GENERALIZED BLOB ALGEBRAS AND ALCOVE GEOMETRY

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Abstract

A sequence of finite-dimensional quotients of affine Hecke algebras is studied. Each element of the sequence is constructed so as to have a weight space labelling scheme for Specht/standard modules. As in the weight space formalism of algebraic Lie theory, there is an action of an affine reflection group on this weight space that fixes the set of labelling weights. A linkage principle is proved in each case. Further, it is shown that the simplest non-trivial example may essentially be identified with the blob algebra (a physically motivated quasihereditary algebra whose representation theory is very well understood by Lie-theory-like methods). An extended role is hence proposed for Soergel's tilting algorithm, away from its algebraic Lie theory underpinning, in determining the simple content of standard modules for these algebras. This role is explicitly verified in the blob algebra case. A tensor space representation of the blob algebra is constructed, as a candidate for a full tilting module (subsequently proven to be so in a paper by Martin and Ryom-Hansen), further evidencing the extended utility of Lie-theoretic methods. Possible generalisations of this representation to other elements of the sequence are discussed.

1. *Introduction*

Affine Hecke algebras are currently the subject of widespread interest in representation theory [1, 31]. It is customary to study them through certain families of quotient algebras, among which the usual choice is the cyclotomic Hecke algebras [2, 10]. These quotients, though finite-dimensional, are still complicated, and complete knowledge of their representation theory remains a significant challenge. Following our consideration of the *physical* context of affine Hecke algebras in the formalism of transfer matrix algebras in statistical mechanics [58], we introduce another class of quotient algebras that are smaller and more accessible. We show that a number of techniques and organisational schemes from algebraic Lie theory work in this context, such as weight spaces and linkage [36] (always) and Soergel's tilting module algorithm [64, 65] (in at least one highly non-trivial case), despite the fact that these algebras lie outside the known regime of applicability of these methods (that is, as underpinned by classical theory).

The blob algebra b_n is a two-parameter generalisation of the Temperley–Lieb algebra $T_n(q)$ that may be defined in terms of a basis of decorated Temperley–Lieb diagrams [55, 57] (hence its name). As with $T_n(q)$, there is also a definition by the generators

$\{e_-, U_1, U_2, \ldots, U_{n-1}\}$ and relations $e_- e_- = e_-$, $U_i U_i = -(q + q^{-1}) U_i$,

$$U_i U_{i\pm 1} U_i = U_i, \tag{1}$$

$$U_1 e_- U_1 = y_e U_1, \tag{2}$$

and generators commuting otherwise, where q and y_e are parameters. (Here, $T_n(q)$ is the subalgebra generated by the U_is.) The representation theory of this algebra is very well understood [15], and it is known to have several beautiful properties evocative of algebraic Lie theory. In this paper we consider b_n as an affine Hecke quotient, and our sequence of affine Hecke quotients is a sequence of generalisations of b_n, chosen so as to be likely to be amenable to the same high level of understanding. As indicated above, we show that this choice succeeds in a number of key areas, culminating in a linkage principle [36]. We further explore the blob algebra's connection to Lie theory, describing in particular an action on a suitable tensor space evocative of the tilting theory and Ringel duality of quantum groups [61, 22, 56]. (Indeed, it has now been shown [53] that this action endows this tensor space with the property of a full tilting module.)

1.1. *Background/overview of the paper*

Soergel has given a beautiful procedure [65, 64] for analysing tilting modules for quantum groups at roots of unity through parabolic Kazhdan–Lusztig polynomials. The procedure itself may be applied formally to an alcove geometry, *without* reference to representation theory. Hence it may be applied, in principle, in cases that are beyond the scope of Soergel's proof of representation-theoretic interpretation. It is interesting, then, to try to find algebras for which the resultant combinatorial data *has* a representation-theoretic interpretation, even though Soergel's proof is not applicable. The output of the usual procedure in type-A may be mapped by Ringel duality [23] to the content of projective modules for certain quotients of ordinary Hecke algebras. (There, it may be understood in terms of idempotent decompositions of 1; see [56].) This leads to a determination of decomposition numbers for standard modules of the Hecke algebras themselves. Here we consider generalising the implementation of the procedure on this Ringel dual side. We do this by constructing generalized Hecke algebra quotients that (mildly) generalize the usual role of alcove geometry.

One example where the formal procedure gives the correct decomposition numbers is the blob algebra b_n (a two-parameter *affine* Hecke algebra quotient); see [55, 57]. We demonstrate the procedure for this example in Section 1.2 below. There is a set of key properties of b_n (see Section 4.2), which it has in common with the ordinary Hecke algebra quotients (see Section 1.3), which may serve to explain the phenomenon. In this paper we discuss generalisations of b_n that also possess these properties.

To generalise b_n suitably, we first place it in the context of affine/cyclotomic Hecke and Ariki–Koike–Levy algebras [2, 10, 52] (although these are *not* themselves the generalisations that we require). The study of these algebras is interesting both abstractly and also since they are useful in studying solutions to the reflection equation in integrable statistical mechanics (see [52] for references). This parallels the role of ordinary Hecke algebras in solving the Yang–Baxter equation. In both cases the 'physical' representation theory focuses attention on specific quotients, and implies that decomposition number data should be organized in a certain specific way. In the ordinary Hecke case this is the 'Soergel' rather than the 'LLT' way [43]. (As complete data sets these are equivalent, but computationally they are not [56].) Thus, while the algebras that we shall construct have representation theory that is accessible in principle by LLT methods [1], this does not remove the need for

a generalized (dual) Soergel approach. In Section 1.3 we discuss quotients of affine Hecke algebras generalizing b_n which, like b_n, realize certain key ingredients of this approach – in particular, they possess a weight space.

In Section 3 we imbue this space with an affine reflection group action and an alcove structure, and verify a linkage principle [36] for each such algebra. This states that, given any two standard modules ($\Delta(\mu)$ and $\Delta(\nu)$, say), if there is a non-trivial homomorphism between any subquotient of $\Delta(\mu)$ and any subquotient of $\Delta(\nu)$, then the weights μ and ν lie in the same orbit of the appropriate affine reflection group.

Each generalized algebra is a quotient of a cyclotomic Hecke algebra by certain primitive and central idempotents (independent of n). In Section 4 we show how the simplest non-trivial such quotient may effectively be *identified* with the blob algebra. The rest of the paper discusses outstanding technical issues in showing the validity of the generalized Soergel procedure for the generalized algebras (the primitive and central idempotents of the relevant cyclotomic Hecke algebras are computed in a convenient form in Section 5). Such issues include:

- the absence of a diagram calculus (indeed, unlike in the blob case, we provide only a conjecture for a basis for each generalisation – see Section 5.3),
- quasi-heredity (if the algebra *is* quasi-hereditary, for example, the linkage principle implies that weights in different orbits of the appropriate affine reflection group are in different blocks), and
- a global limit (recollement between different n values [12, 58]).

One property of the ordinary Hecke quotients *not* possessed by these generalisations is the defining representation on 'tensor space', realizing Ringel duality [22] with a quantum group quotient (compare [3, 20, 63]). Such a faithful tensor space representation is not manifestly necessary for our purpose, but would be very useful in resolving the issues mentioned above. In Section 6 we address this problem, culminating in the construction of some intriguing new concrete representations of b_n, which are candidates. (The serendipitous constructions of a number of other interesting new representations of b_n and its generalisations are outlined in Sections 7 and 8.)

We shall argue that the representation theory of these algebras, while *containing* that of ordinary Lie-theoretic objects, is in a sense more simply described. Given that the representation theory of b_n is itself known for q a root of unity in arbitrary characteristic [15], the possibility that open questions in ordinary Lie representation theory may be accessible by this route makes these algebras particularly interesting objects for study.

1.2. *Alcove geometry and decomposition numbers for b_n*

There is a 'good' parameterisation [57, Section 2] of the blob algebra b_n by q and m, where $y_e = -[m-1]/[m]$ (see also Section 4.2). The decomposition numbers in the 'doubly critical' case (q a primitive lth root of 1; m an integer, $|m| < l$) in characteristic zero are determined in [57, Section 9] by algebraic methods. A formal application of Soergel's procedure to this case works as follows.

First recall, quite generally, that a Euclidean space with reflection hyperplanes removed has a set ($\mathfrak{A}$, say) of connected components, called *alcoves*; see [36, Chapter 6] and [9]. For s a reflection hyperplane and B an alcove, we denote by Bs the image of B in s. Each nonempty intersection of the closure of an alcove with a hyperplane is called a *wall* of the alcove (here we shall confuse each such wall with the hyperplane containing it). We make

no assumption about the relation of hyperplanes to the origin 0, except that the origin lies in the interior of an alcove called the *fundamental alcove* and denoted A^0. Define $|B|$ as the number of hyperplanes between B and 0 (so $|A^0| = 0$).

In algebraic Lie theory, one starts with a set of ordinary (non-affine) reflections generating the ordinary Weyl group. The (q)-group weight space is the underlying Euclidean space with its origin ρ-shifted [36]. In particular, even when an affine reflection is added, the origin is at a fixed position at the base of the dominant region. In our case there is effectively no *ordinary* Weyl group and no dominant region; that is to say, the placement of *all* hyperplanes is controlled by parameters of the algebra. Thus the weight space for the blob algebra, just as for sl_2, is the Euclidean space associated to the A_1 Coxeter system – that is, it is effectively $\mathbb{R}$ (see [57, Section 6]). Now, however, (compare sl_2) *all* integral weights are dominant; that is to say, simple modules may be indexed by $\mathbb{Z}$. (We shall explain this, in the context of our generalised construction, shortly.) In the b_n case, the reflection hyperplanes are just points, and those generating the affine Weyl group lie at $-m$ and $l - m$. (An alcove $B \in \mathfrak{A}$ is a connected component of $\mathbb{R}$ with the reflection points removed.) A reflection is 'upward' if $|B| < |Bs|$ (compare the usual sl_2 situation).

For each alcove A, one defines a map

$$n_A : \mathfrak{A} \to \mathbb{Z}[v],$$

where v is a formal parameter, as follows. (For simplicity, we ignore features of Soergel's procedure that do not arise in our case.) Firstly, $n_A(A) = 1$ and $n_A(B) \neq 0$ implies that $|B| \leqslant |A|$. Note that n_{A^0} is determined immediately by this, and proceed inductively on $>$. For each alcove A, there will be a wall s of A such that $|As| > |A|$. Each alcove B has one wall that is in the affine Weyl orbit of s, and we shall write $B.s$ for the image of B in that wall (thus $As = A.s$). Then, with n_A known, we set

$$n'_{As}(B.s) = \begin{cases} n_A(B) + v^{-1}n_A(B.s), & |B.s| > |B|, \\ v^{-1}n_A(B) + n_A(B.s), & |B.s| < |B|, \end{cases}$$

and we define n_{As} by

$$n_{As}(C) = n'_{As}(C) - \sum_{B : |B|<|As|} n'_{As}(B)|_{v=0}\, n_B(C).$$

(That this procedure is well defined is not trivial [64].)

Evaluating $n_A(B)|_{v=1}$, this construction is (formally) computing the standard module content of tilting modules in a Ringel dual algebra. (Any two weights that are in different affine Weyl orbits [36] are in different blocks [7, Chapter 1]. Thus each block intersects each alcove in at most one weight and, fixing a block, it is the modules with these weights that $n_A(B)|_{v=1}$ describes.) The corresponding data for b_n is, in effect, the standard module content of projective modules. By reading by column instead of by row, as it were, we convert this to the simple module content of standard modules [22]. (Truncation to a finite column interval, such as that pictured in the example which follows, corresponds to localisation to some finite n – see ingredient I2 below.)

As noted, this construct is entirely formal; however, the following proposition holds.

PROPOSITION 1.1. *For $\lambda \in A$ and $\mu \in B$, the b_n standard composition multiplicity*

$$[\Delta(\mu) : L(\lambda)] = \begin{cases} n_A(B)|_{v=1}, & \mu \text{ in the affine Weyl orbit of } \lambda, |\lambda| \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

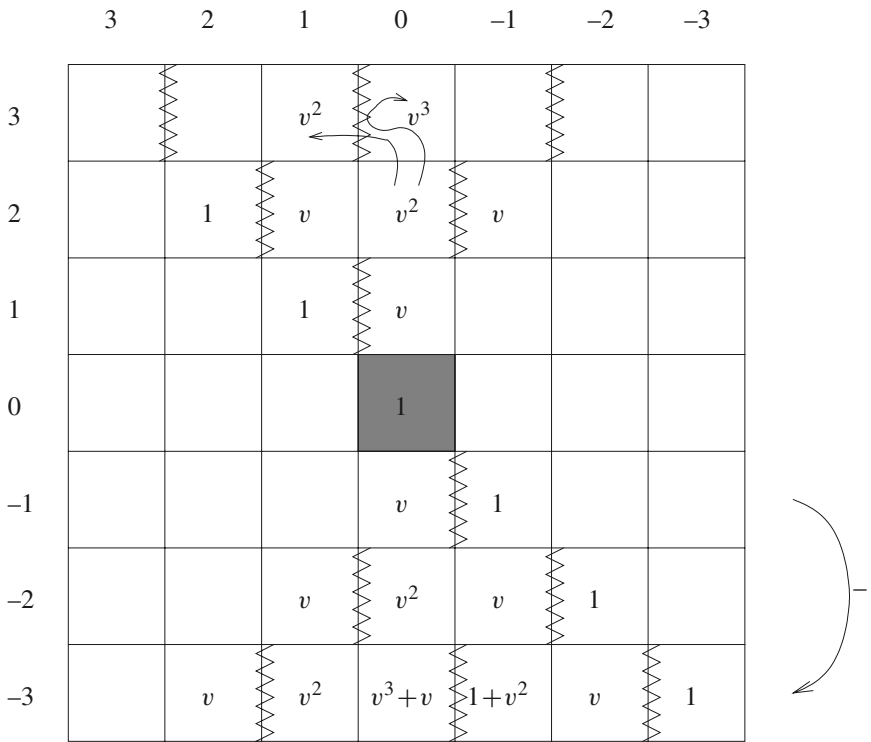*Further, the power of the formal parameter v determines the Loewy layer.*

Figure 1: Computation of the polynomials $n_A(B)$.

Proof. The composition multiplicity data is determined in [57, Section 9]. The computation of the polynomials $n_A(B)$ is illustrated in Figure 1. The row position in the table gives A, and the column position B; and $n_{A^0}(A^0)$ is shaded. (The rows and columns, and hence the alcoves, are labelled using $\mathbb{Z}$. These labels should not be confused with points in the underlying space $\mathbb{Z}$ (each alcove contains $l-1$ such points), or with weights.) The table is complete for the rows shown, except for the top and bottom rows. For the top row, the arrows within the table illustrate the contributions to the n'_{As} from a particular $n_A(B)$ (the shaded lines are the relevant walls for reflection in each case). The arrow outside the table illustrates a required subtraction to obtain n_A in the bottom row (which is complete except for this subtraction). The pattern is clear, and one sees immediately that the formal procedure reproduces the multiplicity and layer data. □

The case $b_n(q, m = 1)$ contains the ordinary Ringel dual, $\text{End}_{U_q sl_2}(V_2^{\otimes n})$, as a quotient. Note that the ordinary Soergel procedure is embedded accordingly in this version (in the 'dominant' region of case $m = 1$).

The 'idempotent splitting' analysis described in [56] applies in principle in this situation, giving a heuristic explanation of why Soergel's procedure is relevant here. Following this paradigm, there is a set of natural generalisations for which an analogous method should work.

The ingredients are (in precis; see also [56, 58]) as follows.

I1 A tower of unital algebras $A_n \subset A_{n+1}$ over a ring R with indeterminates, and a multiplicity-free [69] semisimple specialisation [13] (split, and we shall consider only characteristic 0 here).

For a tower as in I1, let B_{A_-} denote the Bratteli diagram of the semisimple case. Regarded as a set, B_{A_-} will here mean the vertex set of this graph. Now pass to the tower over a field (a not necessarily semisimple specialisation).

I2 (i) A quasi-hereditary global limit via an idempotent $e \in A_m$ (for some m) and isomorphisms [29]

$$eA_{n+m}e \cong A_n \tag{3}$$

(and hence a tower of *recollement* [12]).

(Preferably, R is a unique factorisation domain, e makes sense over the field of fractions, and we simply exclude consideration of specialisations in which e is not well defined – see later.)

(ii) A map $\mathfrak{P}_A$ from B_{A_-} to a global index set Λ, which localises at each n to an index set $\Lambda(n)$ for standard modules $\Delta(\mu)$ of A_n (that is, such that $\Lambda(n) \hookrightarrow \Lambda(n+m)$ via the full embedding of A_n-mod in A_{n+m}-mod consequent on equation (3), while $\dot{\cup}_n \Lambda(n) \cong B_{A_-}$).

Note that here we mean *standard* in the quasi-hereditary sense, but we might also hope that these modules would be 'nice' in some Kazhdan–Lusztig sense [25, 41]. Let $\text{Res}(\mu) \subset \Lambda$ denote the set of weights of standard factors of the restriction $\text{Res}^{A_{n+1}}_{A_n} \Delta(\mu)$, and $\text{Ind}(\mu)$ that of the corresponding induction.

I3 A space V (for definiteness, we shall assume that this is a real Euclidean space) and a map $\Lambda \hookrightarrow V$ with the following properties. The convex hull of $\text{Res}(\mu)$ intersects Λ in a subset of $\{\mu\} \cup \text{Ind}(\mu) \cup \text{Res}(\mu)$ (*locality* of induction and restriction); the set ρ_μ of reflections in V that fix $\{\mu\}$ and $\text{Res}(\mu)$ fixes Λ; the group W generated by $\cup_\mu \rho_\mu$ is an affine Weyl group [34, Section 4.2]; and Λ is a subset of the set of point facets in the alcove geometry induced by W on V.

I4 Control of bases for the algebras and standard modules – including the means in principle to compute Gram matrices in the case of indeterminate parameters.

I5 Explicit forms for the simplest primitive idempotents (in particular any primitive *and* central idempotents).

I6 A linkage principle (see [36]): $\Delta(\mu)$ and $\Delta(\nu)$ are in different blocks if there does not exist any $w \in W_l$ (where W_l is a suitably rescaled version of W, depending on the specialisation) such that $w\mu = \nu$.

The idea is that the structure of A_n for small n would be determined by brute force, and that of subsequent levels would be determined largely by embedding the category of left modules A_n-mod in A_{n+m}-mod using equation (3) (see [12, 29]). The simple modules L of A_{n+m}-mod such that $eL = 0$ are added by hand, and their morphisms are determined partly by expedient use of I4 and I5, and then by using Frobenius reciprocity (with respect to the tower of subalgebras) and I6 to provide an emulation of translation functors [15, 36]. The process is best illustrated with reference to the examples of interest in this paper, to which we now turn.

1.3. *Role and realization of ingredients* 1, 2(ii) *and* 3

Recall (see [**44**]; compare [**42**]) the affine Hecke algebra $H(n)$ defined by generators $\{1, X, g_1, \ldots, g_{n-1}\}$ and relations

$$g_i g_{i\pm 1} g_i = g_{i\pm 1} g_i g_{i\pm 1}, \qquad g_i g_j = g_j g_i, \qquad i \neq j \pm 1, \tag{4}$$

$$g_1 X g_1 X = X g_1 X g_1, \qquad g_j X = X g_j, \qquad j > 1, \tag{5}$$

$$(g_i - q)(g_i + q^{-1}) = 0. \tag{6}$$

The *cyclotomic* Hecke algebra [**10**] $H = H(n, d)$ is the quotient Ψ_d of $H(n)$ by

$$\prod_{i=1}^{d}(X - \lambda_i) = 0. \tag{7}$$

Here $q, \lambda_1, \ldots, \lambda_d, \ldots$ are parameters, which we may begin by regarding as indeterminates. Write $\mathcal{A}$ for $\mathbb{Z}[q, q^{-1}, \lambda_1, \ldots, \lambda_d]$ and $\mathcal{K}$ for the quotient field. Write $H^{\mathcal{A}}(n, d)$ for $H(n, d)$ over $\mathcal{A}$. This is a free module over $\mathcal{A}$ (see [**2**] and Section 2 below). The case $d = 2$, $\lambda_1 = -\lambda_2^{-1}$, is essentially the B-type Hecke algebra (see [**33**, **49**]). For any $d' > d$, let $\Psi_d : H(n, d') \to H(n, d)$ also denote the quotient by equation (7). Denote by $H(-, d)$ the sequence of inclusions

$$H(n, d) \subset H(n + 1, d). \tag{8}$$

Usually, we shall fix an $\mathcal{A}$-algebra k which is a field, and which *as a field* is $\mathbb{C}$, and we shall consider $H(n, d) = H^{\mathcal{A}}(n, d) \otimes_{\mathcal{A}} k$. The semisimple generic structure of $H(n, d)$ over $\mathbb{C}$ is well known, through that of the specialisation to the group algebra of the group $Z_d \wr S_n$ (*confer* [**13**, **33**, **35**] as in [**2**]). We recall it briefly. An *integer partition* μ of degree n is a list $(\mu_1, \mu_2, \ldots)$ of non-negative integers such that $\mu_i \geqslant \mu_{i+1}$ and $\sum_i \mu_i = n$. There is a natural correspondence with *Young diagrams* of degree n. Denote by $\Lambda_n = \Lambda_n^d$ the set of ordered lists of d integer partitions, of summed degree n (called *d-partitions of degree n*). For example,

$$\Lambda_2^2 = \{((2), 0), ((1^2), 0), ((1), (1)), (0, (2)), (0, (1^2))\}.$$

The conjugacy classes of $Z_d \wr S_n$ are readily seen to be indexed by Λ_n^d [**45**, **58**]; thus $H(n, d)$ has simple modules Δ_μ indexed by $\mu \in \Lambda_n^d$. Similarly, the Bratteli diagram $B = B_{H(-,d)}$ of the natural tower of semisimple algebras $H(-, d)$ is determined by

$$\operatorname{Res}_{H(n,d)}^{H(n+1,d)} \Delta_\mu = \bigoplus_{i=1}^{d} \bigoplus_{j} \Delta_{\mu - e_j^i}, \tag{9}$$

where the sum over j is over possible subtractions of one box from the ith Young diagram of μ.

For each $n > 1$, there are $2d$ one-dimensional irreducible representations $R_{\pm l}$ ($l = 1, 2, \ldots, d$) of $H(n, d)$, given by $R_{\pm l}(X) = \lambda_l$ and $R_{\pm l}(g_i) = \pm q^{\pm 1}$. The representation R_l corresponds to the module Δ_μ with multipartition $\mu = (, , \mu_l, ,)$, in which all component integer partitions are empty except the lth partition, which is either (n) (for the case $l > 0$) or (1^n) (case $l < 0$). For each n we may associate a unique primitive (and central) idempotent to each of these representations, in the algebra over generic k [**32**]. We write these idempotents

as $e_n^{\pm l}$. For $l \in \{1, 2, \ldots, d\}$, the element $e_n^{\pm l}$ of $H(n, d)$ uniquely obeys

$$
\begin{aligned}
(g_i \mp q^{\pm 1})e_n^{\pm l} &= 0, \qquad i = 1, 2, \ldots, n - 1, \\
(X - \lambda_l)e_n^{\pm l} &= 0, \\
(e_n^{\pm l})^2 &= e_n^{\pm l}.
\end{aligned}
\tag{10}
$$

The inclusion (8) allows us to regard $e_n^{\pm l}$ as an idempotent in $H(n + 1, d)$, albeit neither primitive nor central in general. Indeed, the idempotent will be expressible as a sum of primitive idempotents in reciprocity with the rule (9). (Let $1 = \sum_\mu e^\mu$ be the unique decomposition of 1 into primitive *central* idempotents of $H(n+1, d)$. Then $e_n^{\pm l} = \sum_\mu e_n^{\pm l} e^\mu$ is this decomposition; that is, it is also unique, even though the decomposition of 1 into primitive idempotents is not.)

In Section 5 of this paper, we give explicit formulae for all $e_n^{\pm l}$ for all d. For now, we shall be concerned particularly with $e^{\pm l} := e_2^{\pm l}$. The reason for this is the desire for a small but significant generalisation of the set of dominant weights and the weight spaces underlying Soergel's procedure for the case A_{m-1} (that is, $U_q sl_m$). Although the induction and restriction rules are straightforward, and satisfy ingredient I1, the 'weight space' of $H(-, d)$ (in which the distance $d(\mu, \lambda)$ is the minimum number of steps on B from μ to λ) is somewhat unmanageable geometrically; compare our desired ingredients 2 and 3.

What is wanted is *something like* an analogue for $H(-, d)$ of the quotients

$$
H_n^m \cong \operatorname{End}_{U_q sl_m}(V_m^{\otimes n})
\tag{11}
$$

of the ordinary Hecke algebra H_n. The Bratteli diagram B_{H_-} of the ordinary Hecke algebra is the Young 'graph', but via (11) the quotients H_n^m are the natural incarnations of the Ringel duals of A_{m-1} quantum groups, and hence may be associated to the same weight spaces, and satisfy ingredients 1–6. Let us briefly review this. Let $e_{m+1}^\pm$ denote the two primitive and central idempotents of H_{m+1} (for simplicity, assume for now that $[m + 1]! \neq 0$). The H_n^m are such that

$$
0 \longrightarrow H_n e_{m+1}^- H_n \longrightarrow H_n \longrightarrow H_n^m \longrightarrow 0
\tag{12}
$$

is exact. For $[m]! \neq 0$ they are quasi-hereditary and satisfy ingredient 2: for example, through

$$
e_m^- H_{n+m}^m e_m^- \cong H_n^m.
\tag{13}
$$

Let $\Lambda^{1,m}$ denote the set of Young diagrams of less than m rows, regarded as a subset of $\mathbb{Z}^{m-1}$. Let $v = (1, 1, \ldots, 1) \in \mathbb{Z}^{m-1}$, and let $\mathbb{Z}^{m-1}/v$ denote the corresponding quotient set. Note that the injective map from $\mathbb{Z}^{m-2}$ into $\mathbb{Z}^{m-1}$ given by $(\mu_1, \mu_2, \ldots, \mu_{m-2}) \mapsto (\mu_1, \mu_2, \ldots, \mu_{m-2}, 0)$ has image a set of representative elements of $\mathbb{Z}^{m-1}$ in $\mathbb{Z}^{m-1}/v$. Denote by $\mathfrak{P}^{m-1}$, the corresponding surjective map from $\mathbb{Z}^{m-1}$ to $\mathbb{Z}^{m-2}$ (and also its restriction to $\Lambda^{1,m}$, whose image is $\Lambda^{1,m-1}$). Note that B_{H^m}, the set of weights for H_n^m for all n, is $\Lambda^{1,m+1}$. In the sense of ingredient 2(ii), the set of isomorphisms (13) collapses B_{H^m} into $\Lambda^{1,m}$, which is the set of dominant weights of sl_m, via $\mathfrak{P}^{m+1}$ [56]. That is, $\mathfrak{P}_{H^m} = \mathfrak{P}^m$, and $\Lambda^{1,m}(n)$, the index set for H_n^m, is the subset of $\Lambda^{1,m}$ of diagrams of degree less than or equal to n and congruent to n modulo m.

Delightfully, we find that only the $n = 2$ idempotents are needed for an analogue of equation (12) for $H(n, d)$ (see also [58]). Denote the sum of the ideals generated by $\{e_2^{-l} \mid \text{all } l\}$ by

$$
\mathcal{D}_d = \mathcal{D}_d(n) := +_{l=1}^d \mathsf{H} e_2^{-l} \mathsf{H}.
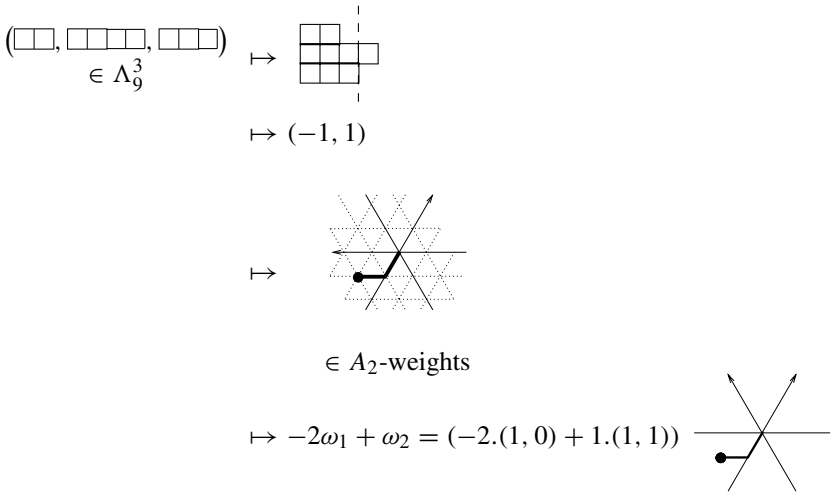$$

(We shall modify this definition very slightly later.) Define algebra $H^{\mathcal{D}} = H^{\mathcal{D}}(n, d)$ by $H^{\mathcal{D}}(1, d) = H(1, d)$ and for $n > 1$ by exactness of the sequence
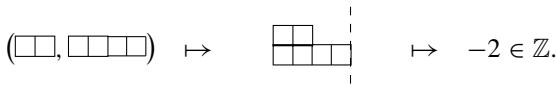
$$0 \longrightarrow \mathcal{D}_d \longrightarrow H \longrightarrow H^{\mathcal{D}} \longrightarrow 0. \tag{14}$$

The idea is to restrict consideration to the subset $\Lambda^{\mathcal{D}}$ of $H(-, d)$-weights in which each integer partition is the trivial partition of that degree. Such an $H(-, d)$-weight is characterised by a sequence of d non-negative integers; that is, the degrees of the component integer partitions (in this way, we have an action of $\mathfrak{P}^d$ on $\Lambda^{\mathcal{D}}$). This sequence need not be ordered as an integer partition, and hence the set of such weights maps onto the set of all weights of A_{d-1} (that is, not just the usual dominant weights).

For example, with $d = 3$, the weight $((2), (4), (3))$ becomes $(2, 4, 3)$, and $\mathfrak{P}^3((2, 4, 3)) = (-1, 1)$, as shown below.

$$\mapsto (-1, 1)$$

$$\in A_2\text{-weights}$$

$$\mapsto -2\omega_1 + \omega_2 = (-2.(1, 0) + 1.(1, 1))$$

(The final row just shows the same weight in terms of fundamental weights.) Meanwhile, with $d = 2$,

$$\mapsto \quad -2 \in \mathbb{Z}.$$

(This last is the weight set used in Section 1.2.) Note also that $\Delta_\mu \in$ H-mod is also in $H^{\mathcal{D}}$-mod if and only if $\mu \in \Lambda^{\mathcal{D}}$ (in which case, as an $H^{\mathcal{D}}$-module, we write it $\Delta(\mathfrak{P}^d(\mu))$). To see this note, from (9), that the restriction of Δ_μ to $n = 2$ contains a copy of one of the excluded one-dimensional modules (not necessarily a *direct* summand in general) if and only if μ contains at least one integer partition with a second part (that is, a Young diagram with a second row).

By construction, then, the tower $H^{\mathcal{D}}(-, d)$ has ingredients 1, 3 (restriction is local via (9) and induction is via Frobenius reciprocity) and, at least formally, the final part of 2. Of course, the construction is most interesting if it can be made to include non-semisimple specialisations (or else the *fundamental alcove* of Section 1.2 is the whole weight space, and the Soergel procedure is trivial). In the remainder of this paper we address ingredients 2, 4, 5 and 6 from this point of view. In particular, we identify the $d = 2$ case with the blob algebra. This has useful implications for all $d > 2$. Note that, fixing k, there are a number of distinct ways to impose a quotient relation of the form of Ψ_d on $H(n, d')$,

corresponding to the choice of factors to be removed in the strengthening of the relation (7). For $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_{d'})$ and λ any subsequence of this of length $d' - d$, let Ψ_d^λ denote the strengthening by omission of the factors $(X - \lambda_i)$ with λ_i in λ. The quotient $\Psi_d^{(\lambda_i)}$ commutes with the quotient to $H^{\mathcal{D}}$, so the following proposition holds.

PROPOSITION 1.2. *Fixing k, there are $d + 1$ ways $(\Psi_d^{(\lambda_i)}, i = 1, 2, \ldots, d + 1)$ to quotient in order to pass from $H^{\mathcal{D}}(n, d + 1)$ to $H^{\mathcal{D}}(n, d)$, where $n > 0$.*

Returning to the example of $d = 3$, we see that as n varies, the image of $\Psi_2^{(\lambda_i)}$ sweeps out a $2\pi/3$ radian arc of weight space, with the union of these arcs over $i = 1, 2, 3$ giving the complete space (see Figure 4, illustrating this, in Section 5.2). If k gives a non-semisimple specialisation, then – as we shall see – there is at least one i such that the corresponding tower of blob algebras (that is, $d = 2$ algebras) is critical (in the sense of [55, 57], or Section 1.2); that is, it has one or more reflection points. As n varies, a given $d = 2$ reflection point sweeps out a straight line in this arc in $d = 3$ weight space (see Section 5.2 for details). This is then a reflection line of the $d = 3$ alcove geometry, to which the Soergel procedure may be applied.

Our approach to ingredient 2 is through representation theory, and the last part of the paper addresses this. It includes a 'walk-though' review of some earlier work in statistical mechanics, which explains our approach to this problem, and concludes by defining certain representations of b_n (and $H^{\mathcal{D}}(n, d)$), which are candidates for 'tensor space' representations. (That is, they would establish ingredient 2 if faithful; see Section 6. The question of faithfulness is *not* resolved here.)

2. Preliminaries

Let the involution $t : \mathbb{Z}[q, q^{-1}] \to \mathbb{Z}[q, q^{-1}]$ be given by $t(q) = -q^{-1}$, and let s act on $\mathbb{Z}[\lambda_1, \ldots, \lambda_d]$ by permuting the indices on the $\{\lambda_i\}$ cyclically. Let

$$[n] = q^{n-1} + q^{n-3} + \cdots + q^{1-n}.$$

For A an algebra, let $Z(A)$ denote the centre of A; and for $B \subseteq A$, let $Z_B(A)$ denote the centralizer of B in A (so that $Z_B(A) \supseteq Z(A)$ and $Z_A(A) = Z(A)$).

For $a = (a_1, a_2, \ldots, a_n)$ an n-tuple of natural numbers, the *Weyl orbit* of a in $\mathbb{N}^n$ is the orbit of the S_n action permuting indices. Let $Y = (Y_1, \ldots, Y_n)$ be an n-tuple of variables in some commutative ring R. Define

$$Y^a = \prod_{i=1}^{n} Y_i^{a_i}$$

and the *monomial symmetric polynomial* (see, for example, [24])

$$(Y^a)^\Sigma = \sum_{a'} Y^{a'},$$

where the sum is over all the elements in the Weyl orbit of a.

Let R be a unique factorisation domain, A a free R-module with basis $\mathbf{A}$, and K the field of fractions of R. Consider any $e \in A \otimes_R K$, and let $a_e \in A$ be such that $\mathbf{e} = a_e e \in A \subset A \otimes_R K$. If there exists an a_e such that the coefficient of some $\mathbf{a} \in \mathbf{A}$ in $\mathbf{e}$ is 1, then a_e and $\mathbf{e}$ are unique with this property, up to a unit. If A is an R-algebra, and e is idempotent, call such an $\mathbf{e}$ a *preidempotent*, and a_e the corresponding *normalisation* of e.

Now let R as above be $\mathcal{A}$, and consider A as a collection of $\mathbb{C}$-algebras by specialisation. We say that a property holds *generically* if it holds on a (Zariski) open subset of parameter space, and that it holds *usually* if the condition for failure to hold may be expressed as a single finite polynomial in a single variable. (For example, [2] is generically and usually invertible; $(q^2\lambda_1 - \lambda_2)$ is generically but not usually invertible.)

Let $e = \mathbf{e}/a'_e$ be *any* explicit expression for an element in A over $\mathcal{K}$ as above. For any point x in parameter space (that is, any k), there is an open region with x in its closure in which the polynomial a'_e has no root, so e may be evaluated as a limit at k. This process does not guarantee a unique finite limit. If e is a primitive and central idempotent, however, then any two finite limits must be the same, since they will have the properties of a unique primitive and central idempotent in A over k. (That is, they will induce the same simple projective module, $A\mathbf{e} \otimes k$.)

Let σ_i denote the elementary transposition $\sigma_i = (i\ i+1) \in S_n$; so $\sigma_i(i) = i+1$, and so on [32]. Let $\mathfrak{B}_n$ be a maximal set of inequivalent reduced words in the generators $\{g_1, \ldots, g_{n-1}\}$ of H_n. For each $w \in \mathfrak{B}_n$, note that there is a natural (reduced expression for an) element of S_n associated to it by substituting $g_i \rightsquigarrow \sigma_i$.

Let

$$X_1 := X \quad \text{and} \quad X_j := g_{j-1}X_{j-1}g_{j-1}. \tag{15}$$

So in $H(n)$,

$$[X_j, X_k] = 0, \tag{16}$$

$$[X_j, g_k] = 0, \qquad j \neq k, k+1 \tag{17}$$

$$g_k X_{k+1} = X_k g_k + (q - q^{-1})X_{k+1},$$
$$g_k X_k = X_{k+1} g_k - (q - q^{-1})X_{k+1}, \tag{18}$$

and so

$$[X_k + X_{k+1}, g_k] = [X_k X_{k+1}, g_k] = 0. \tag{19}$$

Now

$$X_k^j + X_{k+1}^j = \left(X_k^{j-1} + X_{k+1}^{j-1}\right)\left(X_k + X_{k+1}\right) - \left(X_k^{j-2} + X_{k+1}^{j-2}\right)X_k X_{k+1},$$

so

$$\left[g_i, (X^a)^\Sigma\right] = 0$$

for all i and any a.

Let X^Σ denote the algebra of symmetric polynomials in the X_is. Evidently, $X^\Sigma \subseteq Z(H(n))$, and in fact Bernstein has noted that this is an equality (see Appendix A).

It follows from equations (16)–(18) that any product of generators of H(n, d) can be expressed as a $\mathbb{Z}[q, q^{-1}, \lambda_1, \ldots, \lambda_d]$-linear combination of words from the set

$$\mathfrak{C}_n^d = \{X^a w \mid a \in \{0, 1, \ldots, d-1\}^n,\ w \in \mathfrak{B}_n\}.$$

The dimension of this spanning set is clearly $d^n n!$, which is also the dimension of H(n, d); thus the following proposition holds.

PROPOSITION 2.1 (SEE [2]). *The set $\mathfrak{C}_n^d$ is a basis for* $H(n, d)$.

We linearly extend s and t to act on $H(n, d)$, fixing $\mathfrak{C}_n^d$ pointwise.

PROPOSITION 2.2 (SEE [58]). *Let B be a basis for* $H(n - 1, d)$, *let D be a basis for* $\langle X \rangle$, *and let* $G = g_1 g_2 \ldots g_{n-1}$. *Then*

$$\{aGb, g_1 aGb, g_2 g_1 aGb, \ldots, g_{n-1} \ldots g_2 g_1 aGb \mid a \in D, \ b \in B\}$$

is a basis for $H(n, d)$.

Fix d and set $p_n^l = \prod_{i \neq l}(q^{2n-2}\lambda_l - \lambda_i)$. Note that

$$z_n^l = \prod_{k=1}^{n}\left(\prod_{i \neq l}(X_k - \lambda_i)\right)$$

lies in $Z(H(n))$ and obeys $\Psi_d((X - \lambda_l)z_n^l) = 0$. Comparing this with (10), we thus have

$$e_n^{\pm l} = \frac{z_n^l}{\prod_{k=1}^{n} p_k^l} e_n^{\pm}.$$

It follows that $\mathbf{e}_n^{\pm l} = (\prod_{k=1}^{n}([k]p_k^l))e_n^{\pm l}$ is a preidempotent for $e_n^{\pm l}$ (compare with the basis $\mathfrak{C}_n^d$). Now we define

$$\mathcal{D}_d = \mathcal{D}_d(n) := +_{l=1}^{d} H\mathbf{e}_2^{-l}H$$

(a modification of the definition in Section 1.3) and $H^{\mathcal{D}}$ accordingly. We shall give another expression for $e_n^{\pm l}$ shortly.

3. Standard modules and linkage

The generators and relations in equations (4) (and their inverses) define the ordinary braid group, A_n-braid. Denote by B_n-braid the extension by $g_0 = X$ (and its inverse) obeying equations (5) (*confer* [33], [50] and the references therein). Thus $H(n)$ is a quotient of $\mathbb{C}B_n$-braid by the quadratic relation in equation (6).

One realization of B_n-braid is as the group of braids on the cylinder, with g_0 becoming the pure braid in which the first string passes over all the other strings and then around the cylinder. There is a natural 'Young' embedding

$$B_n\text{-braid} \times B_m\text{-braid} \hookrightarrow B_{n+m}\text{-braid}.$$

One places the second cylinder concentrically inside the first, and one then allows the two cylinders to converge in such a way that the nodes of B_n-braid and B_m-braid respectively remain consecutive on the edges of the cylinder (while of course preserving over/under information). There is a corresponding embedding $H(n) \times H(m) \hookrightarrow H(n + m)$. The construction of 'standard' modules (in the sense of [11, 62]) follows from this. The quotient Ψ_d complicates this, in that the spectrum of X_{n+1} (the image of $(1, X_1)$) is not that of X_1 (the image of $(X_1, 1)$).

DEFINITION 3.1. Let $\langle X_i \rangle$ denote the commutative subalgebra $\langle X_i \mid i = 1, \ldots, n \rangle \subseteq H(n, d)$.

Generically, as we shall see, we may determine a *unique* basis of primitive (and of course central) idempotents ϵ_x of $\langle X_i \rangle$ with $\sum_x \epsilon_x = 1$. There will be certain specialisations where

this basis will not make sense (certain idempotents will have preidempotents with vanishing normalisation). In any case, any primitive idempotent decomposition of 1 in $H(n, d)$ will be different, but the unique primitive central idempotent decomposition of 1 in $H(n, d)$ will be expressible as a crudification of the above (albeit depending on k). The generic case will be the least crude (one idempotent per block/isomorphism class of simples), and it will be necessary, formally, to combine certain of these generic idempotents (into non-simple blocks) to make idempotents that make sense over k in non-semisimple cases. Each ϵ_x must obey $X_i \epsilon_x = x_i \epsilon_x$ with x_i some scalar. (Thus each induces a left H-module $\Pi_x := H\epsilon_x$. Since the H_n subalgebra of $H(n)$ maps isomorphically to its image in the quotient, we have $H\epsilon_x = H_n\epsilon_x$, of rank $n!$.)

Evidently, $x_1 \in \{\lambda_i\}$, and with $\pi_j := \prod_{i \neq j}(X - \lambda_i)$ we have $X\pi_j = \lambda_j \pi_j$ and $\pi_j \epsilon_x \propto \epsilon_x$. For each such π_j there exists a minimal polynomial $\pi_{j,-}(X_2) = \prod_k (X_2 - \lambda_{j,k})$ such that $\pi_j \pi_{j,-} = 0$. Set $\pi_{j,k} = \prod_{l \neq k}(X_2 - \lambda_{j,l})$ and $\epsilon_{\cdot(j,k)} = \pi_j \pi_{j,k}$. Then $X_2 \epsilon_{\cdot(j,k)} = \lambda_{j,k} \epsilon_{\cdot(j,k)}$. For each $\epsilon_{\cdot(j,k)}$ there exists a polynomial $\pi_{j,k,-}(X_3)$ such that $\pi_j \pi_{j,k} \pi_{j,k,-} = 0$, and so on. That is, the roots of such a polynomial are certain of the eigenvalues of the X_is.

We can work out these *eigenvalues* of the X_is by looking at the properties of generically irreducible representations as given in Section 1.3. Let $\mu \in \Lambda_n^d$. A 'standard' insertion of $\underline{n} = \{1, 2, \ldots, n\}$ into the boxes of μ is one such that deletion of the boxes containing $\{l, \ldots, n\}$ produces a legitimate d-partition for every l. Let T_μ denote the set of all such standard insertions. For $i \in \underline{n}$ and $w \in T_\mu$ there is a $k \in \{1, 2, \ldots, d\}$ such that i appears in a box in the kth partition in w. Define $w(i) = k$. It will be evident from the restriction rule (9) that T_μ may be used as a basis for Δ_μ, once it has been equipped with a suitable action.

PROPOSITION 3.2. *The action of the X_is on this basis may be taken to be lower triangular, with diagonal elements of X_i on $w \in T_\mu$ given by $\lambda_{w(i)}q^{2w^i}$, where w^i is the distance of i off the main diagonal in the $w(i)$th partition in μ (with distances* below *the diagonal being negative).*

Proof. In the case where $n = 1$ the claim holds, since $R_i(X) = \lambda_i$. Suppose that the claim is true at level $n - 1$. Then the eigenvalues for $X_1, \ldots, X_{n-1}$ at level n are given by restriction, using the rule (9). The eigenvalues for X_n may be determined using equation (10) (in the one-dimensional cases), and then the centrality of $\sum_i X_i$, and so on, together with Schur's lemma. $\square$

Let v_i denote the ith elementary vector in $\mathbb{Z}^d$. Describe a walk on $\mathbb{Z}^d$ (or $\mathbb{Z}^d/(1, 1, \ldots, 1)$) by a word $w = w_1 w_2 \ldots$ in $\underline{d}$ such that the vector between the ith and $(i + 1)$th site visited is v_{w_i}. For a given word w, define

$$\#_l(i) = \#_l^w(i) = \sum_{j=1}^l \delta_{w_j, i}.$$

A reflection hyperplane $(i, j; x)$ is characterised by a pair v_i, v_j ($i \neq j \in \underline{d}$) not parallel to it, and the *signed* distance x in the direction of v_i of this hyperplane from 0 (that is, the x such that $0 + xv_i$ lies on it). A walk touches this hyperplane at l if

$$\#_l(i) - \#_l(j) = x. \tag{20}$$

Let w be a walk that touches hyperplane $(i, j; x)$ at l. The walk w' obtained from w by applying permutation (ij) to every w_t, $t > l$, is called the (*affine*) *reflection* of w in $(i, j; x)$ at l. (Note that the touching point l is not in general uniquely defined by w and $(i, j; x)$ or, indeed, existent.) Every point of the reflection after l is the reflection of this point in $(i, j; x)$ in the usual alcove geometry sense (see [**50**, Chapter 7]).
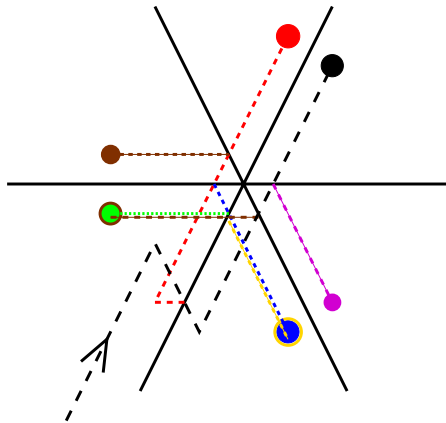
Figure 2: Eight walks in a walk orbit for $d = 3$ (origin in the bottom left-hand corner, endpoints circled).

If w' meets another (not necessarily distinct) hyperplane $(k, m; y)$ at $l' > l$, then of course w meets the image of $(k, m; y)$ in $(i, j; x)$ at the same moment. If w'' is the reflection of w' in $(k, m; y)$ at l', then we say that w, w' and w'' are in the same *walk orbit* of the reflection group generated by these hyperplanes (the ith points of these walks are in the same orbit in the usual sense, for each i). We may think of folding up the space along the set of hyperplanes in the group – the walk orbits are the sets of walks that are mapped into each other by this folding. For G a reflection group generated by hyperplanes, we write $w \sim^G w'$ if walks w and w' are in the same walk orbit of G. Each hyperplane partitions space into two parts (not counting the hyperplane itself). The 'outside' of the hyperplane $(i, j; x)$, $x \neq 0$, is the part not containing 0.

We shall restrict our attention to hyperplanes *not* touching 0. Note that for each walk w from 0 that finishes at a point μ outside some hyperplane, there is not in general a unique walk in its orbit that finishes at the image point of μ, inside it. The orbit structure of walks is more complicated than that of points. For example, the endpoints of elements of the same walk orbit are necessarily in the same point orbit; but the converse does not follow. Further, a walk that stays in the interior of the fundamental alcove is in a singleton orbit; more generally, a walk that touches hyperplanes a total of t times has 2^t elements in its orbit (see Figures 2 and 3).

For G a reflection group generated by a set of reflection hyperplanes S, let $\bar{S} \subset G$ denote the set of simple reflections. It will be convenient to confuse these reflections with the corresponding hyperplanes, noting that $\bar{S} \supset S$ in general. In particular, we may now compose hyperplanes by conjugation in G; that is, $(i, j; x) \circ (j, k; y) = (i, k; x+y)$, and so on. Let $\mathcal{A}' \subset \mathcal{A}$ consist of those elements of the form $(\lambda_i - q^{-2x}\lambda_j)$, for all i, j, x. Define the *factor set* of G, $\mathcal{F}(G) \subset \mathcal{A}'$ such that $(i, j; x) \in \bar{S}$ if and only if $(\lambda_i - q^{-2x}\lambda_j) \in \mathcal{F}(G)$. The elements of $\mathcal{F}(G)$ are called *factors*. Note that there is an equality of ideals

$$\sum_{(i,j;x)\in S} \mathcal{A}(\lambda_i - q^{-2x}\lambda_j) = \sum_{(i,j;x)\in \bar{S}} \mathcal{A}(\lambda_i - q^{-2x}\lambda_j) \qquad (21)$$

since the composition of hyperplanes above corresponds to the addition (up to a unit) of factors.
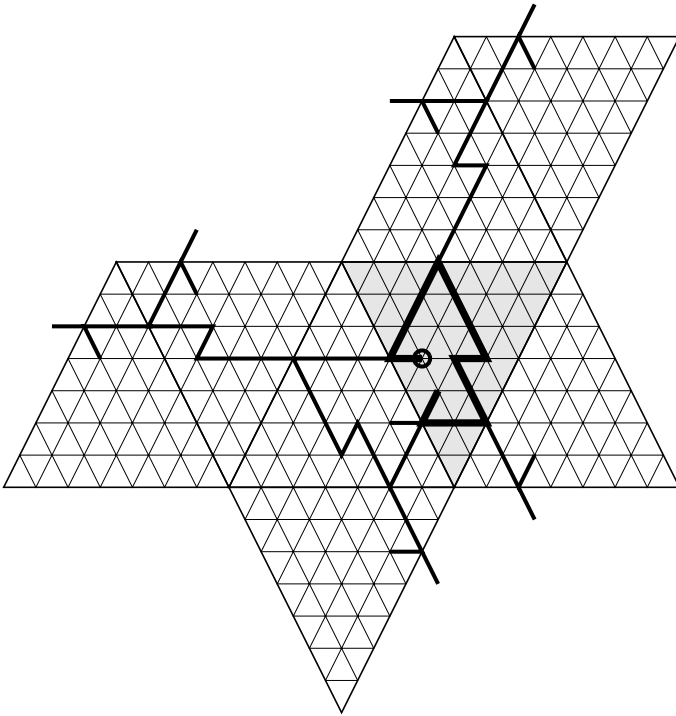
Figure 3: A walk, starting at 0 (circled), in the fundamental alcove (shaded) of a certain $d = 3$ affine reflection group; and all walks in its walk orbit. Note that each of the two endpoints closest to 0 is reached by two different walks.

For a walk w of length n, let $\mu(w) = (\#_n(1), \#_n(2), \dots)$, an *ordered* partition of n. To each walk w of length n, associate an element $\lambda^w \in \mathcal{A}^n$ as follows:

$$(\lambda^w)_i = \lambda_{w_i} q^{2(\#_i(w_i)-1)}.$$

For example, $\lambda^{3331312} = (\lambda_3, \lambda_3 q^2, \lambda_3 q^4, \lambda_1, \lambda_3 q^6, \lambda_1 q^2, \lambda_2)$.

Note that this λ^- gives an injective map from $\bigcup_{\mu \in \Lambda} \mathcal{D}_{(n)} T_\mu$ into $\mathcal{A}^n$.

PROPOSITION 3.3. (i) *Let w' be the reflection of w in $(i, j; x)$ at l; then every non-zero element of $\lambda^w - \lambda^{w'}$ is of the form $\pm q^{2\alpha}(\lambda_i - q^{-2x}\lambda_j)$, for some $\alpha \in \mathbb{Z}$. (Note that there is no mention of l in the implication.)*

(ii) *Let w and w' be two walks. If every element of $\lambda^w - \lambda^{w'}$ is divisible by $(\lambda_i - q^{-2x}\lambda_j)$, then w and w' are related by one (or more) reflections in $(i, j; x)$ for some l (respectively, $l < l' < l'' < \dots$).*

(iii) *Let G be the group generated by a set of simple reflections $\{(i, j; x), (k, l; y), \dots\}$. Walks w and w' are in the same walk orbit of G (that is, they are related by a sequence of reflections in hyperplanes in G) if and only if each non-zero $(\lambda^w - \lambda^{w'})_i$ is divisible by an element of $\mathcal{F}(G)$.*

Proof. (i) Consider the first point at which w and w' differ, which we may take to be $w'_{l+1} = (ij)w_{l+1} = j$.

We have

$$(\lambda^w - \lambda^{w'})_{l+1} = \lambda_i q^{2(\#^w_{l+1}(i)-1)} - \lambda_j q^{2(\#^{w'}_{l+1}(j)-1)}.$$

Now $\#^w_{l+1}(i) = \#^w_l(i) + 1$ and $\#^{w'}_{l+1}(j) = \#^{w'}_l(j) + 1$, so (see equation (20)) the exponents differ by $2x$, as required. Since subsequent points k with $w_k \neq w'_k$ are *all* those at which $\{w_k, w'_k\} \cap \{i, j\} \neq \emptyset$, the *difference* in exponents is preserved.

(ii) This is a special case of (iii).

(iii) *'Only if' part:* Note that, by construction, $(\lambda^w - \lambda^{w'})_i$ is, up to a unit, an element of $\mathcal{A}'$. Each pair $w \sim^G w'$ may be related by a series of simple reflections:

$$w \sim^G w'' \sim^G \dots \sim^G w',$$

and each intermediate $(\lambda^{w^i} - \lambda^{w^{i+1}})_i$ is of the required form, by part (i). Evidently, $(\lambda^w - \lambda^{w'})_i = \sum_i (\lambda^{w^i} - \lambda^{w^{i+1}})_i$. Now confer equation (21).

'If' part: Let $w|_m$ denote the first m steps in walk w. Fix w, and consider w' such that $w \not\sim^G w'$. Suppose that point m is the first at which $w|_m \not\sim^G w'|_m$. Let $\tilde{w}|_{m-1} = w'|_{m-1}$ and let $\tilde{w}_m$ be such that $w|_m \sim^G \tilde{w}|_m$. Thus $\tilde{w}|_m$ is a reflection of $w'|_m$ in some hyperplane h at $m-1$, with $h \notin G$. Thus $\lambda^{w'} - \lambda^{\tilde{w}} = (0, 0, \dots, 0, (q^{2\alpha}(\lambda_i - q^{-2x}\lambda_j)))$ for some α, i, j, x, where $i \neq j$, since w' and $\tilde{w}$ agree at $m-1$. The non-zero factor is not a factor of $\mathcal{F}(G)$, since $h \notin G$. Now consider $\lambda^w - \lambda^{w'} = (\lambda^w - \lambda^{\tilde{w}}) + (\lambda^{\tilde{w}} - \lambda^{w'})$. The mth element of the first summand on the right *is* divisible by some $f \in \mathcal{F}(G)$, by the *only if* part, while that of the second summand is not, by the above argument. Thus the left-hand side is also not divisible. $\square$

For example, consider the sequences 333, 331 and 321 with $\lambda^{333} = (\lambda_3, q^2\lambda_3, q^4\lambda_3)$, $\lambda^{331} = (\lambda_3, q^2\lambda_3, \lambda_1)$ and $\lambda^{321} = (\lambda_3, \lambda_2, \lambda_1)$. We have

$$(3, 1; 2)333 = 331;$$
$$(3, 2; 1)331 = 321;$$
$$\lambda^{333} - \lambda^{321} = (0, q^2(\lambda_3 - q^{-2}\lambda_2), q^4(\lambda_3 - q^{-4}\lambda_1)).$$

PROPOSITION 3.4. *For each $w \in \bigcup_{\mu \in \Lambda^{\mathcal{D}}(n)} T_\mu$ there is an element $\epsilon_{\cdot w}$ of $\langle X_i \rangle$ as described above, obeying the requirement that*

$$X_i \epsilon_{\cdot w} = (\lambda^w)_i \epsilon_{\cdot w}.$$

Proof. For a d-partition μ of form $((\mu_1), (\mu_2), \dots)$ (that is, $\mu \in \Lambda^{\mathcal{D}}$) and $w \in T_\mu$ identified with the corresponding word w, we have $w_i = w(i)$ and $\#_i(w_i) = w^i$. Now see Proposition 3.2. $\square$

Let R_x denote the representation of $\langle X_i \rangle$ corresponding to any $\epsilon_{\cdot x}$ as above. It follows that the $\langle X_i \rangle$-simple character of $\mathrm{Res}^{\mathrm{H}}_{\langle X_i \rangle}(\Delta_v)$ is

$$\left[\mathrm{Res}^{\mathrm{H}}_{\langle X_i \rangle}(\Delta_v) : R_w\right] = \delta_{\mu(w), v}. \tag{22}$$

Note that this is not a *unique* characterisation unless the following corollary holds.

COROLLARY 3.5. *If k is generic (that is, λ^- remains injective on passing from $\mathcal{A}^n$ to $(\mathcal{A} \otimes k)^n$), then $R_w \cong R_{w'}$ implies that $w = w'$, and $\langle X_i \rangle^{\mathcal{D}} \subset H^{\mathcal{D}}$ has enough simples to be semisimple.*

DEFINITION 3.6. We define $\mathcal{G}(k)$, the *reflection group induced by k*, as follows. For each triple $(i, j; x)$ such that $(\lambda_i - q^{2x}\lambda_j) \otimes k = 0$, include the hyperplane $(i, j; x)$ as a generator.

Note that this is sensible inasmuch as $(i, j; x) \circ (j, k; y) = (i, k; x+y)$, while $\lambda_i - q^{2x}\lambda_j = 0$ and $\lambda_j - q^{2y}\lambda_k = 0$ imply that $\lambda_i - q^{2(x+y)}\lambda_k = 0$. If d' of the parameters $\{\lambda_i\}$ are related in this way, then the group generated is $S_{d'}$, unless q is a root of unity – in which case, it is the affine extension. We shall usually refer to the group as the affine Weyl group regardless, to emphasize the fact that even in the finite case, the hyperplanes do not pass through 0. That is, we exclude from consideration any k in which $\lambda_i = \lambda_j$ does not imply that $i = j$.

PROPOSITION 3.7. *Over k, $R_w \cong R_{w'}$ if and only if w and w' are in the same walk orbit of the affine Weyl group induced by k (that is, they are related by some series of reflections).*

Proof. There is an isomorphism if and only if $\lambda^w - \lambda^{w'}$ vanishes over k. Every term is of the form $q^{\alpha}(\lambda_i - q^{2x}\lambda_j)$. Now apply Proposition 3.3. $\square$

PROPOSITION 3.8 (LINKAGE). *If there exists a nontrivial homomorphism $\Delta_\mu \to \Delta_\nu$ over k, then μ and ν lie in the same orbit of the affine Weyl group induced by k.*

Proof. Consider the following diagram.

$$
\begin{array}{ccc}
\Delta_\mu & \longrightarrow & \Delta_\nu \\
\downarrow{\scriptstyle \text{Res}} & \Downarrow & \downarrow{\scriptstyle \text{Res}} \\
\sum_{w \in T_\mu} R_w & \longrightarrow & \sum_{w \in T_\nu} R_w
\end{array}
$$

Now apply Proposition 3.7. $\square$

Note that this strengthens immediately to exclude interorbit maps from any submodule of Δ_μ to any quotient of Δ_ν; that is, $\mu \not\sim^{\mathcal{G}(k)} \nu$ implies that Δ_μ and Δ_ν have no composition factors in common. Thus, under the assumption that, as for a quasi-hereditary algebra, every simple occurs in $\{\text{Head}(\Delta_\mu) \mid \mu \in \Lambda^{\mathcal{D}}\}$, we have linkage in the form of I6.

PROPOSITION 3.9. *Let $w, w' \in \bigcup_{\mu \in \Lambda^{\mathcal{D}}(n)} T_\mu$ be reflections of each other in $(i, j; x)$ at any l. Then over any k in which $(\lambda_i - q^{2x}\lambda_j)$ vanishes, we have an isomorphism of left H-modules*

$$H\epsilon_w \cong H\epsilon_{w'}.$$

Proof. By Proposition 3.7, $\epsilon \cdot_w$ and $\epsilon \cdot_{w'}$ induce isomorphic $\langle X_i \rangle$-modules. $\square$

Given the injectivity of λ^-, it will be convenient to be able to refer to a walk w either directly or via its image λ^w. This unifies the labelling schemes for ϵ_w following Definition 3.1 and in Proposition 3.4.

Note that $R_x(\pi_j) = 0$ for all R_x except those with $x_1 = \lambda_j$. Let S_j be the set of possible values of $R_x(X_2)$ when $x_1 = \lambda_j$ (that is, $S_j = \{\lambda_k \neq \lambda_j, q^2\lambda_j, q^{-2}\lambda_j\}$). Then $R_{(\lambda_j, x_2, \dots)}(\prod_{s \in S_j}(X_2 - s)) = 0$ for any such x, and $\pi_j \prod_{s \in S_j}(X_2 - s)$ lies in the radical of $\langle X_i \rangle$, and hence some power of it vanishes. This tells us, up to multiplicity, the roots of $\pi_{j,-}$, and hence of the preidempotent $\pi_j \pi_{j,k}$ (obtained by omitting a factor $(X_2 - \lambda_{j,k})$, $\lambda_{j,k} \in S_j$, from $\pi_j \pi_{j,-}$). If λ^- is injective, then the radical is $\{0\}$ and all the roots can be distinguished. Iterating this argument, we have the following proposition.

PROPOSITION 3.10. *For w a walk,*

$$\epsilon_w = \prod_{i=1}^{|w|} \prod_{\substack{x \,\in\, \cup_{v,|v|=i} T_v \\ \text{such that} \\ x|_{i-1} = w|_{i-1}, \\ x_i \,\neq\, w_i}} (X_i - R_x(X_i))$$

where the union is over all *multipartitions satisfying the constraints.*

For example, $\epsilon_{111} = \epsilon_{11}(X_3 - q^{-2}\lambda_1)\prod_{i=2}^d (X_3 - \lambda_i)$, and

$$\epsilon_{11122} = \epsilon_{111}\left((X_4 - q^{-2}\lambda_1)(X_4 - q^6\lambda_1)\prod_{2<i\leqslant d}(X_4 - \lambda_i)\right)$$
$$\cdot\left((X_5 - q^{-2}\lambda_1)(X_5 - q^6\lambda_1)(X_5 - q^{-2}\lambda_2)\prod_{2<i\leqslant d}(X_5 - \lambda_i)\right).$$

Note that unless the *omitted* factor in π_j or $\pi_{j,k}$ coincides (under $X_1 \leftrightarrow X_2$) with a factor in the other, then the product $\pi_j\pi_{j,k}$ is automatically in $Z(\mathrm{H}(2,d))$. (Simply multiply in all the factors apparently required for symmetry, and then replace these with scalars using the eigenvector property – the excluded cases are those where one or more such scalars vanish.)

For example, in the case $n = 2, d = 2$, we see that $\langle X_i \rangle$ has rank 6. We have

$$(X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - q^2\lambda_1)(X_2 - \lambda_2) = 0;$$

so, for example,

$$\epsilon_{11} = \epsilon_{\lambda_1,q^2\lambda_1} = (X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - \lambda_2)$$

is a preidempotent with $X_2\epsilon_{\lambda_1,q^2\lambda_1} = q^2\lambda_1\epsilon_{\lambda_1,q^2\lambda_1}$, while

$$\epsilon_{\lambda_1\lambda_2} = (X - \lambda_2)(X_2 - q^{-2}\lambda_1)(X_2 - q^2\lambda_1)$$

is a preidempotent with $X_i\epsilon_{\lambda_1\lambda_2} = \lambda_i\epsilon_{\lambda_1\lambda_2}$. The preidempotent $\epsilon_{\lambda_1,q^2\lambda_1}$ lies in $Z(H)$ (provided that $q^2 \neq 1$), since it can be symmetrized:

$$(X_1 - q^{-2}\lambda_1)\epsilon_{\lambda_1,q^2\lambda_1} = (\lambda_1 - q^{-2}\lambda_1)\epsilon_{\lambda_1,q^2\lambda_1}.$$

On the other hand, $\epsilon_{\lambda_1\lambda_2}$ cannot be rescaled to its symmetrized form because the symmetrizing factor $X_2 - \lambda_2$ kills it.

For $\mathrm{H}(2,d)$ we have

$$\prod_{j\neq 1}(X - \lambda_j)(X_2 - q^2\lambda_1)(X_2 - q^{-2}\lambda_1)\prod_{j\neq 1}(X_2 - \lambda_j) = 0,$$

so similar considerations apply. Indeed, they do for all n. In particular, $\epsilon_{11...1} \in Z(\mathrm{H})$ unless $q^2 = 1$. More generally, we may proceed as follows.

For μ an ordered partition of n, let H_μ denote the corresponding Young subalgebra of $H_n \subset \mathrm{H}(n,d)$. For w a walk, let $e_w \in H_n$ denote the q-Young symmetrizer (see [14] and [26, Section 9.3]) associated to $H_{\mu(w)}$. A walk w is said to be *sorted* if it takes the form $111\ldots 22\ldots$ (more precisely, if $w_i \leqslant w_{i+1}$ for all consecutive pairs of steps in w). There is a unique sorted walk in each T_μ, denoted $w(\mu)$. A walk w is said to be *direct* if all the steps in a given direction are taken consecutively (thus a sorted walk is direct).

PROPOSITION 3.11. *For w sorted, and $(1 + q^2)$ and each λ_i invertible, $\epsilon \cdot_w$ commutes with $H_{\mu(w)}$.*

Proof. Let μ^t denote the tth interval of $\{1, 2, \ldots, n\}$ in the ordered partition $\mu(w)$; that is, the tth set of integers fixed under the action of the Young subgroup $S_{\mu(w)}$ of S_n on $\{1, 2, \ldots, n\}$. We require to show, for each t, that the factors in $\epsilon \cdot_w$ involving $X_{\mu^t} := \{X_i \mid i \in \mu^t\}$ constitute a symmetric polynomial in these variables, and hence commute with the tth factor algebra in $H_{\mu(w)}$. (The remaining factor algebras commute with these variables by equation (17).) Our walk is of the form $w = 111 \ldots 22 \ldots tt \ldots$, and the factors in question are (by Proposition 3.10) those written out explicitly in:

$$\epsilon \cdot_w = \epsilon \cdot_{111\ldots 22\ldots} \prod_{i=t+1}^{d} (X_a - \lambda_i) \left(\prod_{i=1}^{t-1} (X_a - q^{-2}\lambda_i)(X_a - q^{2\mu_i(w)}\lambda_i) \right)$$

$$\prod_{b \in \mu^t \setminus \{a\}} \left((X_b - q^{-2}\lambda_t) \prod_{i=t+1}^{d} (X_b - \lambda_i) \left(\prod_{i=1}^{t-1} (X_b - q^{-2}\lambda_i)(X_b - q^{2\mu_i(w)}\lambda_i) \right) \right) \ldots$$

(where X_a is the first X_i in X_{μ^t}). It will be apparent that this is rendered symmetric by multiplying by $(X_a - q^{-2}\lambda_t)$, but $(X_a - q^{-2}\lambda_t)\epsilon \cdot_w = (\lambda_t - q^{-2}\lambda_t)\epsilon \cdot_w$ so it is *already* symmetric, provided that $(1 - q^{-2})\lambda_t$ is invertible. $\square$

A similar property holds for direct walks.

PROPOSITION 3.12. *If w takes the form $111 \ldots 22 \ldots$, then $\epsilon_w e_w = e_w \epsilon_w$ (and similarly for the preidempotent forms $\epsilon \cdot$ and $\mathbf{e}$). The modules $\Delta_{\mu(w)} = \mathrm{H}e_w \epsilon \cdot_w$ are the left standard modules of H with these weights.*

Proof. Note that $e_w \in H_{\mu(w)}$, and apply Proposition 3.11. $\square$

Write $\mu \geqslant \nu$ if every change of direction in $w(\mu)$ occurs at the same step as one in $w(\nu)$. (Note that $w(\nu)$ may have changes at other points as well.) Note also that $\mu \geqslant \nu$ implies that $e_{w(\mu)} e_{w(\nu)} = e_{w(\mu)}$, and hence $\mathbf{e}_{w(\mu)} \mathbf{e}_{w(\nu)} = \kappa_\nu \mathbf{e}_{w(\mu)}$ for some $\kappa_\nu \in \mathcal{A}$.

PROPOSITION 3.13. *Suppose that $w(\mu) \sim^{\mathcal{G}(k)} w(\nu)$, and $\mu \geqslant \nu$, and $\kappa_\nu \otimes k \neq 0$. Then $\Delta_\mu \hookrightarrow \Delta_\nu$.*

Proof.

$$\Delta_\mu = \mathrm{H}e_{w(\mu)} \epsilon \cdot_{w(\mu)} = \mathrm{H}e_{w(\mu)} \mathbf{e}_{w(\nu)} \epsilon \cdot_{w(\mu)} \hookrightarrow \mathrm{H}e_{w(\nu)} \epsilon \cdot_{w(\mu)} \cong \mathrm{H}e_{w(\nu)} \epsilon \cdot_{w(\nu)} = \Delta_\nu. \ \square$$

4. The case $d = 2$ and the blob algebra

4.1. Idempotents in $\mathrm{H}(n, 2)$

The primitive and central idempotents $e_n^{\pm l}$ corresponding to the four one-dimensional representations of $\mathrm{H}(n, 2)$ over $\mathcal{K}$ may be constructed as follows.

Fixing $d = 2$, define

$$P_n = P_n^{+2} = q^{n-1}[n]\left(q^{2n-2}\lambda_2 - \lambda_1\right);$$

also, $P_n^{-2} = t P_n$, $P_n^{+1} = s P_n$ and $P_n^{-1} = st P_n$.

PROPOSITION 4.1. *Set*

$$\alpha_j = \frac{-\lambda_1}{P_j}, \quad \beta_j = \frac{q\,P_{j-1}}{P_j} \quad and \quad \gamma_j = \frac{q^{2j-2}}{P_j}. \tag{23}$$

Then $e_0^{+2} = 1$ *and*

$$e_{j+1}^{+2} = e_j^{+2} \left(\alpha_{j+1} + \beta_{j+1} g_j + \gamma_{j+1} X_{j+1} \right) e_j^{+2}. \tag{24}$$

Also, $e_j^{-2} = t e_j^{+2}$, $e_j^{+1} = s e_j^{+2}$, *and* $e_j^{-1} = t s e_j^{+2}$.

Proof. See Section 5.1 (or simply consider $g_{j-1} e_j^{+2}$). $\qquad\square$

Examples.

$$e_1^{-2} = e_1^{+2} = \frac{X_1 - \lambda_1}{\lambda_2 - \lambda_1};$$

$$e_2^{-2} = \left(\frac{X - \lambda_1}{\lambda_2 - \lambda_1} \right) \frac{-\lambda_1 + q^{-2} g_1 X g_1 - q^{-1}(\lambda_2 - \lambda_1) g_1}{(1 + q^{-2})(q^{-2}\lambda_2 - \lambda_1)} \left(\frac{X - \lambda_1}{\lambda_2 - \lambda_1} \right). \tag{25}$$

Define

$$\Sigma = (X_1 + X_2 - (\lambda_1 + \lambda_2)) \quad and \quad \Pi = (X_1 X_2 - \lambda_1 \lambda_2).$$

Note that $\mathfrak{D}_2^2 = \{1, \Sigma, \Pi, \Sigma g_1, \Pi g_1\}$ is a basis of $Z(H(2,2))$ (see Appendix A, equation (56)).

In terms of $\mathfrak{C}_2^2$ and $\mathfrak{D}_2^2$, we have

$$e_2^{-2}$$
$$= \frac{\lambda_1^2 - \lambda_1(X + X_2) + X X_2 + q^{-1}\lambda_1(X + X_2 - (\lambda_1 + \lambda_2))g_1 - q^{-1}(X X_2 - \lambda_1\lambda_2)g_1}{(\lambda_1 - \lambda_2)(1 + q^{-2})(q^{-2}\lambda_2 - \lambda_1)}$$
$$= \frac{(-\lambda_1 \Sigma + \Pi)(1 - q^{-1}g_1)}{(\lambda_2 - \lambda_1)(q^{-2}\lambda_2 - \lambda_1)(1 + q^{-2})}, \tag{26}$$

which form manifests the centrality of this idempotent. Note that the preidempotent

$$\mathbf{e}_2^{-2} = (-\lambda_1 \Sigma + \Pi)(1 - q^{-1}g_1)$$

coincides with its *s*-image (up to a unit in $\mathcal{A}$) if and only if $\lambda_1 = \lambda_2$ in k. However,

$$e_2^{-2} + e_2^{-1} = \frac{q^{-2}\Sigma + (1 + q^{-2})\Pi}{(q^{-2}\lambda_2 - \lambda_1)(q^{-2}\lambda_1 - \lambda_2)} \frac{(q - g_1)}{[2]}.$$
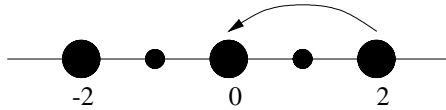
A remark is in order on denominators and idempotent decompositions of 1. The idempotent decomposition of $1 = e_1^{+2} + e_1^{+1} \in H(1,2)$ is not defined in the case where $\lambda_1 = \lambda_2$ over k, and the radical $\mathrm{rad}(H(1,2)) = k\mathbf{e}_1^{+1}$. Obviously, $R_{+1} = R_{\pm2}$ for any n in this case. The decomposition of

$$1 = (e_1^{+2} + e_1^{+1}) + e_{((1),(1))} + (e_1^{-2} + e_1^{-1}) \in H(2,2)$$

thus has the same limitation: the bracketed sums do not split. Similarly, when $q = -q^{-1}$, we have $R_{+i} = R_{-i}$ (for any n). More interesting is the case where $\lambda_2 = q^2\lambda_1$. Here, both e_2^{-2} and e_2^{+1} are undefined. Clearly, $R_{+1} \neq R_{-2}$ (unless $q^2 = -1$, $\lambda_i = 0$), so $e_2^{-2} + e_2^{+1}$ is also undefined, but e_2^{+2} and e_2^{-1} are well defined, so

$$e_2^{-2} + e_2^{+1} + e_{((1),(1))}$$

does not split and R_{+1} and R_{-2} (that is, $\Delta_{((2),)}$ and $\Delta_{(,(1^2))}$) must both be composition factors of $\Delta_{((1),(1))}$. Note that at first sight this seems problematic for our proposed $H^{\mathcal{D}}$ weight space; however, over this k we may identify R_{-2} with the simple head of $\Delta_{((1),(1))}$ and label it accordingly (rather than by its H(n, d) label, which is $(, (1^2))$). This is a good paradigm for the subtleties with labels in realising ingredient I2(ii) (*confer* [59]). In terms of the A_1 integral weight set $\mathbb{Z} \subset \mathbb{R}$ (recall that $\mathfrak{P}^2((, (2))) = -2$, $\mathfrak{P}^2(((1), (1))) = 0$ and $\mathfrak{P}^2(((2),)) = 2$), we depict the standard modules and morphism in the following manner.

It will be convenient to note the equality

$$(X - \lambda_2)(-\lambda_1 \Sigma + \Pi) = 0 \tag{27}$$

and its s image; and hence that

$$(-\lambda_1 \Sigma + \Pi)e_2^{-2} = (-\lambda_1 \Sigma + \Pi)\frac{g_1 - q}{-q^{-1} - q} \tag{28}$$

and its s image.

Now consider the algebras $H^{\mathcal{D}}(n, 2)$ obtained by quotienting by $\mathbf{e}_2^{-1} = 0$ and $\mathbf{e}_2^{-2} = 0$.

First consider $n = 2$. A basis of the ideal $\mathcal{D}_2(2)$ is simply $\{\mathbf{e}_2^{-1}, \mathbf{e}_2^{-2}\}$, for $\lambda_1 \neq \lambda_2$. The basis $\mathfrak{C}_2^2$ of H$(2, 2)$ is

$$\{1, X_1, X_2, X_1X_2, g_1, X_1g_1, X_2g_1, X_1X_2g_1\}.$$

We can think of using the relation $e_2^{-2} = 0$ to eliminate $X_1X_2g_1$, and then using its s-partner to eliminate g_1, to obtain a basis for $H^{\mathcal{D}}(2, 2)$ (see also Section 5.3). Specifically, it is convenient to represent the quotient relations as

$$\begin{aligned}(X_1 + X_2 - (\lambda_1 + \lambda_2))(g_1 - q) &= 0; \\ (X_1X_2 - \lambda_1\lambda_2)(g_1 - q) &= 0.\end{aligned} \tag{29}$$

Note that these two relations generate the *same* ideal.

Alternatively, rewriting $g_i - q =: u_i$ (so $u_i^2 = -(q+q^{-1})u_i$) and $X - \lambda_1 =: v$, we have

$$vu_1vu_1 = (\lambda_1q^{-1} - \lambda_2q)vu_1 \quad \text{and} \quad u_1vu_1v = (\lambda_1q^{-1} - \lambda_2q)u_1v, \tag{30}$$

and similarly for the image under s. Combining (or by applying $(g_1 - q)$ to the first equation in (29) from the left), we obtain

$$u_1vu_1 = (\lambda_1q^{-1} - \lambda_2q)u_1 \tag{31}$$

(and its s image).

4.2. $H^{\mathcal{D}}(n, 2)$ *and the blob algebra*

The blob algebra b_n [55] may be defined as in Section 1. (That the presentation in Section 1 is equivalent to the usual definition in terms of a basis of decorated Temperley–Lieb diagrams may be verified by a straightforward (if tedious) generalisation of the corresponding exercise for the ordinary Temperley–Lieb algebra as in [50].) The properties I1–I6 are established for the blob algebra in [57].

In the parametrisation of b_n by q and m, where $y_e = -[m-1]/[m]$, it is convenient to replace e_- by the rescaled generator $U_0 = -[m]e_-$, or rather to replace e_- with U_0, with relations $U_0^2 = -[m]U_0$ and

$$U_1 U_0 U_1 = [m-1]U_1. \tag{32}$$

The variant with these relations is isomorphic to the original, except (obviously) when $[m] = 0$.

PROPOSITION 4.2. *Let m, q, λ_1 and λ_2 be such that*

$$[m-1](\lambda_1 - \lambda_2) = (\lambda_1 q^{-1} - \lambda_2 q)[m].$$

(Note that $\lambda_1 \neq \lambda_2$ unless $q^2 = 1$ and $m \to \infty$, or $[m] = 0$; otherwise, putting $\lambda_1/\lambda_2 = q^{-2r}$ and $m = r$, we may take $\lambda_1 = q^m/(q - q^{-1})$, and then $\lambda_1 - \lambda_2 = [m]$). Then there is an isomorphism

$$\phi : H^{\mathcal{D}}(n, 2) \to b_n$$

given by $\phi(u_i) = U_i$ and $\phi(v) = U_0$. There is a homomorphism $\psi : b_n \to H^{\mathcal{D}}(n, 2)$, given by $\psi(U_i) = u_i$, $\psi(U_0) = v$.

Proof. It is straightforward to verify the cyclotomic Hecke relations under ϕ. A direct calculation also shows that the image under ϕ of the *numerator* of e_2^{-2}, as in equation (25), vanishes identically if we use the form of λ_1 given in the proposition. The image of e_2^{-1} vanishes similarly. Thus ϕ is a surjective homomorphism (except possibly when $q^2 = 1$). A dimension count reveals that this surjective map is generically an isomorphism.

For ψ, equation (31) verifies relation (32). It is now enough to check that $u_1 u_2 u_1 = u_1$ (that is, in H(3, 2), that $u_1 u_2 u_1 - u_1 = u_1(u_2 u_1 - 1) = [3]!e_3^- \in \mathcal{D}_2$). Write $f = u_1 u_2 u_1 - u_1$ and consider $f \mathcal{D}_2 f$. (If $f \in \mathcal{D}_2$, then $f \in f \mathcal{D}_2 f$, at least if $[3]!$ is invertible, and this is a much smaller and simpler object to work with.) This (pre)idempotent subalgebra includes

$$f(X_1 + X_2 - (\lambda_1 + \lambda_2))f = f((1 + q^{-2})X - (\lambda_1 + \lambda_2))f,$$
$$f(X_1 X_2 - \lambda_1 \lambda_2)f = f(-q^{-1}Xg_1 X - \lambda_1 \lambda_2)f$$

and

$$f Xg_1 g_2 (X_1 + X_2 - (\lambda_1 + \lambda_2))f = f(Xg_1 g_2 X + Xg_1 g_2 g_1 Xg_1 - Xg_1 g_2 (\lambda_1 + \lambda_2))f$$
$$= f(-q^{-1}Xg_1 X + Xg_2 g_1 g_2 Xg_1 - q^{-2}(\lambda_1 + \lambda_2)X)f$$
$$= f(-(q^{-1} + q^{-3})Xg_1 X - q^{-2}(\lambda_1 + \lambda_2)X)f.$$

A linear combination of these is

$$[3]!((1 + q^{-4})\lambda_1 \lambda_2 - q^{-2}(\lambda_1^2 + \lambda_2^2))f,$$

so $f \in f \mathcal{D}_2 f$, at least in an open subset of the parameter space. The coefficient may be rewritten as $[3]!q^{-2}\lambda_1 \lambda_2 (q - q^{-1})^2 [m+1][m-1]$, so this covers most of the interesting cases. The remainder require a more tedious calculation. $\square$

COROLLARY 4.3. *The finite characteristic b_n decomposition matrices in [15] are a subset of type-B Hecke decomposition matrices, via the correspondence given in Section 1.3.*

Recall that $m = \pm 1$ implies the existence of a quotient algebra isomorphic to the Temperley–Lieb algebra (in addition to the noted Temperley–Lieb subalgebra). The quotient identifies $X = 1$ (or q^2).

5.   *On general d*

5.1.   *Idempotents in* $H(n, d)$

The primitive and central idempotents corresponding to the $2d$ one-dimensional representations of $H(n, d)$ over $\mathcal{K}$ may be constructed as follows. Fixing d, define

$$P_n = P_n^{+1} = q^{n-1}[n]\prod_{i \neq 1}\left(q^{2n-2}\lambda_1 - \lambda_i\right). \tag{33}$$

$P_n^{-1} = t P_n$, $P_n^{+2} = s P_n$, $P_n^{+3} = s^2 P_n$, and so on. Define $\lambda^{(0)} = 1$ and

$$\lambda^{(i)} = \sum_{d \geqslant j_1 > j_2 > \cdots > j_i > 1}\left(\prod_{l=1}^{i} -\lambda_{j_l}\right)$$

(a sum over descending positive integer sequences $(j_1, j_2, \ldots, j_i)$, with $j_1 \leqslant d$).

PROPOSITION 5.1. *Set*

$$\beta_{j+1} = \frac{q P_j}{P_{j+1}}; \tag{34}$$

$$\alpha_j^0 = \frac{\prod_{i \neq 1}(-\lambda_i)}{P_j}; \tag{35}$$

$$\alpha_{j+1}^{d-i} = \frac{q^{2j}\lambda^{(i-1)} + \sum_{l=2}^{i} q^{(i-1)(2j-2)}(q^{2j} - 1)\lambda_1^{l-1}\lambda^{(i-l)}}{P_{j+1}} \qquad (1 \leqslant i < d).$$

Then $e_0^{+1} = 1$,

$$e_{j+1}^{+1} = e_j^{+1}\left(\beta_{j+1}g_j + \sum_{i=0}^{d-1}\alpha_{j+1}^i X_{j+1}^i\right)e_j^{+1}, \tag{36}$$

$e_j^{-1} = t e_j^{+1}$, *and so on.*

For example, $\beta_1 = 0$, $\alpha_1^i = \lambda^{(d-1-i)}$, and

$$e_1^{+1} = \frac{\prod_{i \neq 1}(X_1 - \lambda_i)}{P_1}.$$

(There are more examples in Section 5.2.)

Proof of Proposition 5.1. We work by induction on j, with the example above as base. Firstly, note that

$$e_{j+1}^{+1}e_j^{+1} = e_j^{+1}e_{j+1}^{+1} = e_{j+1}^{+1};$$

so, from Proposition 2.2, equation (36) gives a correct *form* for the R_{+1} idempotent up to coefficients. Then note that in this form it is sufficient (see equation (10)) to check the identity $(g_j - q)e_{j+1}^{+1} = 0$, and the normalisation (that is, the idempotency). The former is a direct calculation, and the latter may be checked by evaluating in R_{+1}; that is, by substituting q for g_i and λ_1 for X (in which case e_{j+1}^{+1} must evaluate to 1). To begin, we rewrite the claimed

expression for $(g_j - q)e_{j+1}^{+1}$ as

$$(g_j - q)\left(e_j^{+1}\left(\beta_{j+1}g_j\right)e_j^{+1} + \left(\sum_{i=0}^{d-1}\alpha_{j+1}^i X_{j+1}^i\right)e_j^{+1}\right)$$

using commutation properties. By the inductive assumption, this is equal to

$$(g_j - q)\left(e_{j-1}^{+1}\left(\beta_j g_{j-1} + \sum_{i=0}^{d-1}\alpha_j^i X_j^i\right)(\beta_{j+1}g_j)e_j^{+1} + \left(\sum_{i=0}^{d-1}\alpha_{j+1}^i X_{j+1}^i\right)e_j^{+1}\right)$$

$$= e_{j-1}^{+1}\beta_{j+1}\left(\beta_j(g_j - q)g_{j-1}g_j + (g_j - q)\left(\sum_{i=0}^{d-1}\alpha_j^i X_j^i\right)g_j\right)e_j^{+1}$$

$$+ (g_j - q)\left(\sum_{i=0}^{d-1}\alpha_{j+1}^i X_{j+1}^i\right)e_j^{+1}$$

$$= e_{j-1}^{+1}\beta_{j+1}\left((g_j - q)\left(\sum_{i=0}^{d-1}\alpha_j^i X_j^i\right)g_j\right)e_j^{+1} + (g_j - q)\left(\sum_{i=0}^{d-1}\alpha_{j+1}^i X_{j+1}^i\right)e_j^{+1}.$$

Now, using the commutation rules on the first summand to bring the factor g_j forward through the X_j^is, we find that the expression equals

$$(g_j - q)\left(\left(g_j\beta_{j+1}\alpha_j^0 + \alpha_{j+1}^0\right)\right.$$

$$\left. + \sum_{i=1}^{d-1}\left(\beta_{j+1}\left(-q\alpha_j^i - (q - q^{-1})\left(\sum_{l=i+1}^{d-1}\alpha_j^l (X_j)^{l-i}\right)\right) + \alpha_{j+1}^i\right)X_{j+1}^i\right)e_j^{+1}.$$

But $X_j e_j^{+1} = q^{2j-2}\lambda_1 e_j^{+1}$, so we see that this is equal to

$$(g_j - q)\left(\left(-q^{-1}\beta_{j+1}\alpha_j^0 + \alpha_{j+1}^0\right)\right.$$

$$\left. + \sum_{i=1}^{d-1}\left(\beta_{j+1}\left(-q\alpha_j^i - (q - q^{-1})\left(\sum_{l=i+1}^{d-1}\alpha_j^l\left(q^{2j-2}\lambda_1\right)^{l-i}\right)\right) + \alpha_{j+1}^i\right)X_{j+1}^i\right)e_j^{+1}.$$

Then, equating coefficients to zero, we get

$$\alpha_{j+1}^0 = q^{-1}\alpha_j^0\beta_{j+1},$$
$$\alpha_{j+1}^{d-1} = q\alpha_j^{d-1}\beta_{j+1},$$
$$\alpha_{j+1}^{d-2} = \left(q\alpha_j^{d-2} + (q - q^{-1})\alpha_j^{d-1}(q^{2j-2}\lambda_1)\right)\beta_{j+1},$$

and

$$\alpha_{j+1}^i = \left(q\alpha_j^i + (q - q^{-1})\sum_{l=i+1}^{d-1}\alpha_j^l(q^{2j-2}\lambda_1)^{l-i}\right)\beta_{j+1} \qquad (0 < i < d).$$

Without regard for normalization, any one coefficient may be chosen arbitrarily, so (without loss of generality) try $\beta_{j+1} = q\,P_j/P_{j+1}$. Then $\alpha_{j+1}^{d-1} = \rho_{j+1}^{d-1}/P_{j+1}$, where

$$\rho_{j+1}^{d-1} = q^{2j}, \quad \alpha_{j+1}^0 = \frac{\lambda^{(d-1)}}{P_{j+1}}, \quad \text{and} \quad \alpha_{j+1}^{d-2} = \frac{\rho_{j+1}^{d-2}\lambda_1 + \rho_{j+1}^{d-1}\lambda^{(1)}}{P_{j+1}},$$

for some ρ_{j+1}^{d-2}. Then

$$\rho_{j+1}^{d-2} = q^2\rho_j^{d-2} + (q^2-1)q^{2j-2}q^{2j-2} = q^{2j-2}(q^{2j}-1).$$

Similarly,

$$\alpha_{j+1}^{d-3} = \frac{\rho_{j+1}^{d-3}\lambda_1^2 + \rho_{j+1}^{d-2}\lambda^{(1)}\lambda_1 + \rho_{j+1}^{d-1}\lambda^{(2)}}{P_{j+1}},$$

and, for $1 \leqslant i < d$,

$$\alpha_{j+1}^{d-i} = \frac{\sum_{l=1}^{i}\rho_{j+1}^{d-l}\lambda_1^{l-1}\lambda^{(i-l)}}{P_{j+1}}$$

for some ρ_{j+1}^-. Then

$$\rho_{j+1}^{d-3} = q^2\rho_j^{d-3} + (q^2-1)(q^{2j-2}q^{2j-4}(q^{2j-2}-1) + (q^{2j-2})^3) = q^{4j-4}(q^{2j}-1)$$

$$\rho_{j+1}^{d-4} = q^2\rho_j^{d-4}$$
$$+ (q^2-1)((q^{2j-2})q^{4j-8}(q^{2j-2}-1) + (q^{2j-2})^2q^{2j-4}(q^{2j-2}-1) + (q^{2j-2})^4)$$
$$= q^{6j-6}(q^{2j}-1),$$

and

$$\rho_{j+1}^{d-i} = q^{(i-1)(2j-2)}(q^{2j}-1) \qquad (1 < i < d).$$

Finally, the normalization condition,

$$\lambda^{(d-1)} + \sum_{i=1}^{d-1}\left(\left(q^{2j}\lambda^{(i-1)} + \sum_{l=2}^{i}q^{(l-1)(2j-2)}(q^{2j}-1)\lambda_1^{l-1}\lambda^{(i-l)}\right)\left(q^{2j}\lambda_1\right)^{d-i}\right)$$
$$= P_{j+1} - q^2P_j$$

is verified by direct computation. $\qquad\square$

By inspection of these idempotents, we see that the algebras will not be generic in specialisations in which $\lambda_i/\lambda_j = q^{2r}$ for some $i, j, r \in \mathbb{N}$, nor in certain specialisations in which q is a root of unity (this also follows immediately from [1, 59]). Let us disregard, for the moment, the cases in which X is degenerate or non-invertible. Then, noting that rescaling all the λ_is by the same factor produces an isomorphic algebra, we can fix $\lambda_1 = 1$ (say) and adopt as parameters q and λ_i/λ_1; see [59]. It is illuminating to proceed by example (and *confer* [58, Section 5]).

5.2. *The case $d = 3$*

The primitive and central idempotents corresponding to the six one-dimensional representations of H$(n, 3)$ may be constructed as follows. Fixing $d = 3$, define $P_n = P_n^{+1}$ and

$P_n^{\pm i}$ as in equation (33). For example,

$$P_1 = P_1^{\pm 1} = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3);$$
$$P_2 = q[2](q^2\lambda_1 - \lambda_2)(q^2\lambda_1 - \lambda_3).$$

As before, $\beta_j = q\,P_{j-1}/P_j$, and here

$$\alpha_j^0 = \frac{\prod_{i \neq 1}(-\lambda_i)}{P_j};$$

$$\alpha_j^1 = \frac{q^{2j-4}(q^{2j-2} - 1)\lambda_1 - q^{2j-2}(\lambda_2 + \lambda_3)}{P_j}; \tag{37}$$

$$\alpha_j^2 = \frac{q^{2j-2}}{P_j}.$$

Compute $e_j^{\pm l}$ by $e_0^{+1} = 1$, and then

$$e_{j+1}^{+1} = e_j^{+1}\left(\beta_{j+1}g_j + \alpha_{j+1}^0 + \alpha_{j+1}^1 X_{j+1} + \alpha_{j+1}^2 X_{j+1}^2\right)e_j^{+1}, \tag{38}$$

$e_j^{-1} = t e_j^{+1}$, and so on. For example

$$e_1^{+1} = \frac{X_1^2 - (\lambda_2 + \lambda_3)X_1 + \lambda_2\lambda_3}{P_1}$$

and

$$e_2^{+1} = e_1^{+1}\left(\frac{q P_1 g_1 + \lambda_2\lambda_3 + ((q^2 - 1)\lambda_1 - q^2(\lambda_2 + \lambda_3))X_2 + q^2 X_2^2}{P_2}\right)e_1^{+1}$$

$$= q\,\frac{(X_1^2 - (\lambda_2 + \lambda_3)X_1 + \lambda_2\lambda_3)}{P_1}\,\frac{(X_2^2 - (\lambda_2 + \lambda_3)X_2 + \lambda_2\lambda_3)(g_1 + q^{-1})}{P_2}.$$

Given these results, let us consider the generalised Soergel procedure for $d = 3$ corresponding to that in Section 1.2 for $d = 2$. For illustration, we consider a field k in which the equations $\lambda_i/\lambda_1 = q^{-n_i}$ ($i = 2, 3$) and $\lambda_1/\lambda_3 = q^{-n_1}$ are solved for positive integer n_i *only* in the case $n_2 = 2, n_3 = 4$ (in particular, in this instance q is not a root of unity). Figure 4 illustrates the location of the corresponding reflection hyperplanes (shown as thick lines) in the $H^\mathcal{D}$ (that is, A_2) weight space. The Kazhdan–Lusztig polynomials for this geometry are given in the figure on [**58**, p. 1289]. The claim, then, is that if μ is a weight in the fundamental alcove A^0 (such as 0), then $\Delta(\mu)$ has the following simple content (and Loewy structure).

$$
\begin{array}{ccc}
 & L_\mu & \\
L_{\mu.s} & & L_{\mu.t} \\
L_{\mu.st} & & L_{\mu.ts} \\
 & L_{\mu.sts} &
\end{array}
$$

(Here s and t are the walls of A^0 – we are abusing the notation $A.s$ of Section 1.2 to apply to weights in the obvious way.) Of course, localising at small n, some of these simple modules will vanish. Let us consider $n = 0, 1, 2, 3$.
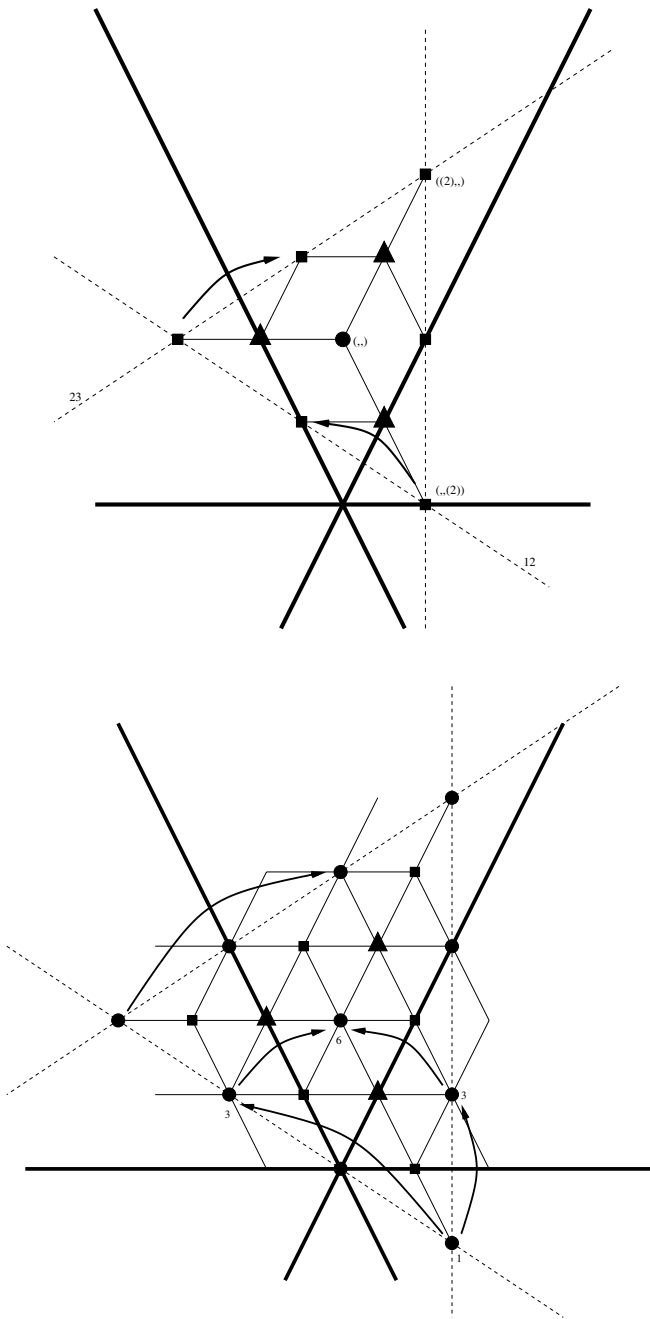
Figure 4: Weights of $H^{\mathcal{D}}(n,3)$ in weight space formalism: (a) for $n = 0, 1, 2$; and (b) for $n = 0, 1, 2, 3$. Weight set $\Lambda(0)$ consists of the empty weight (shown as a black circle marked (,,) in (a); $\Lambda(1)$ consists of the three weights marked with triangles, and so on (see the main text).

(For any k) we have $H^{\mathcal{D}}(0, 3) \cong k$, with basis $\{1\}$. The weight for the corresponding simple module is the innermost dot in the figures (marked (, ,) in Figure 4(a)). For $H^{\mathcal{D}}(1, 3)$, the idempotents e_1^{+i} are all well defined, and $1 = \sum_{i=1}^{3} e_1^{+i}$ is an idempotent decomposition into primitive idempotents. (The corresponding simples are marked as triangles.)

The set of weights for $H^{\mathcal{D}}(2, 3)$ are marked with squares. (The weights corresponding to the multipartitions $((2), ,)$ and $(, , (2))$ have been explicitly labelled, to fix the coordinate system.) Note that e_2^{+i} is divergent in the case $i = 2, 3$. The easiest way to see what is happening at $n = 2$ is to recall the $d = 3$ version of Proposition 1.2.

PROPOSITION 5.2. *Fixing k, there are three ways to quotient to pass from $H^{\mathcal{D}}(n, 3)$ to $H^{\mathcal{D}}(n, 2)$, where $n > 0$.*

The corresponding subsets of the set of weights for $H^{\mathcal{D}}(2, 3)$ lie in three straight lines: the dashed lines shown in Figure 4(a). Since we have shown that the $d = 2$ algebra is isomorphic to a blob algebra in each of these cases, we can give a complete description. The lines marked 12 and 23 correspond to singly critical blob algebras, with reflection points the intersection points of these lines with the thick lines shown. Accordingly, there are injective standard module morphisms ('reflection' morphisms) as indicated by arrows. The other blob line is a semisimple quotient (from the parameters, it is nominally a singly critical case, but the reflection point lies at the outside edge of localisation of weight space to $n = 2$, so there is no image of any $n = 2$ weight in it). It is straightforward to show that there are no other non-trivial morphisms. Since the $d = 3$ alcove structure is determined by the three $d = 2$ quotients, and since the morphisms indicated are all also blob module morphisms, their location is necessarily consistent with the $d = 3$ Soergel procedure.

For $H^{\mathcal{D}}(3, 3)$, the three blob quotients correspond to the three dashed lines shown in Figure 4(b). The reflection morphisms shown within these lines are again simply blob morphisms. (Note that the reflection point on the third line now lies within the localisation of weight space.) The only question, then, concerns the morphisms into the module with A_2-weight 0. This weight is marked in the figure by the rank of the module, $\mathrm{Rank}(\Delta(0)|_{n=3}) = 6$. (This weight coincides with the weight for the unique simple of $H^{\mathcal{D}}(0, 3)$ in our scheme, since that module is a localisation of $\Delta(0)|_{n=3}$.) A straightforward Frobenius reciprocity argument (using the morphisms at level $n = 2$ and the generic restriction rules) shows that there are at least two homomorphisms into this module; but it does not uniquely determine the domain in either case. To confirm the indicated morphisms, consider the walk basis of $\Delta(0)|_{n=3}$ and the walk orbits of these walks – see Figure 5. Note from the last of these pictures that $321 \sim^{\mathcal{G}(k)} 333$. It follows that there is a homomorphism between the corresponding standard modules, by Proposition 3.13. (Note that walk 321 is not sorted, but that the weaker condition of all steps in a given direction being taken consecutively is sufficient for this construction.) The other claimed morphisms follow similarly.

It is worth noting that $\Delta(0)|_{n=3}$ may generically be defined as $\Delta(0)|_{n=3} = H^{\mathcal{D}}(3, 3)e_{(1^3)}$, where $e_{(1^3)} = e_3^-$ is the usual q-antisymmetriser [58] (that is, $\Delta(0)|_{n=3}$ is the globalisation of the unique simple of $H^{\mathcal{D}}(0, 3)$). This is an illuminating construct to consider in any case. The module $H(3, 3)e_{(1^3)}$ is, of course, much larger. By Proposition 2.1 it has basis $\{X^a e_{(1^3)} \mid a \in \{0, 1, 2\}^3\}$. To determine a basis in our case, one must use the vanishing of e_2^{-i} (which generates a significant part of $He_{(1^3)}$). This problem is dealt with elsewhere; see [16].
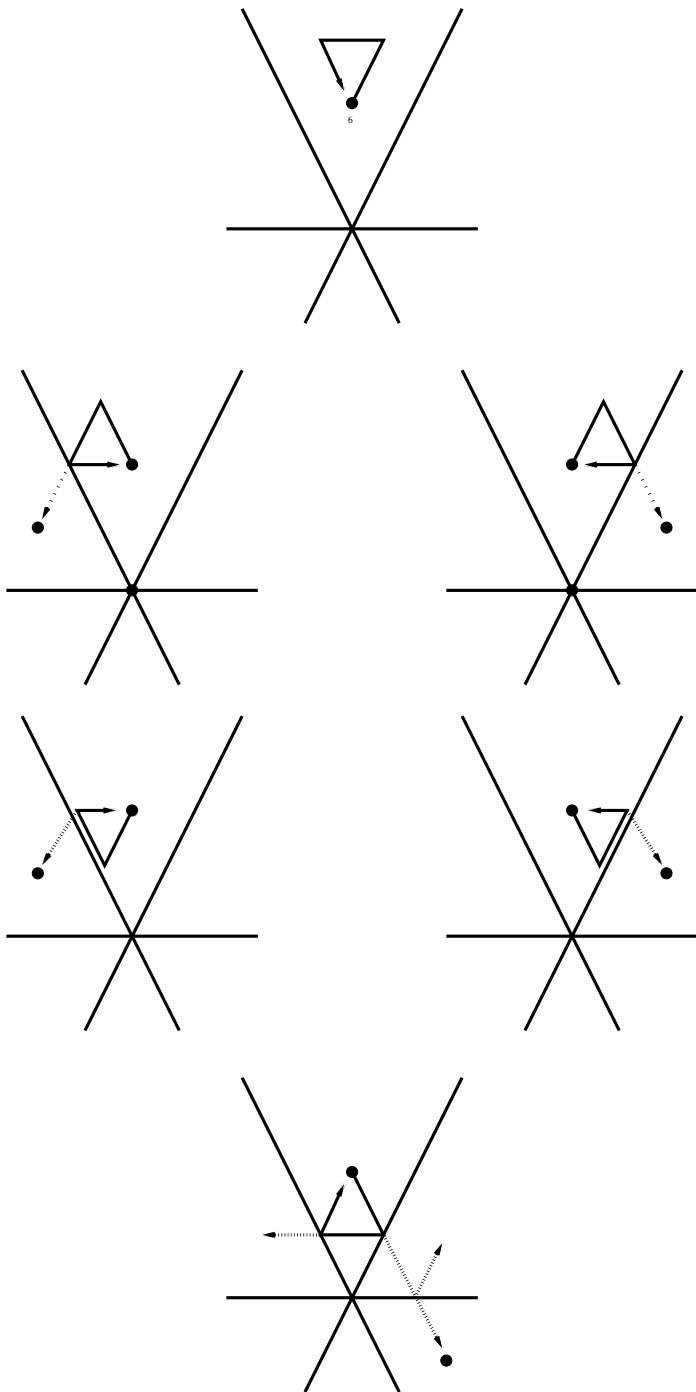
Figure 5: The six walks in $T_0(3)$, and their walk orbits.

5.3. *Conjectured basis for $H^{\mathcal{D}}(n, d)$*

Consider $w \in S_n$, a permutation, and let $\mathsf{S}(w)$ be a corresponding reduced word in the Coxeter generators $\{\sigma_i\}$. The set map from $\mathsf{S}(S_n)$ to $\mathfrak{B}_n$ given by $\sigma_i \rightarrow g_i$ is an isomorphism [34]. Thus from $w \in \mathfrak{B}_n$ we may read off a permutation (which we shall also call w). Given such a permutation, define a symmetric relation $(w) \subset \underline{n} \times \underline{n}$ by: $i(w)j$ if $i > j$ implies that $w(i) < w(j)$ (and symmetry). That is, $i(w)j$ if the lines i and j cross in the diagram of w.

CLAIM 5.3. $\{X^{(a_1, a_2, \dots)} w \in \mathfrak{C}_n \mid i(w)j \text{ implies that } a_i \neq a_j\}$ *is a basis for* $H^{\mathcal{D}}(n, d)$.

Idea of the proof. Note that the dimensions are right (consider the Robinson–Schensted correspondence in the form given in [67], or, for example, in [66]). Thus it is enough to show spanning or linear independence. For the latter, it is convenient to have a representation $\mathcal{R}$ of $H^{\mathcal{D}}$ (if it is linear independent in $\mathcal{R}$, then the claim is proven *and* $\mathcal{R}$ is faithful).

The remainder of the paper is concerned with representations of $H^{\mathcal{D}}$.

6. *On representations of b_n and $H^{\mathcal{D}}(n, d)$*

The ordinary Temperley–Lieb algebra has a powerful diagram calculus (see [50] for a review). One motivation for the introduction of the blob algebra [55] was to bring the utility of such a calculus to the representation theory of the periodic/affine algebras studied in [54] (see also [50, Chapter 8], as well as [60, 38, 27, 30] and the references therein). In fact, all the finite irreducible representations of these infinite-dimensional algebras may be constructed using [55]. (Although completeness is not shown there, see [28, 57].) From the point of view of lattice statistical mechanics, b_n also renders the 'seam' boundary conditions (as in [5]; see also [47], for example) of the ice-type model [4] into the algebraic formalism of Temperley and Lieb [68]. Indeed, there are a number of mathematical and physical reasons (in addition to the pursuit of our ingredient I2 for $H^{\mathcal{D}}(-, d)$) why a faithful 'tensor space' representation of the blob would be useful (*confer* [55, Section 4], as well as [60, 6, 22]).

By a *tensor space representation* of b_n (or $H^{\mathcal{D}}(n, d)$), we mean a representation *for each n* with underlying module of form $V(n) = V_{\text{aux}} \otimes V^{\otimes n}$ as a vector space, with V_{aux} and V finite-dimensional vector spaces (*confer* [40]), on which the generators act 'locally' (in particular, $b_{n-1} \subset b_n$ acts trivially on the last factor V, so the restriction of $V(n)$ is a manifest direct sum of dim V copies of $V(n-1)$); and a representation that is well defined in arbitrary specialisations.

Let $e \in H^{\mathcal{D}}(n+d, d)$ be idempotent. A module $V(n+d)$ for a tensor space representation over K is *globalisable by e* if e projects the last d tensor factors $V^{\otimes d} \rightarrow K$, and acts trivially on other tensor space factors.

To establish ingredient I2 *generically* for $H^{\mathcal{D}}(-, d)$ (that is, to establish that there exists idempotent $e_d \in H^{\mathcal{D}}(n+d, d)$ such that

$$e_d H^{\mathcal{D}}(n+d, d) e_d \cong H^{\mathcal{D}}(n, d); \tag{39}$$

confer [12, 21]), certain generalised *braid diagrams* may be used [58]. However, the following proposition holds.

PROPOSITION 6.1. *Suppose that $V(n+d)$ is the module for a faithful tensor space representation of $H^{\mathcal{D}}(n+d, d)$ over $\mathbb{C}$, and is globalisable by e_d. Then equation* (39) *holds.*

Proof. To see this, note that under these assumptions the actions of e_d and $H^{\mathcal{D}}(n, d)$ on $V(n + d)$ commute, and hence they commute in $H^{\mathcal{D}}(n + d, d)$. Thus

$$H^{\mathcal{D}}(n, d)e_d \subseteq e_d H^{\mathcal{D}}(n + d, d)e_d \subseteq H^{\mathcal{D}}(n + d, d)$$

is a sequence of inclusions of algebras, and $H^{\mathcal{D}}(n, d)e_d$ is isomorphic to $H^{\mathcal{D}}(n, d)$. Thus the action of $e_d H^{\mathcal{D}}(n+d, d)e_d$ on $V(n+d)$ would be isomorphic to the $H^{\mathcal{D}}(n, d)$ action on $V(n)$. Since the latter has a trivial kernel, this would establish equation (39) in general. □

Any faithful tensor space representation in which X acts non-trivially only in the first normal tensor factor V (and an otherwise redundant factor V^{aux}) would be a likely candidate, because of the way in which ingredient I2 works in the A_n case; see [**37**].

Let us briefly review this.

DEFINITION 6.2. The ordinary H_n action on $V_N^{\otimes n}$ (dual to that of $U_q sl_N$, as in equation (11)) is

$$M_N^q : A_n\text{-braid} \to \mathrm{End}(V_N^{\otimes n}),$$
$$g_i \mapsto 1 \otimes 1 \otimes \cdots \otimes \mathcal{M}^q \otimes 1 \otimes 1 \otimes \cdots \otimes 1,$$

where 1 denotes the unit matrix; and $\mathcal{M}^q$ (the ith factor) is given by

$$\mathcal{M}^q|_{N=2} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

$$\mathcal{M}^q|_{N=3} = \begin{pmatrix} q & 0 & 0 & 0 & & & & & \\ 0 & -q^{-1}+q & 0 & -1 & 0 & & & & \\ 0 & 0 & -q^{-1}+q & 0 & 0 & 0 & -1 & 0 & \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1}+q & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix},$$

and so on.

In particular, each such representation obeys a quadratic local relation with coefficients in $R = \mathbb{Z}[q, q^{-1}]$. (Specifically, $M_N^q((g_1 - q)(g_1 - (-q^{-1}))) = 0$.)

Putting $U_i = g_i - q$, we have

$$M_2^q(U_i U_{i\pm 1} U_i - U_i) = 0, \tag{40}$$

so M_2^q factors through $T_n(q)$. (In fact, it is faithful on $T_n(q)$; that is, $T_n(q) = H_n^2(q)$.)

Recall from Section 1.3 that e_N^- denotes the H_N q-symmetriser (normalisable as an idempotent, provided that $[N]!$ is invertible in K [**50**]). For $n \geqslant N$, V_N^n is the tensor space module for $H_n^N(q)$. It is easy to check that V_N^n is globalisable by e_N^-.

Recall also that the T_n action on V_2^n breaks up directly, over any field, into summands $\mathcal{P}_\lambda$ of fixed 'charge' or weight $\lambda \in \mathbb{N}_0$, and then

$$\mathcal{P}_\lambda = +_{\mu \geqslant \lambda} \Delta_\mu \tag{41}$$

(generically a direct sum).

It is desirable not only to have representations of b_n that act on tensor space, but also that they preserve some version of this charge conservation – that is, they are a direct sum of analogues of permutation representations. (Note that the tensor space representation in [**55**, Section 4] is neither full tilting [**22**] in general, nor charge-conserving.)

One way to proceed is to search for maps from b_n to $T_{n'}$ (for some n') (or, respectively, from $H^{\mathcal{D}}(n, d)$ to $H_{n'}^d$), and hence to obtain b_n-modules by restriction of M_2^q. Another possibility is to enrich suitable $T_{n'}$-modules with the property of a b_n-module by determining an action of the blob operator on them. We begin by investigating the latter.

6.1. *Generalised bialgebra construction*

Let $(M, \circ, e)$ be a finite monoid, and let A be a K-algebra with basis M and multiplication defined on this basis by $m_1 m_2 = k_{m_1, m_2}(\mathbf{q}) m_1 \circ m_2$, where $\mathbf{q}$ is some set of parameters and $k_{12}(\mathbf{q}) \in K$. (The possibilities for the coefficients will in general by constrained by M as follows:

$$m_1(m_2 m_3) = m_1 k_{m_2, m_3}(\mathbf{q})(m_2 \circ m_3) = k_{m_2, m_3}(\mathbf{q}) k_{m_1, m_2 \circ m_3}(\mathbf{q}) m_1 \circ (m_2 \circ m_3)$$
$$= k_{m_1, m_2}(\mathbf{q})(m_1 \circ m_2) m_3 = k_{m_1, m_2}(\mathbf{q}) k_{m_1 \circ m_2, m_3}(\mathbf{q})(m_1 \circ m_2) \circ m_3.$$
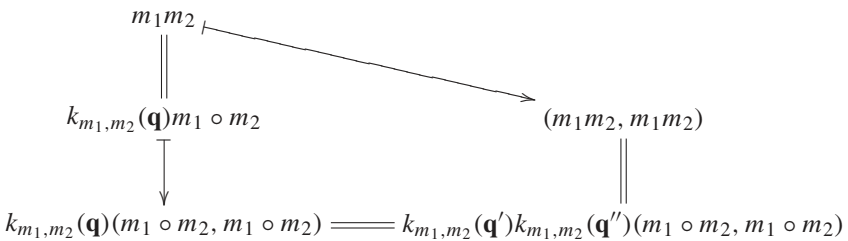
So $k_{m_2, m_3}(\mathbf{q}) k_{m_1, m_2 \circ m_3}(\mathbf{q}) = k_{m_1, m_2}(\mathbf{q}) k_{m_1 \circ m_2, m_3}(\mathbf{q})$, but there are plentiful solutions – for example, any finite group algebra.) Suppose that there is a triple of points in parameter space for which

$$k_{m_1, m_2}(\mathbf{q}') k_{m_1, m_2}(\mathbf{q}'') = k_{m_1, m_2}(\mathbf{q}) \tag{42}$$

for all m_1, m_2. Then A has a kind of generalised coproduct:

$$A \hookrightarrow A' \times A'',$$
$$m \mapsto (m, m),$$

making it a kind of generalised bialgebra, since

(*confer* [**39**, Section 1.1.3(iv)], for example). In particular, the coproduct is an algebra morphism, and if a submanifold S of parameter space can be found for which each pair $\mathbf{q}', \mathbf{q}'' \in S$ has a $\mathbf{q} \in S$ satisfying equation (42), then the sum over all $\mathbf{q} \in S$ of categories of (left) modules is closed under tensor product. (The example of group algebras is the usual bialgebra and tensor product.)

It is easy to show, using the diagram calculus (or via a mild generalisation of the Temperley–Lieb diagram variant of cabling [**48**, Section A(iii)]), that $T_n(q)$ is an algebra of this type, with $\mathbf{q} = \{q\}$. The diagram in Figure 6 illustrates the coproduct on $U_1 U_2 \in T_4(q)$, using lines of different thickness for different q. The cabling-like visualisation of the two factors, in which they are embedded in a single pseudodiagram, is possible because the thin and thick lines are arranged into subdiagrams that never meet in any composition.
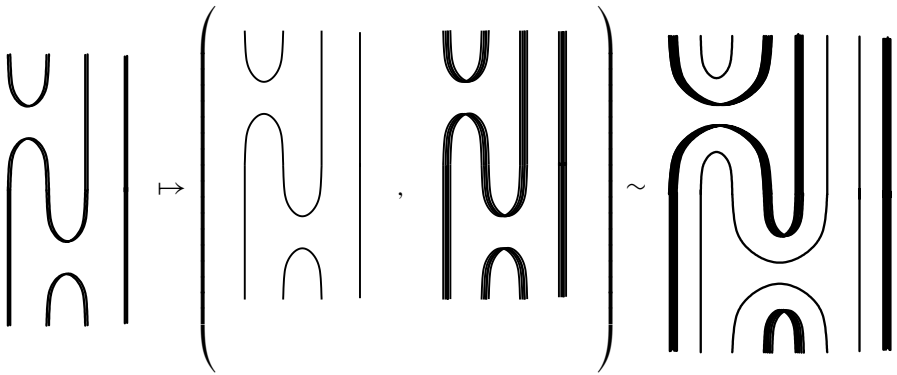
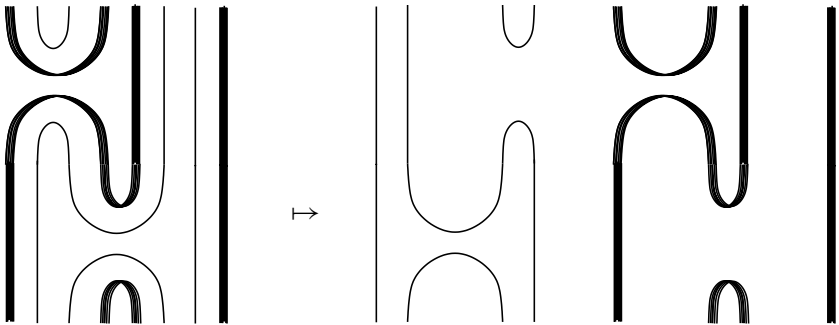Figure 6: Coproduct on $U_1 U_2 \in T_4(q)$.

Figure 7: Reflected visualisation.

It will be evident that the set of conditions (42) include $-[2]_q = [2]_{q'}[2]_{q''}$ in this case (consider a composition in which a closed loop arises, such as $U_1 U_1$); and that this is the only non-trivial condition. Through the cabling picture we may pass to another visualisation, in which the thinner lines have been reflected in a vertical line at the left edge of the diagram (*confer* [19]), as illustrated in Figure 7. There is no significant difference between these two visualisations, except that it is perhaps slightly easier to describe the construction of certain tensor product representations *explicitly* using the reflected form, as we shall see. Again, in the reflected form we may view the picture as a single pseudodiagram (but again there is no sense in which the right- and left-hand sides ever touch).

For example, we may tensor together two tensor space representations in the form:

$$\rho : T_n(q) \longrightarrow \mathrm{End}(V_2^{2n});$$
$$\rho : U_i \mapsto M_2^s(U_{n-i})M_2^t(U_{n+i}).$$

Here, the set of conditions (42) reduces to $-[2]_q = [2]_s[2]_t$ via, for example,

$$
\begin{array}{ccc}
\rho(U_i U_i) & \!\!=\!\! & \rho(U_i)\rho(U_i) \\
\| & & \| \\
-[2]_q\rho(U_i) & \!\!=\!\! & [2]_s[2]_t\rho(U_i).
\end{array}
$$

Set

$$
\mathcal{U}^q(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & x \end{pmatrix}
$$

and $\mathcal{U}^q = \mathcal{U}^q(0)$ (*confer* [18]). Just as $M_2^r(U_i)$ is a matrix acting trivially on every tensor factor except the ith and $(i+1)$th, where it acts as $-\mathcal{U}^r$, so let $M_2^{r,x}(U_i)$ denote a matrix differing from this only in acting like $-\mathcal{U}^r(x)$ in that position.

Note that

$$
(\mathcal{U}^s \otimes \mathcal{U}^t)(1 \otimes \mathcal{U}^r(x) \otimes 1)(\mathcal{U}^s \otimes \mathcal{U}^t) = \left(\frac{r}{st} + \frac{st}{r} + x\frac{t}{s}\right)(\mathcal{U}^s \otimes \mathcal{U}^t) \tag{43}
$$

for any r, s, t, x (an explicit calculation).

PROPOSITION 6.3. *Fix q and m, put $q = e^{i\mu_q}$ and $u_0 = [m]_q e_-$, and choose r, s and t such that*

$$
-\cos(\mu_q) = 2\cos(\mu_s)\cos(\mu_t); \tag{44}
$$

$$
-\frac{\sin((m-1)\mu_q)}{\sin(m\mu_q)} = \frac{\cos(\mu_s + \mu_t - \mu_r)}{\cos(\mu_r)}. \tag{45}
$$

(Note: exclude $m = 0$, and beware of $q = 1$. A convenient realisation is $r = i(-q)^m$, $s = -i\sqrt{iq}$, $t = -\sqrt{iq}$; that is, $\mu_r = m(\mu_q + \pi) + \pi/2$, $\mu_s = (\mu_q + \pi)/2 - 3\pi/4$, $\mu_t = (\mu_q + \pi)/2 + 3\pi/4$ – so rational μ_q/π and m give rational μ_r/π, μ_s/π, μ_t/π.) Then there is an algebra homomorphism

$$
\rho_0 : b_n(q, m) \longrightarrow \operatorname{End}(V_2^{2n})
$$

given by

$$
\rho_0 : e_- \mapsto \frac{1}{[2]_r} M_2^r(U_n); \tag{46}
$$

$$
\rho_0 : U_i \mapsto M_2^s(U_{n-i}) M_2^t(U_{n+i}).
$$

Proof. We may readily verify that $\rho_0(e_-e_- = e_-)$.

The relations for $T_n(q) \subset b_n$ may be checked in ρ; that is, as above. (Note that $[m]_q = \sin(m\mu_q)/\sin(\mu_q)$.) There remains $U_1 e_- U_1 \propto U_1$ (relation (2)). This is validated by the explicit calculation in equation (43):

$$
U_1 e_- U_1 = \frac{[m-1]_q}{[m]_q} U_1 \mapsto -\frac{\left(\frac{r}{st} + \frac{st}{r}\right)}{[2]_r}\rho_0(U_1). \qquad \square
$$

Another way to see this is to note that, in tensor space, equation (43) allows us to make sense of an extension of the pseudodiagrams in Figure 7, in which the left- and right-hand sides meet as in Figure 8. (Specifically, this figure may be replaced by a scalar multiple of one in which the loop composed of mixed line segments is omitted.)
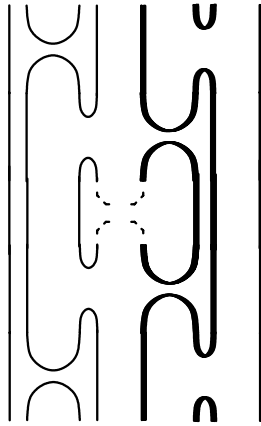
Figure 8: Extension of the reflected visualisation.

There is a similar homomorphism ρ_s, which simply replaces equation (46) with

$$\rho_s : e_- \mapsto \frac{1}{[2]_r} M_2^{r,[2]_r}(U_n) \tag{47}$$

and equation (45) with

$$-\frac{\sin((m-1)\mu_q)}{\sin(m\mu_q)} = \frac{\cos(\mu_s + \mu_t - \mu_r)}{\cos(\mu_r)} + e^{i(\mu_t - \mu_s)}. \tag{48}$$

This mild complication has the merit that the blob/box symmetry of the algebra [55] maps this representation to one of the same type.

The isomorphism of b_n to its opposite (note the symmetry of the relations under writing back to front) provides an automorphism that is fixed by ρ_0 (the representations of the generators are symmetric matrices); thus ρ_0 is contravariant self-dual. The same is true of ρ_s.

These very exciting representations merit further study. Martin has recently shown [51] that ρ_0 is faithful, and Martin and Ryom-Hansen have shown [53] that it is full tilting in the quasihereditary cases. The intriguing question now is: "How do they generalize to higher d?" This question is not trivial. We have made considerable use of the Temperley–Lieb diagram calculus here, and there is no such powerful tool in evidence for higher d, short of the braid group itself. The remainder of the paper is essentially concerned with addressing this question.

6.2. *Other constructions*

The blob algebra $b_n(q, m)$ is (at least) *singly critical* when $m \in \mathbb{Z}^+$, see Section 1.2: if q is not a root of unity, then the procedure there still works, but with l set unboundedly large. This means that the usual Pascal triangle of standard modules [55] is complicated by at least one wall of 'reflection' homomorphisms from outside to inside across the alcove wall at m, as exemplified in Figure 9. There is a suggestive combinatorial coincidence with T_n manifest in the dimension of heads of the blob standard modules in certain singly critical (that is, $m \in \mathbb{Z}$) cases. This starts with the $m = \pm 1$ cases, where there is a b_n quotient
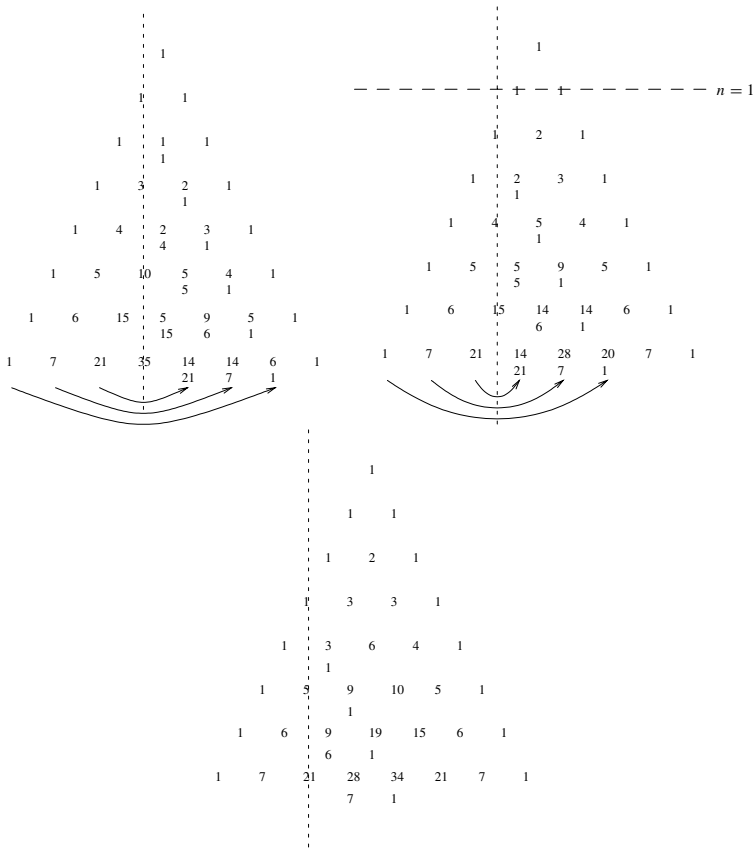
Figure 9: The Pascal triangle of b_n standard modules (layers $n = 0, 1, \ldots, 7$ are shown) complicated at $m \in \mathbb{Z}$ by reflection homomorphisms. Here we show, by their dimensions, the composition factors of each standard in cases $m = -1, -2, -3$.

given by $e_- \mapsto 0$ (or, respectively, $e_- \mapsto 1$). It follows immediately from the relations that this quotient is isomorphic to T_n (and so the coincidence is explained). Figure 9 exhibits a similar phenomenon at $m = -2$. These may be regarded as special cases of a more general 'braid construction'.

In the next sections we describe this braid construction, and review aspects of the connection between the blob algebra and periodic (and affine) systems that lead to other useful maps from b_n (and other $\mathrm{H}(n, d)$ quotients) into ordinary Hecke quotients, and hence into tensor space.

7. Braids and the blob approach to periodic systems

In this paper, the 'A-type braid group' A-braid is the group of braidings of a row of initially vertical strings numbered from the top left: $1, 2, \ldots$, which braidings are trivial on all but finitely many strings. The subgroup A_n-braid acts trivially on all but the first n strings (so A_0-braid $= A_1$-braid $\subset A_2$-braid $\subset \ldots$). Thus H_n is a quotient of $\mathbb{C}A_n$-braid, and g_i is the braid in which string i crosses over string $i + 1$. Evidently, such elements, and their
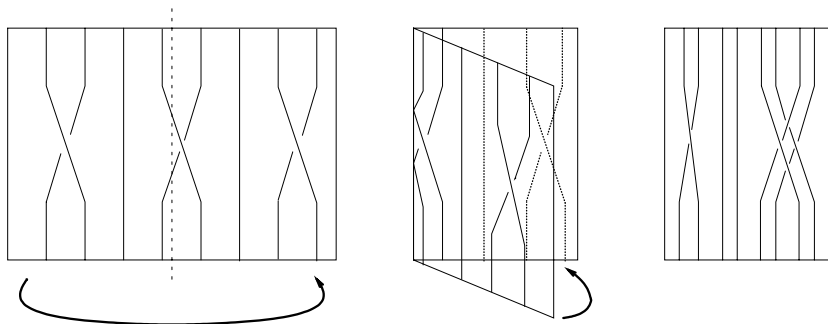
Figure 10: Illustration of the folding injection of A_n-braid into itself.

inverses, generate the group [**46, 8**]. Let $(-) : A_n$-braid $\to A_n$-braid be the automorphism given by $g_i \mapsto g_{n-i}$; see [**50**, Section 5.7.2]; *confer* [**19**].

Occasionally, we shall need to refer to the group of braidings of precisely n strings (as, for example, in the Young subgroup construction – see [**50**, Section 13.1, p. 323]). This group is obviously isomorphic to A_n-braid, and we shall distinguish them only by context. Let 1^m denote the identity element on precisely m strings, and let

$$1_C^m : A_n\text{-braid} \to A_{mn}\text{-braid}$$

be the corresponding *cabling* morphism (replace each string with m parallel strings).

Let $\Delta : A_n$-braid $\to A_n$-braid $\times A_n$-braid be the group comultiplication. Let $Y : A_n$-braid $\times A_n$-braid $\hookrightarrow A_{2n}$-braid be the natural 'Young' embedding, extended to (a version of) the full braid group by extending the numbering of strings to $\mathbb{Z} \setminus \{0\}$ – that is, essentially the full line, not the half line – and placing the second copy of A_n-braid on the 'minus' side. Let F be the map back to the full braid group proper, obtained by *folding* the left-hand side of the plane over on to the right-hand side at a point slightly shifted from the origin (so that each negatively numbered string starting point lies just to the left of its positively numbered version, and the system is bounded on the left again), and then renumbering – see Figure 10. Let S be the map back to A-braid, obtained by renumbering $i \mapsto i + n + 1$ ($i < 0$) and $i \mapsto i + n$ (for $i > 0$) and discarding all strings numbered less than 1. Now define a map 1_F^2 from A_n-braid $\to A_{2n}$-braid by commutativity of:

$$
\begin{array}{ccc}
A_n\text{-braid} & \xrightarrow{\quad 1_F^2 \quad} & A_{2n}\text{-braid} \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\uparrow{\scriptstyle F} \\
A_n\text{-braid} \times A_n\text{-braid} \xrightarrow{\ 1 \times (-)\ } A_n\text{-braid} \times A_n\text{-braid} & \xrightarrow{\ Y\ } & A_{2n}\text{-braid}
\end{array}
$$

and similarly for 1_S^2. The map 1_F^2 is similar also to the $m = 2$ cabling morphism, in that each string now has a partner running parallel to it, but the over/under information is not the same.

We now recall certain constructions from [**50**, Section 5.7] (some changes of notation will be necessary). Let $\mathbb{C}\hat{A}_n$-braid denote the algebra associated to affine graph $\hat{A}_n$ there (strictly speaking, only every other such graph has a pregraph, but this need not concern us), let $\hat{A}_n$-braid denote the underlying group (that is, the affinization of the ordinary braid

group), and let $g.$ denote the 'extra' generator associated to the affinizing vertex in $\hat{A}_n$ compared to A_{n-1}. Define

$$G = g_1 g_2 \ldots g_{n-1}$$

in A_n-braid (the element in which string 1 crosses over strings 2 to n), and note that $G g_i G^{-1} = g_{i+1}$ ($i < n - 1$).

PROPOSITION 7.1 (see [**50**]). *There is a homomorphism $\phi_0 : \hat{A}_n$-braid $\mapsto A_n$-braid given by identification on the A_n-braid subalgebra and $g. \mapsto G g_{n-1} G^{-1}$.*

As noted for example in [**48**, Section 3], there are actually a number of closely related ways of building representations of the affine (or periodic) case, corresponding to the choice of periodic boundary conditions (the 'cohomological seam') in a physical system. This was systematized, in [**55**, Section 3], by the introduction of the idempotent blob generator e_-. Using this, we can build an invertible generator $g_- = 1 + ae_-$ (for a a suitable constant) obeying

$$g_- g_1 g_- g_1 = g_1 g_- g_1 g_- \tag{49}$$

(see also Proposition 4.2), and we define $G_- = g_- G$. Then

$$g. \mapsto G_- g_{n-1} (G_-)^{-1} \tag{50}$$

defines a generalisation ϕ_a of ϕ_0 for each suitable choice of a, y_e (see [**55**, equation (25)]).

Note that neither equation (6) nor the blob construction for g_- are needed to verify the map in equation (50); only equation (49) is necessary. Thus the map generalises to $H(n)$, and even to the level of braids.

The connection between the B-type and periodic systems now follows, inasmuch as B_n-braid may be realized as the group of braids on the cylinder, whereupon $\pi = g_{n-1} g_{n-2} \ldots g_1 g_0$ is the braid obtained from the identity braid by turning the bottom edge of the cylinder through one vertex clockwise (that is, so as to take vertex 1 into vertex 2, and so on); π', the corresponding generalisation of G_- (that is, with g_- replaced by g_0), is the anticlockwise turn. Thus B_n-braid may be thought of as having affine $\hat{A}_n$-braid as a subgroup, with $g. = \pi g_1 \pi^{-1} = \pi' g_{n-1} (\pi')^{-1}$; *confer* [**55**, Section 3].

7.1. *Homomorphisms of B_n-braid to $A_{n'}$-braid*

Let $A = A_n$-braid, and let $\underline{n} = \{1, 2, \ldots, n\}$. For $b \in A$ and $i \in \underline{n}$, define $b(i)$ to be the final position of string i in b.

Let p be any partition of (or, equivalently, any equivalence relation on) $\underline{n}$. Then for each such p, there is a subset of $A = A_n$-braid such that $b \in A$ implies that $b(i) \sim i$. This subset is a subgroup: call it $p - A_n$-braid, or just p-braid. For example, if p is the 'trivial' relation $(\{\underline{n}\})$, then p-braid $= A$; if p is the identity relation, then p-braid is the pure braid group (the normal subgroup whose quotient is the permutation action on $\underline{n}$, $b(i)$, described above).

For convenience when dealing with general n, we shall describe a partition for which each $i > m$, for some m, is in the same part by only giving the other parts. Thus {}-braid $= \{\underline{n}\}$-braid $= A$; and {{1}}-braid$=: A'$ is the group that is pure on string 1.

Let $L_i \in A_n$-braid be the pure braid that takes string i behind all earlier strings, and then back in front of them (that is, $L_1 = 1$ and $L_{i+1} = g_i L_i g_i$); see Figure 11.
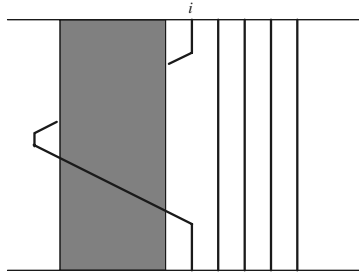
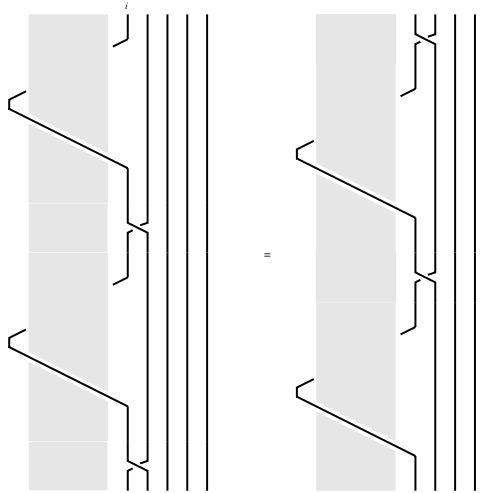Figure 11: L_i; here, the shaded area represents the identity braid on the first $i - 1$ strings.

Figure 12: Schematic for generalisations of $L_i g_i L_i g_i = L_i L_{i+1} = L_{i+1} L_i = g_i L_i g_i L_i$ as in f_b (b lives in the shaded region).

Note that in the Hecke algebra quotient, L_i is the image of $X_i \in H(n, 1)|_{\lambda_1=1}$, that $L_i L_j = L_j L_i$, and that $C_n = \prod_{i=1}^{n} L_i$ is the ('clockwise' or g_i-built) pure twist element of A_n-braid, denoted M^2 in [**50**, Section 5.7.2].

PROPOSITION 7.2. *Let b be any element of A_{m-1}-braid. Then there is an injective group homomorphism*

$$f_b^m : B_n\text{-braid} \longrightarrow \{m - 1\} - A_{n+m-1}\text{-braid};$$
$$g_0 \mapsto C_m b;$$
$$g_i \mapsto g_{i+m-1}.$$

Proof. Consider Figure 12. This checks the key relation explicitly. □

For example, f_1^2 is $g_0 \mapsto g_1 g_1$, $g_1 \mapsto g_2$, and so on. Likewise, f_1^1 is $g_0 \mapsto 1$, $g_i \mapsto g_i$ ($i > 0$).

There is a group homomorphism extending the 1_C^2 cabling morphism (*confer* [**50**, Chapter 13]; note also [**17**]):

$$1_C^2 : B_n\text{-braid} \longrightarrow A_{2n}\text{-braid};$$
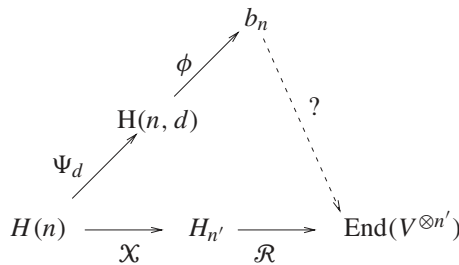
$$g_0 \mapsto g_1;$$

$$g_i \mapsto 1_C^2(g_i).$$

There is a similar homomorphism extending the 1_S^2 morphism:

$$1_S^2 : B_n\text{-braid} \longrightarrow A_{2n}\text{-braid};$$

$$g_0 \mapsto g_n;$$

$$g_i \mapsto g_{n-i}\, g_{n+i}$$

(*confer* Figure 10), and another extending 1_F^2.

8. *On tensor space representations of* H(n, d)

The constructions above allow us to build representations of B_n-braid from type-A representations (and if these are charge-conserving tensor space representations, then these properties will be preserved, in some sense, as will globalisability). Our strategy now in searching for maps from b_n to T_n (and generalisations to $d > 2$) may be summarized by the following picture.

The Northeast-pointing maps are the canonical quotients from Sections 1 and 4 (we shall call the combined map η), the map $\mathcal{R}$ is the ordinary representation on tensor space, and the dotted line is the desired map occurring if $\mathcal{R} \circ \mathcal{X}$ factors through b_n. The exercise is to find among the maps $\mathcal{X}$ those for which $\mathcal{R} \circ \mathcal{X}$ factors thus. (As we shall see, the candidates that we noted in Section 6.2 for maps from b_n to T_n are just special cases of the simple representations f_b^m of B_n-braid above.)

By Definition 6.2, each B-braid quotient of form

$$B_n\text{-braid} \xrightarrow{f_b^m} A_{m+n}\text{-braid} \xrightarrow{M_N} V_N^{m+n} \tag{51}$$

('auxiliary space' construction) or

$$B_n\text{-braid} \xrightarrow{1^2} A_{2n}\text{-braid} \xrightarrow{M_N} V_N^{2n}$$

('cabling-related' construction) factors through a partial specialisation of H(n, d) (some d) in which the g_0 (that is, $\mathcal{X}$) eigenvalues are determined in terms of q, but q is indeterminate.

Obviously, the $M_2 \circ f_1^2$ quotient obeys a quadratic $f(g_0) = 0$, so $d = 2$, and so it factors through some specialisation of $\text{Hecke}_B(n)$; but $f(g_0)$ has coefficients in $\mathbb{Z}[q, q^{-1}]$, so this is still not generic (Hecke_B has two parameters, q and Q, say).

Since all our maps $M_2 \circ \mathcal{X}$ map into the T_n action on V_2^n (for some n), their image breaks up at least as far as in equation (41).

Possible next steps here are: (i) to investigate the generic irreducible content of the f_b representations (of whatever specialisation of $H(n, d)$ they might provide); and (ii) to investigate what portion of the parameter space is actually accessible by this construction (that is, what eigenvalues of g_0 we can realise by varying b). Fixing q, this portion is discrete (that is, of measure zero), but then so is the (at least singly critical) portion that we are most interested in, so it is not necessarily too restricted.

Note that if we want to access a dense subset of the parameter space, this *cannot* be via $M_N \circ f_b^m$, since this depends continuously only on q. Instead, we could look at maps ending on, say, $\otimes_{i=1}^d V_N^n|_{q=q_i}$.

8.1. *On cabling-related maps*

Underlying the map 1_F^2 is the full range of direct product representations of A_n-braid. In particular, we can regard $\otimes_i M_N^{q_i}$ as a representation of A_n-braid. In general, we may not assume that these representations will factor through any particular Hecke quotient, but if one does, then it could provide a generalisation of the extension of 1_F^2 to B_n-braid. We form

$$R^{\otimes}(g_i) = M_N^q(g_{n-i}) \otimes M_N^r(g_{n+i})$$

and try, say, $R^{\otimes}(g_0) = 1 \otimes M_N^s(g_{n-i}) \otimes 1$, and we compute $R^{\otimes}(g_0 g_1 g_0 g_1 - g_1 g_0 g_1 g_0)$. In this particular case, with $N = 3$, the image vanishes only when $q = r = s$ (a brute-force calculation). An analogous deformation of the extended 1_C^2 map at $N = 3$ fails in the same way.

The 1_C^2 cabling map with $N = 2$ does not in general factor through b_n either. Further investigations are hindered by the magnitude of the computations required, but these negative results serve well to illustrate the extraordinary nature of the ρ-representations in Proposition 6.3.

8.2. *On $N = 2$ auxiliary spaces, b_n, and 'the coincidence'*

In (51) we require that $M_N \circ f_b^m(x)$ be also a representation (R, say) of $\eta(x)$ for all x. Firstly,

$$\eta : g_i \mapsto q + U_i,$$

so equation (40) verifies equation (1). We also require that $R(U_1 e_- U_1 - k_- U_1) = 0$ for some k_- (the relation (5) is not sufficient to ensure this). Whenever we *find* a map, the other question is: "Is it faithful?"

Since

$$\eta : g_0 \mapsto \alpha 1 + \beta e_-,$$

the spectrum of $M_2(f_b^m(g_0))$ must be quadratic if the map is to factor through the blob as it stands. By equation (41), this spectrum may be determined from the action on the zero charge sector $\mathcal{P}_0$ of V_2^n. The following lists are the eigenvalues with multiplicities in this

sector, arranged by standard T_n-module factor with the 'spine' module on the left, and so on.

$M_2(L_1)$: $\{1\}$;

$M_2(L_2 = C_2)$: $\{q^2, q^{-2}\}$;

$M_2(L_3)$: $\{q^2, q^{-2}, q^{-4}\}$;

$M_2(C_3)$: $\{1, 1, q^{-6}\}$;

$M_2(L_4)$: $\{1, 1, q^2, q^{-4}, q^{-4}, q^{-6}\}$;

$M_2(C_4)$: $\{1, 1, q^{-4}, q^{-4}, q^{-4}, q^{-12}\}$;

$M_2(C_5)$: $\{q^{-4}, q^{-4}, q^{-4}, q^{-4}, q^{-4}, q^{-10}, q^{-10}, q^{-10}, q^{-10}, q^{-20}\}$;

$M_2(C_6)$: $\{q^{-8}, \ldots, q^{-8}, q^{-12}, \ldots, q^{-12}, q^{-20}, \ldots, q^{-20}, q^{-32}\}$.

The pattern for C_i will be obvious.

Note, therefore, that C_1, C_2 and C_3 are the only possibilities here (unless we further specialize to q a root of unity). 'Null' twist C_1 corresponds to the $m = \pm 1$ case already discussed. For the other cases, it remains to check that the candidates for images of the generators obey $U_1 e_- U_1 = k_- U_1$ for some scalar k_-.

Example: $M_2 \circ f_1^2$. (Let u_1 and u_0 denote the canonical preimages along η of U_1 and e_-.) An elementary calculation finds a value of k_- for which

$$M_2(f_1^2(u_1 u_0 u_1 - k_- u_1)) = 0.$$

This value then determines that the blob parameter $m = -2$ here.

Since $\text{span}(1, g_1) = \text{span}(1, g_1 g_1)$ here (and V_2^n is a faithful T_n-module), the image of B_n-braid here is the whole of T_{n+1}. Thus equation (41) determines the structure of V_2^{n+1} as a B_n-braid-module. For $n = 2$, it is

$$V_2^3 = 1 + (2 + 1) + (2 + 1) + 1 \tag{52}$$

(representing summands by their dimensions), which is the structure as a T_3-module. Since we hit the whole of T_3, equation (52) is the irreducible decomposition with q generic. Let us call the two inequivalent modules here M^1 and M^2. The generic simples of H(2, 2), as indexed by their 2-partitions (see Section 1), are:

2-partition	$((2),)$	$((1), (1))$	$(, (2))$	$((1^2),)$	$(, (1^2))$
dimension	1	2	1	1	1

all but the last two of which survive the quotient to b_2. Note, therefore, that if the T_3 standard M^2 breaks up no further (that is, q-generically) it *is* a blob representation for $m = -2$, but that $M_2 \circ f_1^2$ cannot be a faithful b_n-module.

For B_3-braid, we have $V_2^4 = 1 + (3 + 1) + (2 + 3 + 1) + (3 + 1) + 1$ (as a T_4-module) as follows; *confer* Hecke$_B$.

2-partition	$((3),)$	$((2, 1),)$	$((1^3),)$	$((2), (1))$
dimension	1*	2	1	3*

2-partition	$((1^2), (1))$	$((1), (2))$	$((1), (1^2))$	$(, (3))$	...
dimension	3	3*	3	1*	...

(Blob representations are indicated with a *.)

Recall that at $m = -2$, blob standards break up as shown in Figure 9. In particular, at $n = 3$,

$$\begin{matrix} 1 \\ \longrightarrow \end{matrix} \quad \begin{matrix} 2\ 3\ 1 \\ 1 \end{matrix} \ .$$

Note that here, and for all n, the heads of the b_n standards to the right of the $m = -2$ line may indeed be identified with the (generic) irreducible T_{n+1}-modules. This is neat, but it follows that none of these representations is faithful.

To summarize the last two sections, we have not been successful in generalising the ρ-representations. The search for full tilting modules for general d continues, and we report these negative results partly in order to avoid unnecessary duplication later. More positively, the representations that we have found are interesting from the point of view of Yang–Baxter equations in physics [**4**], but we shall discuss these applications elsewhere.

We commend the following recent papers on related topics to the reader: [**63, 31**].

Appendix A.   *On the Bernstein centre $Z(H(n))$ and $Z(\mathrm{H}(n, d))$*

Some of the manipulations of ideals in Section 4 and thereafter are not trivial. The following mechanical exposition of the Bernstein centre and its image in $Z(\mathrm{H}(n, d))$ may help the reader to see where they come from.

Following on from equation (19), we define $\hat{g}_i = [X_i, g_i]$ (thus $\hat{g}_i = (X_i - X_{i+1})g_i + (q - q^{-1})X_{i+1}$). Then

$$\hat{g}_i X_j = X_{\sigma_i(j)}\hat{g}_i \tag{53}$$

and

$$\begin{aligned}
\hat{g}_i \hat{g}_{i+1} \hat{g}_i &= X_i g_i (X_{i+1}g_{i+1} - g_{i+1}X_{i+1})X_i g_i + \dots \\
&= X_i g_i X_i (X_{i+1}g_{i+1})g_i - X_i g_i (g_{i+1}g_i X_i g_i)X_i g_i + \dots \\
&\ \ \vdots \\
&= X_{i+1}g_{i+1}X_i g_i^{-1}X_{i+1}g_{i+1} - g_{i+1}X_{i+1}X_i g_i^{-1}g_{i+1}X_{i+1} + \dots \\
&= X_{i+1}g_{i+1}X_i(g_i - (q - q^{-1}))X_{i+1}g_{i+1} \\
&\quad - g_{i+1}X_{i+1}X_i(g_i - (q - q^{-1}))g_{i+1}X_{i+1} + \dots \\
&= X_{i+1}g_{i+1}X_i g_i X_{i+1}g_{i+1} - g_{i+1}X_{i+1}X_i g_i g_{i+1}X_{i+1} + \dots \\
&= \hat{g}_{i+1}\hat{g}_i \hat{g}_{i+1}.
\end{aligned} \tag{54}$$

Let K be our ground ring (an integral domain), let $K[X_-]$ be the ring of polynomials in the X_is, and let $K(X_-)$ be the field of fractions. Note that we can write

$$\alpha_i \hat{g}_i = g_i + \beta_i,$$

where both α_i and β_i lie in $K(X_-)$. (Note that in our quotient Ψ_d, the image of $K[X_-]$

itself generically, but not always, contains α_i and β_i. For example, when $d = 1$ and $q^2 = 1$, then $X_i - X_{i+1}$ is not invertible.) It then follows from equation (53) that any element of the extension of $H(n)$ by $K(X_-)$ can be expressed in the form

$$h = \sum_{w \in \mathfrak{B}_n} c_w \hat{g}_w,$$

where $c_w \in K(X_-)$ and $\hat{g}_w$ is obtained by putting hats on the generators in w. Indeed, the extension may be decomposed as

$$\bigoplus_{w \in \mathfrak{B}_n} K(X_-)\hat{g}_w \tag{55}$$

(an induction on the usual *length* function on $\mathfrak{B}_n$).

Suppose that h has at least one $w \neq 1$, with $c_w \neq 0$ (that is, $h \notin K(X_-)$ subalgebra). Then there is at least one i such that $w(i) \neq i$ (under the obvious generalisation of the σ_i action in equation (53)), and

$$hX_i = \sum_w c_w \hat{g}_w X_i = \sum_w c_w X_{w(i)} \hat{g}_w,$$

so the $\hat{g}_w$ component of $hX_i - X_i h$ is $c_w(X_{w(i)} - X_i) \neq 0$. Thus

$$Z_{K[X_-]}(H(n)) = K[X_-] \supseteq Z(H(n)).$$

But with $c \in K[X_-]$, then $\hat{g}_w c - c\hat{g}_w = (c^w - c)\hat{g}_w$, so $Z(H(n)) = X^\Sigma$, as Bernstein says.

Naturally, $Z(\mathrm{H}(n, d)) \supseteq \Psi_d(X^\Sigma)$, depending in principle on the ground ring. The argument above mostly works in this case (to show equality), although the possible specialisations of the ground ring become more restricted ($\lambda_i \neq 0$, plus the restrictions already mentioned, for example). Note also that $\Psi_d(K[X_-])$ is finite-dimensional. This makes it interesting to study $\Psi_d(X^\Sigma)$ – an algebra that is, in a sense, more complicated than X^Σ itself. For example, $\mathrm{H}(2, 1) \cong H_n$ and $\dim(Z(H_2) = H_2) = 2$, so $\dim(\Psi_1(X^\Sigma)) = 2$. Here a basis for $\Psi_1(X^\Sigma)$ is

$$\{1, X + X_2 = \lambda_1(1 + g_1^2)\}.$$

(Note that this example illustrates the problem with $q^2 = 1$.)

For another example, recall that $\dim(Z(\mathrm{H}(2, 2))) = 5$ generically, so $\dim(\Psi_1(X^\Sigma)) = 5$. Here a basis for $\Psi_2(X^\Sigma)$ is

$$\{1, X + X_2, XX_2, X^2 + X_2^2, (X + X_2)XX_2\}$$

(this is not supposed to be obvious!). Another basis, convenient for comparison with the basis of $\mathrm{H}(2, 2)$, is

$$\{1, X + X_2, XX_2, (X + X_2 - (\lambda_1 + \lambda_2))g_1, (XX_2 - \lambda_1\lambda_2)g_1\}. \tag{56}$$

The set of monomial symmetric polynomials in two variables X_1 and X_2 (a basis for $Z(H(2))$) may be indexed by the set Λ^2 of Young diagrams of not more than two rows. (Write $m^a = (X^a)^\Sigma$; then $m^{(0)} = 1$ and $m^{(1)} = X_1 + X_2$, and so on.) Note that the set of such polynomials in which the degree of no individual variable exceeds $d - 1$, is *not* a basis for $\Psi_d(X^\Sigma)$ in general, as we see from the examples above.

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