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GENERALIZED BERNSTEIN-CHLODOWSKY POLYNOMIALS

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ABSTRACT. For given positive integers n and m, the following generalization of Bernstein-Chlodowsky polynomials is studied,

$$B_{n,m}(f,x) = (1+(m-1)\frac{x}{b_n}) \sum_{k=0}^{[n/m]} f\left(\frac{b_n k}{n-(m-1)k}\right) \cdot C^k_{n-(m-1)k}\left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-mk},$$

where b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty}(b_n/n) = 0$, $0 \le x \le b_n$ and $[p]$, as usual, denotes the greatest integer less than p. A theorem about convergence of $B_{n,m}(f,x)$ to $f(x)$ as $n \to \infty$ in weighted space of functions f continuous on positive semiaxis and satisfying the condition $\lim_{x\to\infty}(f(x)/(1+x^2)) = K_f < \infty$ is established.

1. Let $\rho(x) = 1+x^2$, $-\infty < x < \infty$ and B_ρ be the set of all functions f defined on the real axis and satisfying the condition $|f(x)| \le M_f \rho(x)$ with some constant M_f, depending only on f. By C_ρ we denote the subspace of all continuous functions belonging to B_ρ. Obviously, we may convert C_ρ and B_ρ into normed linear space by introducing the following ρ-norm

$$\|f\|_\rho = \sup_x \frac{|f(x)|}{\rho(x)}.$$

Also, let C^0_ρ be the subspace of all functions $f \in C_\rho$ for which $\lim_{|x|\to\infty}(f(x)/\rho(x))$ exists finitely.

The properties of linear positive operators acting from C_ρ to B_ρ and the Korovkin type theorems for them have been studied by the first author who has established the following basic theorem, see [**3, 4**].

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Theorem A. *Let the sequence of linear positive operators L_n, acting from C_ρ to B_ρ, satisfy the conditions*

$$\lim_{n\to\infty} \|L_n(t^\nu, x) - x^\nu\|_\rho = 0, \quad \nu = 0, 1, 2.$$

Then, for any function $f \in C_\rho^0$,

$$\lim_{n\to\infty} \|L_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^0$ such that

$$\lim_{n\to\infty} \|L_n f^* - f^*\|_\rho \geq 1.$$

Note that, in [**4**], the theorems of this type are studied for more general functions $\rho(x)$.

Now let α_n be the sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = \infty$ and

$$\|f\|_{\rho,[0,\alpha_n]} = \sup_{0\leq x\leq \alpha_n} \frac{|f(x)|}{\rho(x)}.$$

Also let T_n be a sequence of linear positive operators such that

$$\lim_{n\to\infty} \|T_n(t^\nu, x) - x^\nu\|_{\rho,[0,\alpha_n]} = 0, \quad \nu = 0, 1, 2.$$

Then applying Theorem A to operators

$$L_n(f, x) = \begin{cases} T_n(f, x) & \text{if } 0 \leq x \leq \alpha_n \\ f(x) & \text{if } x > \alpha_n, \end{cases}$$

we obtain

Theorem B. *Let the sequence of linear positive operators L_n, acting from C_ρ to B_ρ, satisfy the conditions*

$$\lim_{n\to\infty} \|L_n(t^\nu, x) - x^\nu\|_{\rho,[0,\alpha_n]} = 0, \quad \nu = 0, 1, 2.$$

Then for any function $f \in C_\rho^0$,

$$\lim_{n\to\infty} \|L_n f - f\|_{\rho,[0,\alpha_n]} = 0.$$

We propose the construction of a sequence of linear positive operators of Bernstein-Chlodowsky type, see [**8**], and study its convergence in ρ-norm using Theorem B.

There are many original constructions of sequences of linear positive operators, satisfying the conditions of Korovkin's theorem and therefore converging to continuous functions on a finite interval. One of this type of general constructions of operators was given in [**5**][1] and some new properties of these operators were then investigated in [**9**]. The approximation problem on unbounded sets has also been investigated by many authors. But all of these results give the condition of convergence of special operators (Szasz or Bernstein type) on any finite interval of the real axis or at any fixed point, see, for example, [**6, 11, 7**].

As may be seen, Theorem B gives the conditions of convergence of sequences of linear positive operators in weighted spaces, when the interval of convergence grows as $n \to \infty$. The consideration of general constructions of operators, satisfying the conditions of Theorem A and B is interesting. Our construction of generalized Bernstein-Chlodowsky operators is an example of such types of operators. Moreover, this is a polynomial-type construction, approximating unbounded continuous functions on unbounded sets.

2. For given positive integers n and m, let

$$B_{n,m}(f,x) = \left(1+(m-1)\frac{x}{b_n}\right) \cdot \sum_{k=0}^{[n/m]} f\left(\frac{b_n k}{n-(m-1)k}\right) C_{n-(m-1)k}^{k} \left(\frac{x}{b_n}\right)^{k} \cdot \left(1-\frac{x}{b_n}\right)^{n-mk} \tag{1}$$

where b_n is the increasing sequence of real numbers such that

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0, \tag{2}$$

$0 \le x \le b_n$, and $[n/m]$, as usual, denotes the greatest integer less than n/m.

For $m = 1$, $B_{n,1}(f,x)$ is the well known Bernstein-Chlodowsky polynomial, see, for example, [**8**]. The generalization of this type for classical Bernstein polynomials is mentioned in [**2**].

First we study some properties of the polynomials $B_{n,m}(f,x)$. Let

$$A_{n,m}(x) = \frac{B_{n,m}(1,x)}{1+(m-1)(x/b_n)}. \tag{3}$$

Proposition 1. *For $n \geq m$ and any $x \in [0, b_n]$*

$$A_{n,m}(x) = \left(1 - \frac{x}{b_n}\right) A_{n-1,m}\left(\frac{xb_{n-1}}{b_n}\right) + \frac{x}{b_n} A_{n-m,m}\left(\frac{xb_{n-m}}{b_n}\right). \tag{4}$$

Proof. By the definition of $A_{n,m}(x)$ we obtain

$$\left(1 - \frac{x}{b_n}\right) A_{n-1,m}\left(\frac{xb_{n-1}}{b_n}\right) = \sum_{k=0}^{[(n-1)/m]} C^k_{n-1-(m-1)k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-mk},$$

and

$$\frac{x}{b_n} A_{n-m,m}\left(\frac{xb_{n-m}}{b_n}\right) = \sum_{k=1}^{[(n-m)/m]+1} C^{k-1}_{n-1-(m-1)k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-mk}.$$

Obviously, $[(n-m)/m] = [n/m] - 1$,

$$\left[\frac{n-1}{m}\right] = \begin{cases} [n/m] - 1 & \text{if } n/m \text{ is integer} \\ [n/m] & \text{if } n/m \text{ is fractional,} \end{cases}$$

and $C^k_{n-1-(m-1)k} + C^{k-1}_{n-1-(m-1)k} = C^k_{n-(m-1)k}$.

From this (4) follows after simple calculations. □

Proposition 2. *Let*

$$K_{n,m}(x) = A_{n,m}(x) + \frac{x}{b_n}\sum_{k=1}^{m-1} A_{n-k,m}\left(\frac{xb_{n-k}}{b_n}\right). \tag{5}$$

Then $K_{n,m}(x) \equiv 1$ *for any* $x \in [0, b_n]$.

Proof. By the induction method. It is easy to see that due to (3) and (5),

$$K_{2,2}(x) = A_{2,2}(x) + \frac{x}{b_2}A_{1,2}\left(\frac{xb_1}{b_2}\right) \equiv 1.$$

Let, for any $x \in [0, b_n]$ and any $m > 2$, $K_{m,m}(x) \equiv 1$.

Then since $b_m \le b_{m+1}$ we have

$$K_{m,m}\left(\frac{xb_m}{b_{m+1}}\right) \equiv 1. \tag{6}$$

It follows from (3) that for $x \in [0, b_n]$,

$$A_{m,m}\left(\frac{xb_m}{b_{m+1}}\right) = \left(1 - \frac{x}{b_{m+1}}\right)^m + \frac{x}{b_{m+1}}, \tag{7}$$

and

$$A_{m-k,m}\left(\frac{xb_{m-k}}{b_m}\right) = \left(1 - \frac{x}{b_m}\right)^{m-k}, \quad k = 1, 2, \dots, m. \tag{8}$$

Now by (5), (4), (6), (7) and (8) we obtain

$$\begin{aligned} K_{m+1,m+1}(x) &= A_{m,m+1}\left(\frac{xb_m}{b_{m+1}}\right) + \frac{x}{b_{m+1}}A_{0,m+1}\left(\frac{xb_0}{b_{m+1}}\right) \\ &\quad + \frac{x}{b_{m+1}}\sum_{k=1}^{m-1} A_{m-k,m+1}\left(\frac{xb_{m-k}}{b_{m+1}}\right) \\ &= \left(1 - \frac{x}{b_{m+1}}\right)^m + \frac{x}{b_{m+1}}\sum_{k=1}^{m} A_{m-k,m+1}\left(\frac{xb_{m-k}}{b_{m+1}}\right) \\ &= \left(1 - \frac{x}{b_{m+1}}\right)^m + \frac{x}{b_{m+1}} + \frac{x}{b_{m+1}}\sum_{k=1}^{m-1}\left(1 - \frac{x}{b_{m+1}}\right)^{m-k} \\ &= K_{m,m}\left(\frac{xb_m}{b_{m+1}}\right) \equiv 1, \end{aligned}$$

and for $n = m \geq 2$ the proof is completed.

Now let in general $K_{n,m}(x) \equiv 1$ for some $n > m$ and $x \in [0, b_n]$. From (5) and (4)

$$K_{n+1,m}(x) = K_{n,m}\left(\frac{xb_n}{b_{n+1}}\right), \quad 0 \leq x \leq b_n,$$

and therefore since $b_n \leq b_{n+1}$ we have $K_{n+1,m}(x) \equiv 1$. This completes the proof. □

Proposition 3. *Let*

$$\Delta_{n,m}(x) = A_{n,m}(x) - A_{n-1,m}\left(\frac{xb_{n-1}}{b_n}\right). \tag{9}$$

Then for all $x \in [0, b_n]$ *and* $n > m > 1$, *the inequality*

$$|\Delta_{n,m}(x)| \leq \left\{(m-1)\frac{x}{b_n}\right\}^{[(n-m)/(m-1)]} \tag{10}$$

holds.

Proof. From (4) and (9),

$$\Delta_n(x) = \frac{x}{b_n}\left\{A_{n-1,m}\left(\frac{xb_{n-1}}{b_n}\right) - A_{n-m,m}\left(\frac{xb_{n-m}}{b_n}\right)\right\},$$

and consequently we have

$$\Delta_n(x) = \frac{x}{b_n}\sum_{k=1}^{m-1}\left[A_{n-k,m}\left(\frac{xb_{n-k}}{b_n}\right) - A_{n-k-1,m}\left(\frac{xb_{n-k-1}}{b_n}\right)\right]. \tag{11}$$

On the other hand, from (9) we obtain

$$\Delta_{n-k}(x) = A_{n-k,m}(x) - A_{n-k-1,m}\left(\frac{xb_{n-k-1}}{b_{n-k}}\right),$$

where $k < n - m$. This gives

$$\Delta_{n-k}\left(\frac{xb_{n-k}}{b_n}\right) = A_{n-k,m}\left(\frac{xb_{n-k}}{b_n}\right) - A_{n-k-1,m}\left(\frac{xb_{n-k-1}}{b_n}\right).$$

Using this formula in (11), we have

$$\Delta_n(x) = \frac{x}{b_n} \sum_{k=1}^{m-1} \Delta_{n-k}\left(\frac{xb_{n-k}}{b_n}\right). \tag{12}$$

Taking in (12) $n-1, n-2, \dots, n-m+1$ instead of n, we successively obtain that

$$\Delta_{n-k}(x) = \frac{x}{b_{n-k}} \sum_{l=1}^{m-1} \Delta_{n-k-l}\left(\frac{xb_{n-k-l}}{b_{n-k}}\right),$$

and

$$\Delta_{n-k}\left(\frac{xb_{n-k}}{b_n}\right) = \frac{x}{b_n} \sum_{l=1}^{m-1} \Delta_{n-k-l}\left(\frac{xb_{n-k-l}}{b_n}\right), \tag{13}$$

where $k = 1, 2, \dots, m-1$. Applying (13) in (12) p-times, where $p = [(n-m)/(m-1)]$, and taking into account that $|\Delta_n(x)| \le 1$ (since $A_{n,m}(x) \ge 0$ and (5) holds), we obtain (10). □

Corollary 1. *Let γ_n be an increasing sequence of positive numbers and $\gamma_n < b_n/(m-1)$, $m > 1$. Then*

$$A_{n-1,m}\left(\frac{xb_{n-1}}{b_n}\right) = A_{n,m}(x) + \alpha_n(x), \tag{14}$$

$$A_{n,m}(x) = \frac{1}{1 + (x/b_n)(m-1)} + \beta_n(x), \tag{15}$$

where $\alpha_n(x)$ and $\beta_n(x)$ uniformly converge to zero as $n \to \infty$ on the closed interval $[0, \gamma_n]$.

Proof. Equation (14) follows from (10). Because of (14), the equality (5) may be written (for a large n) in the form

$$A_{n,m}(x) + \frac{x}{b_n}(m-1)\{A_{n,m}(x) + \delta_n(x)\} \equiv 1,$$

where $\delta_n(x)$ uniformly converges to zero on $[0, \gamma_n]$. From this it follows that

$$A_{n,m}(x) = \frac{1}{1 + (x/b_n)(m-1)} - \frac{(x/b_n)(m-1)}{1 + (x/b_n)(m-1)}\delta_n(x),$$

which coincides with (15). □

3. In this part we give the following theorem about the convergence of a generalized Bernstein-Chlodovsky polynomials (1) in ρ-norm

Theorem 1. *For any function $f \in C_\rho^0$ the sequence $B_{n,m}f$ converges to f in ρ-norm, that is,*

$$\lim_{n\to\infty} \|B_{n,m}f - f\|_{\rho,[0,\gamma_n]} = 0,$$

where $m > 1$ and $\gamma_n < b_n/(m-1)$.

Proof. Using Theorem B we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty} \|B_{n,m}(t^\nu, x) - x^\nu\|_{\rho,[0,\gamma_n]} = 0, \quad \nu = 0, 1, 2. \tag{16}$$

By (3) and (15)

$$\begin{aligned}\|B_{n,m}(1, x) - 1\|_{\rho,[0,\gamma_n]} &= \sup_{0\le x\le\gamma_n} \frac{1 + (m-1)(x/b_n)}{1 + x^2}|\beta_n(x)| \\ &\le 2 \sup_{0\le x\le\gamma_n} |\beta_n(x)|,\end{aligned}$$

where $\beta_n(x)$ uniformly converges to zero on $[0, \gamma_n]$. Hence the first condition of (16) is fulfilled ($\nu = 0$). Now it is easy to see that

$$B_{n,m}(t, x) = \left(1 + (m-1)\frac{x}{b_n}\right) x A_{n-m,m}\left(\frac{x b_{n-m}}{b_n}\right),$$

and due to (14) for a large n,

$$B_{n,m}(t, x) = \left[1 + (m-1)\frac{x}{b_n}\right] x \left[A_{n,m}(x) + \alpha_n(x)\right].$$

Therefore, by (15) we have

$$\begin{aligned}\|B_{n,m}(t,x)-x\|_{\rho,[0,\gamma_n]} &= \sup_{0\le x\le\gamma_n}\frac{x}{\rho(x)}\left[1+(m-1)\frac{x}{b_n}\right][|\alpha_n(x)|+|\beta_n(x)|]\\ &\le 2\sup_{0\le x\le\gamma_n}[|\alpha_n(x)|+|\beta_n(x)|],\end{aligned}$$

and the condition (16) holds also for $\nu=1$.

To verify this condition for $\nu=2$, consider $B_{n,m}(t^2,x)$. Letting

$$p_{n,m}(x)=\sum_{k=1}^{[n/m]}\frac{1}{n-(m-1)k}C_{n-(m-1)k-1}^{k-1}\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-mk},$$

$$q_{n,m}(x)=\sum_{k=2}^{[n/m]}\frac{k-1}{n-(m-1)k}C_{n-(m-1)k-1}^{k-1}\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-mk},$$

we see that

$$B_{n,m}(t^2,x)=\left(1+(m-1)\frac{x}{b_n}\right)b_nx(p_{n,m}(x)+q_{n,m}(x)). \tag{17}$$

Since for $k\le[n/m]$

$$\frac{1}{n-(m-1)k}\le\frac{1}{n-(m-1)[n/m]}<\frac{1}{n-(m-1)(n/m)}=\frac{m}{n},$$

and $[(n-m)/m]=[n/m]-1$, we obtain

$$p_{n,m}(x)<\frac{m}{n}A_{n-m,m}\left(\frac{xb_{n-m}}{b_n}\right). \tag{18}$$

Also, since

$$\begin{aligned}\frac{k-1}{n-(m-1)k}C_{n-(m-1)k-1}^{k-1} &= \frac{n-(m-1)k-1}{n-(m-1)k}C_{n-(m-1)k-2}^{k-2}\\ &< C_{n-(m-1)k-2}^{k-2},\end{aligned}$$

we have

$$q_{n,m}(x)<\frac{x}{b_n}A_{n-2m,m}\left(\frac{xb_{n-2m}}{b_n}\right). \tag{19}$$

From (17), (18) and (19), in view of (14) and (15), we obtain that, for $x \in [0, \gamma_n]$,

$$
\begin{aligned}
B_{n,m}(t^2, x) &< \left(1 + (m-1)\frac{x}{b_n}\right) \\
&\quad \cdot b_n x \left\{ \frac{m}{n}[A_{n,m}(x) + \xi_n(x)] + \frac{x}{b_n}[A_{n,m}(x) + \eta_n(x)] \right\} \\
&= \left(1 + (m-1)\frac{x}{b_n}\right) b_n x \\
&\quad \cdot \left\{ \frac{m}{n}\left[\frac{1}{1 + (m-1)(x/b_n)} + \beta_n(x) + \xi_n(x)\right] \right. \\
&\qquad \left. + \frac{x}{b_n}\left[\frac{1}{1 + (m-1)(x/b_n)} + \beta_n(x) + \eta_n(x)\right] \right\} \\
&\le x\frac{b_n}{n}m + 2mx\frac{b_n}{n}(\beta_n(x) + \xi_n(x)) \\
&\quad + 2x^2(\beta_n(x) + \eta_n(x)) + x^2
\end{aligned}
$$

holds, where $\xi_n(x)$, $\eta_n(x)$, $\beta_n(x)$ tend to zero uniformly on $[0, \gamma_n]$ as $n \to \infty$.

From this, setting

$$\phi_{n,m}(x) = x\frac{b_n}{n}m + 2mx\frac{b_n}{n}(\beta_n(x) + \xi_n(x)) + 2x^2(\beta_n(x) + \eta_n(x)),$$

we can write

$$
\begin{aligned}
\|B_{n,m}(t^2, x) - x^2\|_{\rho,[0,\gamma_n]} &\le \sup_{0 \le x \le \alpha_n} \frac{\phi_{n,m}(x)}{\rho(x)} \\
&< \frac{b_n}{n}m + 2m\frac{b_n}{n} \sup_{0 \le x \le \alpha_n}(|\beta_n(x)| + |\xi_n(x)|) \\
&\quad + 2 \sup_{0 \le x \le \alpha_n}(|\beta_n(x)| + |\eta_n(x)|).
\end{aligned}
$$

This means that the condition (1) holds also for $\nu = 2$ and by Theorem B the proof is completed. □

Note. For a function $f(x) \equiv 1$ we have from (1)

$$B_{n,m}(1, x) = \left(1 + (m-1)\frac{x}{b_n}\right) \sum_{k=0}^{[n/m]} C_{n-(m-1)k}^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-mk}.$$

From this it is easy to see that, at the point $x = b_n$, $B_{n,m}(1, b_n) = m$ if n/m is an integer, $B_{n,m}(1, b_n) = 0$ if n/m is not an integer. Consequently, at $x = b_n$, $B_{n,m}(1, b_n)$ diverges.

ENDNOTES

[1]In translations from Russian, A.D. Gadjiev, A.D. Gadzhiev, A.D. Gadjiev, and A.D. Gadžiev are the same person.

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