under certain conditions (see [15], p. 133) that, if γ and θ are analytic functions and

$$w = v + \gamma(w), \tag{16}$$

then

$$\theta(w) = \theta(v) + \textstyle\sum_{r=1}^{\infty} D_v^{\,r-1}[\theta'(v)(\gamma(v))^r]/r!. \tag{17}$$

Cornish and Fisher ([3], p. 316) in effect used the early terms of the Lagrange formula when inverting their normalizing expansion. Riordan [11] applied (17) with $\theta(w) \equiv w$ to derive general relations between the polynomials occurring in the two expansions.

Although the solution can be established by inverting the series (10) or (12), it appears more instructive to apply the Lagrange formula directly to equation (1), rewritten in the form

$$\Phi(x) = \Phi(u) - Z_n(x). \tag{18}$$

If new variables v and w are defined by

$$v = \Phi(u), \qquad w = \Phi(x), \tag{19}$$

then equivalently

$$w = v - Z_n(\Phi^{-1}(w)) \tag{20}$$

where Φ^{-1} denotes the inverse function of Φ. Since (20) is of the form (16), this functional equation can be solved for $\Phi^{-1}(w) = x$ by taking $\theta \equiv \Phi^{-1}$ in (17):

$$\Phi^{-1}(w) = \Phi^{-1}(v) + \textstyle\sum_{r=1}^{\infty} (-1)^r(r!)^{-1}D_v^{\,r-1}\{[Z_n(\Phi^{-1}(v))]^r/\phi(\Phi^{-1}(v))\}, \tag{21}$$

or, on reverting to the original variables:

$$x = u - \textstyle\sum_{r=1}^{\infty} (r!)^{-1}(-[\phi(u)]^{-1}D_u)^{r-1}[(Z_n(u))^r/\phi(u)]. \tag{22}$$

In cases where Z_n is a multiple of ϕ as in (11), (22) takes the form

$$x = u - \textstyle\sum_{r=1}^{\infty} D_{(r)}(z_n(u))^r/r!, \tag{23}$$

where $D_{(1)}$ denotes the identity operator and

$$D_{(r)} = (\psi(u) - D_u)(2\psi(u) - D_u) \cdots ((r-1)\psi(u) - D_u), \quad (r = 2, 3, \cdots). \tag{24}$$

As in the case of the standardizing expansion the rth term in the general inverse expansion (23) is seen to be $O(n^{-r})$.

EXAMPLE 1. *For normal* Φ, $\psi(u) = u$, and if z_n is expressed in terms of cumulants, (23) becomes the Cornish-Fisher inverse expansion as shown in the following sections.

EXAMPLE 2. *The Wilks likelihood ratio criterion.* Let **X**, **Y** be $p \times m$ $(m > p)$ and $p \times q$ matrices respectively with the joint probability density function

$$(25)\quad (2\pi)^{-\frac{1}{2}p(m+q)}|\mathbf{\Sigma}|^{-\frac{1}{2}(m+q)} \exp\{-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1}[\mathbf{XX}' + (\mathbf{Y} - \boldsymbol{\mu})(\mathbf{Y} - \boldsymbol{\mu})']\}$$

where $\mathbf{\Sigma}$ is a $p \times p$ positive definite matrix. Then the likelihood ratio criterion for testing the hypothesis $\boldsymbol{\mu} = 0$ is

$$(26)\qquad \Lambda = \det(\mathbf{XX}')/\det(\mathbf{XX}' + \mathbf{YY}').$$

Let

$$(27)\qquad n = m - \tfrac{1}{2}(p - q + 1).$$

Then $-n \log \Lambda$ is asymptotically distributed as χ^2 with $\nu = pq$ degrees of freedom, and Rao ([10], see also [1] Theorem 8.6.2) has developed the distribution function in an asymptotic expansion of the form (11) to order n^{-4}. Let x and u be corresponding quantiles for $-n \log \Lambda$ and χ_ν^2. Schatzoff [12] has tabulated exact values of the correction factor x/u for $q = 4, 6, 8, 10$ and p such that $pq \leqq 70$. Applying the first three terms of (23) to Rao's expansion with ψ given by (15), it is found that

$$(28)\quad \begin{aligned} x/u \sim 1 &+ n^{-2}2\gamma_2(u + (\nu + 2))[\nu(\nu + 2)]^{-1} \\ &+ n^{-4}\{2\gamma_4[u^3 + (\nu + 6)u^2 + (\nu + 4)(\nu + 6)u \\ &+ (\nu + 2)(\nu + 4)(\nu + 6)][\nu(\nu + 2)(\nu + 4)(\nu + 6)]^{-1} \\ &- \gamma_2^2[u^3 + (\nu - 2)u^2 + (\nu + 2)(\nu - 6)u + (\nu - 2)(\nu + 2)^2 \\ &\quad \cdot[\nu^2(\nu + 2)^2]^{-1}\} \\ &+ O(n^{-6}), \end{aligned}$$

where

$$(29)\quad \begin{aligned} \gamma_2 &= pq(48)^{-1}(p^2 + q^2 - 5), \\ \gamma_4 &= \tfrac{1}{2}\gamma_2^2 + pq(1920)^{-1}[3p^4 + 3q^4 + 10p^2q^2 - 50(p^2 + q^2) + 159]. \end{aligned}$$

At the upper 5% and 1% levels, formula (28) gives results agreeing with those of Schatzoff to within 0.1% for n ranging from $n = 4$ for $p = 3$, $q = 4$ to $n = 10$ for $p = 7$, $q = 10$. Presumably, similar accuracy would apply for q odd.

EXAMPLE 3. *Hotelling's generalized* T_0^2. Similarly, Ito's expansion of Hotelling's generalized T_0^2 statistic in terms of χ^2 quantiles ([8] equation (3.33)) may be obtained directly from his expansion of the cumulative distribution function ([8] equation (4.3)), by applying the first three terms of (23) with ψ defined by

(15). This expansion includes the case of the F-ratio, which is asymptotic to χ^2 when the number of degrees of freedom in the denominator is large.

4. Expansion in terms of cumulants. If the cumulants of F_n and Φ are $\{\kappa_r\}$ and $\{\gamma_r\}$, respectively, then ([14], Section 3) F_n may be formally expanded about Φ in the Charlier differential series

$$F_n(x) = \exp [\textstyle\sum_{r=1}^{\infty} \lambda_r(-D_x)^r/r!]\Phi(x), \tag{30}$$

where

$$\lambda_r = \kappa_r - \gamma_r . \tag{31}$$

In developing the exponential series of (30), it is convenient to denote by π a partition of the positive integer m into l positive integers:

$$\pi = [s_1^{\rho_1}, \cdots, s_k^{\rho_k}], \qquad m = \textstyle\sum_{i=1}^k \rho_i s_i , \qquad l = \textstyle\sum_{i=1}^k \rho_i , \tag{32}$$

and let $a(\pi)$ denote the elementary partition function:

$$a(\pi) = m! [(s_1 !)^{\rho_1} \cdots (s_k !)^{\rho_k} \rho_1 ! \cdots \rho_k !)]^{-1}. \tag{33}$$

Then defining

$$\lambda_\pi = a(\pi)\lambda_{s_1}^{\rho_1} \cdots \lambda_{s_k}^{\rho_k}/m! , \tag{34}$$

terms of like degree in the exponential series may be collected:

$$F_n(x) = \Phi(x) - \phi(x) \textstyle\sum_\pi \lambda_\pi\{[\phi(x)]^{-1}(-D_x)^{m-1}\phi(x)\}, \tag{35}$$

where the summation is extended over all partitions π of all positive integers.

The functions

$$\psi_m(x) = [\phi(x)]^{-1}(-D_x)^{m-1}\phi(x) \tag{36}$$

satisfy the recurrence relation

$$\psi_m(x) = (\psi(x) - D_x)\psi_{m-1}(x), \tag{37}$$

where, (see also (6)),

$$\psi_1(x) \equiv 1, \qquad \psi_2(x) = \psi(x) = -\phi'(x)/\phi(x) = D_x \log (1/\phi(x)). \tag{38}$$

From (11) and (35) a series in terms of cumulants is obtained:

$$z_n(x) = - \textstyle\sum_\pi \lambda_\pi \psi_m(x) \tag{39}$$

and a similar expression may be sought for $(z_n(x))^r/r!$

For (non-empty) partitions $\pi_1, \cdots, \pi_r$ of positive integers $m_1, \cdots, m_r$ having as their union the partition π of the integer m,

$$\pi = [s_1^{\rho_1}, \cdots, s_k^{\rho_k}], \qquad \pi_j = [s_i^{\rho_{ij}}, \cdots, s_k^{\rho_{kj}}], \qquad \rho_i = \textstyle\sum_{j=1}^r \rho_{ij}, \qquad (j = 1, \cdots, r); \tag{40}$$

we define the partition function

$$p(\pi_1, \cdots, \pi_r) = \textstyle\prod_{i=1}^k \binom{\rho_i}{\rho_{i1}, \cdots, \rho_{ir}}, \tag{41}$$

where $\binom{\rho_i}{\rho_{i1},\cdots,\rho_{ir}}$ denotes the multinomial coefficient. Then using the definition (33) of $a(\pi)$, it is found that

$$(42)\qquad (z_n(x))^r/r! = (-1)^r \textstyle\sum_\pi \lambda_\pi \psi_\pi^{(r)}(x),$$

where

$$(43)\qquad \psi_\pi^{(r)} = (r!)^{-1} \textstyle\sum_{\pi_1+\cdots+\pi_r=\pi} p(\pi_1, \cdots, \pi_r)\psi_{m_1} \cdots \psi_{m_r}, \quad (\psi_\pi^{(1)} = \psi_m).$$

The summation is extended over all distinct *arrangements* of π_i's having union π; i.e. if the π_i's are identical in groups of sizes $\sigma_1, \cdots, \sigma_R$ then $p(\pi_1, \cdots, \pi_r)\psi_{m_1} \cdots \psi_{m_r}$ occurs $\binom{r}{\sigma_1,\cdots,\sigma_R}$ times. It follows also that

$$(44)\qquad \psi_\pi^{(r)} = \textstyle\sum_{\pi_1\cup\cdots\cup\pi_r=\pi} q(\pi_1, \cdots, \pi_r)\psi_{m_1} \cdots \psi_{m_r},$$

where

$$(45)\qquad q(\pi_1, \cdots, \pi_r) = p(\pi_1, \cdots, \pi_r)/\sigma_1! \cdots \sigma_R!,$$

and the summation in (44) is extended over all distinct *combinations* of π_i's with union π.

Substituting (42) in (12), the *standardizing expansion* takes the form

$$(46)\qquad u = x + \textstyle\sum_\pi \lambda_\pi \sum_{r=1}^{l} (-1)^r c_r(x)\psi_\pi^{(r)}(x),$$

and similarly the *inverse expansion* (23) becomes

$$(47)\qquad x = u - \textstyle\sum_\pi \lambda_\pi \sum_{r=1}^{l} (-1)^r D_{(r)}\psi_\pi^{(r)}(x).$$

These expansions relate the quantiles u and x in terms of cumulant differences and functions derived by application of differential operators of the type $r\psi \pm D$, where ψ is determined by the frequency function of the limiting distribution.

5. The Cornish-Fisher expansions. In the case of Cornish-Fisher expansions, $\Phi(x)$ is the unit normal distribution and

$$(48)\qquad \lambda_2 = \kappa_2 - 1, \qquad \lambda_i = \kappa_i \ (i \neq 2), \qquad \psi(x) = x,$$
$$\psi_r(x) = h_r(x) = e^{x^2/2}(-D_x)^{r-1}e^{-x^2/2}, \qquad (r = 1, 2, \cdots),$$

where h_r is Hermite's $(r-1)$th polynomial. The Cornish-Fisher *normalizing expansion* may be obtained from (46):

$$(49)\qquad u = x + \textstyle\sum_\pi \lambda_\pi N_\pi(x),$$

where the polynomials N_π are defined by:

$$(50)\qquad N_\pi = \textstyle\sum_{r=1}^{l} (-1)^r c_r h_\pi^{(r)},$$

and the polynomials c_r are to be derived from equations (8) and (13). Similarly (47) becomes the Cornish-Fisher *inverse expansion*:

$$(51)\qquad x = u + \textstyle\sum_\pi \lambda_\pi P_\pi(u),$$

where the polynomials P_π are given by:

$$(52)\qquad P_\pi = \textstyle\sum_{r=1}^{l} (-1)^{r-1} D_{(r)} h_\pi^{(r)},$$
$$D_{(r)} \equiv (u - D)(2u - D) \cdots ((r-1)u - D).$$

The Cornish-Fisher assumption that

$$(53)\quad \lambda_1 = O(n^{-\frac{1}{2}}), \qquad \lambda_2 = O(n^{-1}), \qquad \lambda_r = O(n^{-r/2+1}), \quad (r = 3, 4, \cdots),$$

leads to a classification of the λ_π into successive "adjustments": if $\lambda_\pi = O(n^{-M/2})$ then λ_π (and hence N_π and P_π) belongs to the Mth adjustment. The Cornish-Fisher polynomials, whose numeric coefficients were determined in the case of leading terms in the formulae [3], [5], are seen to involve sums of products of Hermite polynomials. Indeed, all components, including the Hermite polynomials, in the general forms of the Cornish-Fisher polynomials can be derived from $c_1 = h_1 = 1$ by application of differential operators of the type $nx \pm D$.

6. Properties of Cornish-Fisher polynomials. In the case $\pi = [s^m]$ it is easily shown from (44) that

$$(54)\qquad h_{[s^m]}^{(r)} = \Sigma a(\pi)(h_{t_1s})^{\rho_1} \cdots (h_{t_ks})^{\rho_k},$$

where the summation is extended over all partitions $\pi = [t_1^{\rho_1}, \cdots, t_k^{\rho_k}]$ of m into r parts. In particular,

$$(55)\qquad h_{[s^m]}^{(m)} = (h_s)^m, \qquad h_{[s^m]}^{(m-1)} = \binom{m}{2}(h_s)^{m-2}h_{2s}.$$

Next consider $h_{\pi,s}^{(r)}$, where π, s denotes the partition obtained by adjoining the singleton $[s]$ to the arbitrary partition π. From (43) it is seen that a given arrangement $(\pi_1, \cdots, \pi_r)$ is effectively given the multiplicity $p(\pi_1, \cdots, \pi_r)$ in the sum defining $h_\pi^{(r)}$. But (41) shows that $p(\pi_1, \cdots, \pi_r)$ is the number of ways of constructing this arrangement if each set of s_i's is considered as being composed of ρ_i distinct individuals. Hence,

$$(56)\quad h_{\pi,s}^{(r)} = h_s h_\pi^{(r-1)} + (r!)^{-1}\textstyle\sum_{\pi_1+\cdots+\pi_r=\pi} p(\pi_1, \cdots, \pi_r)$$
$$\cdot[h_{m_1+s}h_{m_2} \cdots h_{m_r} + \cdots + h_{m_1} \cdots h_{m_{r-1}}h_{m_r+s}],$$

where the term $h_s h_\pi^{(r-1)}$ corresponds to the sum over all partitions of π, s containing s as a singleton and the second term corresponds to all other partitions.

The well known relations for Hermite polynomials:

$$(57)\qquad (x - D_x)h_s = h_{s+1},$$

$$(58)\qquad h_{s+2} = xh_{s+1} - sh_s,$$

may now be generalized for the $h_\pi^{(r)}$ by means of (56):

$$(59)\qquad (rx - D_x)h_\pi^{(r)} = h_{\pi,1}^{(r)} - h_\pi^{(r-1)}, \qquad (h_\pi^{(0)} \equiv 0),$$

$$(60)\qquad h_{\pi,2}^{(r)} = xh_{\pi,1}^{(r)} - mh_\pi^{(r)}$$

for all partitions π of m.

By inspection of the expressions obtained by Cornish and Fisher for the leading adjustments in their expansions the polynomials may be observed to satisfy the following identities:

$$(61)\qquad -N_{[s]} = P_{[s]} = h_s,$$

for $[s]$ a singleton, while for all partitions π of m:

$$N_{\pi,1} = -D_x N_\pi ; \tag{62}$$

$$N_{\pi,2} = xN_{\pi,1} - mN_\pi = -x^{1-m}D_x(x^m N_\pi); \tag{63}$$

$$P_{\pi,1} \equiv 0; \tag{64}$$

$$P_{\pi,2} = -(m-1)P_\pi . \tag{65}$$

These identities follow from the expressions (50) and (52) of N and P in terms of symmetric sums of products of Hermite polynomials.

To prove (62), equation (59) and the defining relation (8) of the c_r may be used; clearly

$$\begin{aligned} D_x N_\pi &= \Sigma_r(-1)^r[-c_r(rx - D_x)h_\pi^{(r)} + h_\pi^{(r)}(rx + D_x)c_r] \\ &= \Sigma_r(-1)^r[-c_r h_{\pi,1}^{(r)} + c_r h_\pi^{(r-1)} + c_{r+1}h_\pi^{(r)}] \\ &= -N_{\pi,1} . \end{aligned} \tag{66}$$

Equation (63) is a trivial consequence of (60) and (62), while (64) follows immediately from (59) and (52). Lastly, in virtue of (59) and (60):

$$P_{\pi,2} = \Sigma_r(-1)^r[-D_{(r+1)}(xh_\pi^{(r)}) + D_{(r)}(xh_\pi^{(r-1)}) + (1-m)D_{(r)}h_\pi^{(r)}], \tag{67}$$

which implies (65).

The identities (62) and (64) reflect the fact that changes in the location parameter of x affect λ_1 only. Equivalent identities apply to symmetric sums associated with $\lambda_{\pi,1}$ in the general expansions (46) and (47), but the identities (63) and (65) for elements involving π, 2 arise from properties of Hermite polynomials and hold only in asymptotic expansions about normal Φ. In practice, terms involving λ_1 and λ_2 can be excluded by relating quantiles of u to quantiles of $x = (x' - \mu)/\sigma$, for which $\lambda_1 = \lambda_2 = 0$, and treating x as an intermediate variate, whose quantiles are linearly related to the quantiles of x'.

Since the Hermite polynomial h_s is an odd or even function according as $(s - 1)$ is odd or even, the parities of the polynomials $h_\pi^{(r)}$, P_π and N_π, where π is a partition of m, are those of the integers $(m - r)$, $(m - 1)$ and $(m - 1)$, respectively. The order of the adjustment to which λ_π belongs is clearly of the same parity as m. Hence, the polynomials P_π, N_π in odd order adjustments are even, whereas those in even order adjustments are odd.

These results are illustrated in the Table, which presents polynomials for the first three adjustments of the normalizing and inverse expansions. The first four adjustments listed by Cornish and Fisher ([3] pp. 316–317) and the first six adjustments of the inverse expansion listed by Fisher and Cornish [5] have been checked against the expressions presented in Section 5. Using these formulations, the first twelve normalizing and inverse adjustments have been tabulated [7] by means of a computer program, which used algorithms for generating the partitions (40) and partition functions (45) and for polynomial operations, including the application of the operator $nx \pm D$, arising in (8), (52) and (57).

TABLE 1

Cornish-Fisher Polynomials

M	π	λ_Π	$h_\pi^{(1)}$	$h_\pi^{(2)}$	$h_\pi^{(3)}$	n_Π	p_Π
1	[1]	λ_1	1	0	0	-1	1
	[3]	$\lambda_3/6$	h_3	0	0	$-(x^2-1)$	x^2-1
2	[2]	$\lambda_2/2$	h_2	0	0	$-x$	x
	[4]	$\lambda_4/24$	h_4	0	0	$-(x^3-3x)$	x^3-3x
	[1^2]	$\lambda_1^2/2$	h_2	1	0	0	0
	[1, 3]	$\lambda_1\lambda_3/6$	h_4	x^2-1	0	$2x$	0
	[3^2]	$\lambda_3^2/72$	h_6	$(x^2-1)^2$	0	$2(4x^3-7x)$	$-2(2x^3-5x)$
3	[5]	$\lambda_5/120$	h_5	0	0	$-(x^4-6x^2+3)$	x^4-6x^2+3
	[1, 2]	$\lambda_1\lambda_2/2$	h_3	x	0	1	0
	[1, 4]	$\lambda_1\lambda_4/24$	h_5	x^3-3x	0	$3(x^2-1)$	0
	[2, 3]	$\lambda_2\lambda_3/12$	h_5	x^3-x	0	$5x^2-3$	$-2(x^2-1)$
	[3, 4]	$\lambda_3\lambda_4/144$	h_6	x^5-4x^3+3x	0	$11x^4-42x^2+15$	$-(6x^4-5x^2+2)$
	[1^2, 3]	$\lambda_1^2\lambda_3/12$	h_5	$3x^3-7x$	x^2-1	-2	0
	[1, 3^2]	$\lambda_1\lambda_3^2/72$	h_7	$3x^5-18x^3+21x$	$(x^2-1)^2$	$-2(12x^2-7)$	0
	[3^3]	$\lambda_3^3/1296$	h_9	$3(x^7-11x^5+25x^3-15x)$	$(x^2-1)^3$	$-2(69x^4-187x^2+52)$	$4(12x^4-53x^2+17)$

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