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GENERALIZED BOUNDARY VALUE PROBLEMS WITH ABSTRACT SIDE CONDITIONS AND THEIR ADJOINTS I

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1. INTRODUCTION

Suppose $[a, b]$ is a compact interval of the reals. Let $AC_m \equiv AC_m[a, b]$ and $L^p_m \equiv L^p_m[a, b]$, $1 \leqq p < \infty$ denote the spaces of functions $y : [a, b] \to C^m$ having absolutely continuous or L^p-integrable components where C^m is the complex m-dimensional space (elements in C^m are regarded as column vectors) under the usual Euclidean norm. L^∞_m is the space of functions $y : [a\ b] \to C^m$ essentially bounded on $[a, b]$. For $1 \leqq p \leqq \infty$ and $n = 1, 2, \ldots$

$$W^{n,p}_m := \{y : y^{(n-1)} \in AC_m;\ y^{(n)} \in L^p_m\}\ (W^{1,1}_m = AC_m)\,.$$

Consider the generalized boundary value problem

$$\ell y := A_0 y' + Ay = f\,, \qquad Hy = r\,, \tag{1,1}$$

where $y \in W^{1,p}_u$, $H : W^{1,p}_m \to F$ is a linear continuous operator into a locally convex topological vector space F, and A_0, A are $k \times m$ $(k \geqq m)$ matrices with columns in L^∞_k and L^p_k.

In this paper we will study adjoints and Fredholm Alternatives for the system (1,1). In particular we extend results of several recent papers on generalized boundary value problems (see the monograph [22] or the survey paper [16] for a list) where F is finite dimensional.

There have been basically two approaches in recent years to the adjoint theory of differential systems like (1,1). In the first place, if we take the norm on $W^{n,p}_m$ to be

$$\|y\|_{W_m^{n,p}} = \|y\|_{n,p} := \sum_{j=0}^{n-1} |y^{(j)}(a)| + \|y^{(n)}\|_{L^p} \tag{1,2}$$

each $W^{n,p}_m$ $(1 \leqq p \leqq \infty,\ n = 1, 2, \ldots)$ becomes a Banach space and (1,1) determines

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a continuous operator $\mathscr{L} : W_m^{n,p} \to L_k^p \times F$. Thus in the case $1 \leqq p < \infty$, $\mathscr{L}$ possesses a unique adjoint $\mathscr{L}^* : L_k^q \times F^* \to W_m^{n,q}$, where $1/p + 1/q = 1$ if $p > 1$ and $q = \infty$ if $p = 1$. By determining $\mathscr{L}^*$ and showing that the range of $\mathscr{L}$ is closed, conditions for the solvability of (1,1) can be stated in the form of Fredholm Alternatives. This continuous approach was initiated by D. WEXLER [28] and has since been developed by VEJVODA and TVRDÝ [22]–[27] for differential and integro-differential systems under Stieltjes boundary conditions. On the other hand, $W_m^{1,p}$ is also a dense subspace of L_m^p with respect to the norm of the space L_m^p and (1,1) generates an unbounded operator $L : y \in D(L) \subset L_m^p \to \ell y \in L_k^p$ defined on the set $D(L)$ of all $y \in W_m^{1,p}$ fulfilling $Hy = 0$. An adjoint operator or relation (if $D(L)$ is not dense in L_m^p) $\mathscr{L}^*$ exists in $L_k^q \times L_m^q$. This approach which generalizes the classical theory of ordinary differential operators in Hilbert spaces was first applied to operators with general boundary conditions by A. M. KRALL in 1968 [14]. Recent papers stressing this point of view include CODDINGTON and DIJKSMA [8], [9], Brown [2]–[5], Brown and Krall [6], [7], [17], Krall [14]–[16] and PARCHIMOVIČ [19].

Of these approaches the first is the most general. It is not difficult to prove the normal solvability of $\mathscr{L}$ and to characterize its adjoint $\mathscr{L}^*$. However, $\mathscr{L}^*$ is an integral operator which is different from the formal Lagrange adjoint of an ordinary differential expression. On the other hand in the second "unbounded" setting L^* amounts to an extension of the adjoint of an ordinary "maximal operator" to a larger domain of possibly nonsmooth functions. Further, from this point of view there exists significant applications of the adjoint to spline and optimization problems (cf. [4], [6]) and also to the theory of selfadjoint extensions of symmetric differential operators defined on nondense domains (cf. [8], [9]). Unfortunately L^* has proved difficult to construct. Previous methods in the papers cited above have been both complicated and not as general as we would like, requiring for example either the existence of a Green's function or the finite dimensionality of F. This paper is an attempt at synthesis: we show that the bounded and unbounded approaches are more closely related than has been hitherto realized.

We conclude this section with a brief outline of the paper. Notation and preliminary facts concerning the representation of the operator H and the theory of adjoints of nondensely defined operators are developed in § 2. This furnishes the tools by which reasoning valid when $\dim F < \infty$ can be generalized. In § 3 we develop the continuous theory. Sufficient conditions for the normal solvability of $\mathscr{L}$ are given (Theorem 3.12) and its adjoint $\mathscr{L}^*$ is characterized (Theorem 3.4). § 4 defines the "minimal" and "maximal" operators L_0 and L determined on L_m^p by (1,1) and finds their adjoints, while their normal solvability is obtained easily from Theorem 3.12. We restrict ourselves to $p < \infty$.

In the further parts of the paper we shall extend both the bounded and unbounded approaches to higher order operators and to the case $p = \infty$. The final section of the third part of the paper will sketch some applications of adjoint theory to the theory of splines and optimal control.

2. PRELIMINARIES

2.1. Basic notation. We now introduce the following additional notation which will be used throughout the paper.

R^m is the space of column real m-vectors under the Euclidean norm.

If B is an $m_1 \times m_2$-matrix, then $|B|$ denotes its operator norm and B^* is its conjugate transpose. If $B(t)$ is a matrix valued function defined on $[a, b]$, its L^p-norm $\|B\|_p$ is defined by

$$(2,1) \qquad \|B\|_p := \left(\int_a^b |B|^p \, dt\right)^{1/p} \quad \text{for} \quad 1 \leqq p < \infty\,,$$

$$\|B\|_\infty := \sup_{t \in [a,b]} \text{ess}\, |B(t)|\,.$$

In particular, when restricted to functions with values in C^m, $\|\cdot\|_p$ is a norm in the Banach space L^p_m $(1 \leqq p \leqq \infty)$. We interpret equality between such functions in the almost everywhere sense. $C_m \equiv C_m[a, b]$ is the Banach space of continuous functions $x : [a, b] \to C^m$ equipped with the norm $\|\cdot\|_\infty$.

If X, Y are locally convex topological vector spaces, $X \times Y$ denotes their cartesian product equipped with the usual topology. If X, Y are normed spaces, then the norm on $X \times Y$ is given by

$$(2,2) \qquad \|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y\,,$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the norms on X, Y, respectively. The notation $T : X \to Y$ means that T is an operator defined on the whole space X and with range in Y. X^* is the dual space of X and by $[\cdot, \cdot]_X$ we denote the pairing on $X \times X^*$ defined by $[x, u]_X := u(x)$ for $x \in X$ and $u \in X^*$. If X is a normed space, the topology on X^* is the uniform operator norm topology. In a general case (when X need not be normed) the topology on X^* is supposed to be such that for each $x \in X$ the linear functional $u \in X^* \to [x, u]_X$ is continuous. Simultaneously we shall consider the weak*-topology on X^* which is the weakest topology on X^* possessing this property. For $M \subset X$ and $N \subset X^*$ their closures are denoted respectively by $\overline{M}$ and $\overline{N}$. The weak*-closure of $N \subset X^*$ is denoted by $\text{cl}^*(N)$. Recall that $u \in \text{cl}^*(N)$ iff for every subset $Q \subset X$ with at most a finite number of elements there exists a sequence $\{u_n^Q\} \subset N$ such that $[x, u_n^Q - u]_X \to 0$ for each $x \in Q$. The pairing on $(X \times Y) \times$ $\times (X^* \times Y^*)$ is given by

$$(2,3) \qquad [(x, y), (u, v)]_{X \times Y} := [x, u]_X + [y, v]_Y \quad \text{for} \quad (x, y) \in X \times Y\,,$$

$$(u, v) \in X^* \times Y^*\,.$$

Let us recall that the dual spaces to C^m, L^p_m and $W^{n,p}_m$ $(1 \leqq p < \infty,\ n = 1, 2, \ldots)$ may be identified with the spaces C^m, L^q_m and $W^{n,q}_m$, respectively, where $1/p + 1/q = 1$

if $p > 1$ and $q = \infty$ if $p = 1$, while

$$(2,4) \qquad [c, d]_{C^m} := d^*c = \sum_{i=1}^{m} d_i c_i \quad \text{for} \quad c, d \in C^m ,$$

$$(2,5) \qquad [x, u]_{L_m{}^p} := [x, u]_L := \int_a^b u^* x \, dt \quad \text{for} \quad x \in L_m^p \quad \text{and} \quad u \in L_m^q ,$$

$$(2,6) \qquad [y, v]_{W_m{}^{n,p}} := [y, v]_W := \sum_{i=0}^{n-1} (v^{(r)}(a))^* \, y^{(r)}(a) + [y^{(n)}, v^{(n)}]_L .$$

We shall often omit the subscripts from the pairing or norm notations and rely on the context to indicate precisely which pairing is meant.

Given $M \subset X$ and $N \subset X^*$, we denote

$$M^\perp := \{u \in X^* : [x, u] = 0 \text{ for all } x \in M\} ,$$
$$^\perp N := \{x \in X : [x, u] = 0 \text{ for all } u \in N\} .$$

Clearly, $M^\perp$ is a weakly*-closed subspace of X^* and $^\perp N$ is a closed subspace of X. Moreover (cf. RUDIN [21], 4.7), if M and N are linear subspaces of X and X^*, respectively, then

$$(2,7) \qquad ^\perp(M^\perp) = \overline{M} , \quad (^\perp N)^\perp = \text{cl}^*(N) .$$

Notice that $N \subset \overline{N} \subset \text{cl}^*(N)$ for every $N \subset X^*$.

2.2. Adjoint and pre-adjoint relations. Let X, Y be locally convex topological vector spaces and let T be a linear operator having the domain $D(T)$ in X and the range $R(T)$ in Y. $N(T)$ and $G(T)$ denote its null space $\{x \in X : Tx = 0\}$ and its graph $\{(x, Tx) \in X \times Y : x \in D(T)\}$, respectively. If $\overline{D(T)} = X$ and T is continuous, then it is known that T possesses a unique adjoint T^* which is a linear continuous operator defined on the whole Y^* with values in X^* and such that

$$(2,8) \qquad [Tx, v]_Y = [x, T^*v]_X \quad \text{for all} \quad x \in D(T) \text{ and } \quad v \in Y^* .$$

In a general case when T need not be continuous and $D(T)$ need not be dense in X, we define

$$(2,9) \qquad G(T^*) := \{(u, v) \in X^* \times Y^* : [Tx, v]_Y = [x, u]_X \text{ for all } x \in D(T)\} ,$$

$$(2,10) \qquad D(T^*) := \{v \in Y^* : \text{there exists } u \in X^* \text{ such that } (u, v) \in G(T^*)\} ,$$

$$(2,11) \qquad R(T^*) := \{u \in X^* : \text{there exists } v \in Y^* \text{ such that } (u, v) \in G(T^*)\}$$

and

$$(2,12) \qquad T^*v := \{u \in X^* : (u, v) \in G(T^*)\} \quad \text{for} \quad v \in D(T^*) .$$

Since $G(T^*) = G(-T)^\perp$, $G(T^*)$ is a linear weakly*-closed (hence also closed) manifold in $X^* \times Y^*$ for every T. The multivalued mapping T^* of $D(T^*) \subset Y^*$ into X^* is called the adjoint relation to T, $D(T^*)$ is its domain, $R(T^*)$ its range and $G(T^*)$ its graph. The null space $N(T^*)$ of T^* is defined as

$$N(T^*) := \{v \in Y^* : [Tx, v]_Y = 0 \text{ for all } x \in D(T)\} . \tag{2,13}$$

Obviously $N(T^*) = R(T)^\perp$. Consequently $\overline{R(T)} = {}^\perp(R(T)^\perp) = {}^\perp N(T^*)$. In particular, if $R(T)$ is closed, then $R(T) = {}^\perp N(T^*)$ (T is normally solvable). T^* is an operator (T^*v has a unique value for every $v \in D(T^*)$) if and only if $T^*(0) = \{0\}$. Further since

$$T^*(0) = \{u \in X^* : [x, u] = 0 \text{ for all } x \in D(T)\} = D(T)^\perp \tag{2,14}$$

and ${}^\perp T^*(0) = \overline{D(T)}$, T^* is an operator if and only if $D(T)$ is dense in X. Obviously in this case T^* is the adjoint operator to T.

If $G(T)$ is closed (T is closed), then $N(T) = {}^\perp R(T^*)$ and $\text{cl}^*(R(T^*)) = N(T)^\perp$, while $\text{cl}^*(R(T^*)) = R(T^*)$ if and only if $\overline{R(T^*)} = R(T^*)$ and this occurs if and only if $R(T)$ is closed in Y. Obviously the closedness of $G(T)$ implies the closedness of $G(-T)$. Moreover, since $y \in {}^\perp D(T^*)$ if and only if $(0. y) \in {}^\perp G(T^*) = \overline{G(-T)} = G(-T)$, we have ${}^\perp D(T^*) = \{0\}$ and hence $\text{cl}^*(D(T^*)) = D(T^*) = Y^*$.

If X^{**}, Y^{**} are dual spaces to X^* and Y^*, respectively, the linear relation $(T^*)^* = T^{**}$ with graph

$$G(T^{**}) := G(-T^*)^\perp = (G(T)^\perp)^\perp \subset X^{**} \times Y^{**} ,$$

domain

$$D(T^{**}) := \{w \in X^{**}: \text{ there is } z \in Y^{**} \text{ such that } (w, z) \in G(T^{**})\}$$

and range

$$R(T^{**}) := \{z \in Y^{**}: \text{ there is } w \in X^{**} \text{ such that } (w, z) \in G(T^{**})\}$$

is called the second adjoint of T. By (2,14), $T^{**}(0) = D(T^*)^\perp$. Since $\overline{D(T^*)} = Y^*$, $T^{**}(0) = \{0\}$ so that T^{**} is an operator. If $X^{**} = X$ and $Y^{**} = Y$ (X, Y are reflexive) we may also write $G(T^{**}) = {}^\perp G(-T^*) = {}^\perp(G(T^\perp) = \overline{G(T)}$. Hence T is closed if and only if $T^{**} = T$.

Let S be a linear operator having domain in Y^* and range in X^*. The linear relation $^*S$ with graph

$$G(^*S) = {}^\perp G(-S) \subset X \times Y ,$$

domain

$$D(^*S) = \{x \in X: \text{ there is } y \in Y \text{ such that } (x, y) \in G(^*S)\}$$

and range

$$R(^*S) = \{y \in Y\text{: there is } x \in X \text{ such that } (x, y) \in G(^*S)\}$$

is called the pre-adjoint of S. In the same way as for adjoints we have

$$\overline{D(^*S)} = X \quad \text{and} \quad \text{cl}^*(R(S)) = N(^*S)^{\perp},$$

where $N(^*S) = \{x \in X : (x, 0) \in G(^*S)\}$. Since $({}^{\perp}G(S))^{\perp} = \text{cl}^*(G(S))$, S is weakly*-closed if and only if $G((^*S)^*) := G(-^*S)^{\perp} = ({}^{\perp}G(S))^{\perp} = G(S)$, i.e. if $(^*S)^* = S$. If S is weakly*-closed, then $\overline{R(^*S)} = {}^{\perp}N(S)$, while $R(S)$ is weakly*-closed if and only if it is closed and this occurs if and only if $R(^*S)$ is closed. Moreover, if $S = T^*$ for some linear operator T with $\overline{D(T)} = X$ and $R(T) \subset Y$, then $G(^*S) = G(^*(T^*)) = \overline{G(T)}$ and T is closed if and only if $^*(T^*) = T$.

Obviously, if the linear operators $T_1 : X \to Y$ and $T_2 : X \to Y$ are such that $G(T_1) \subset G(T_2)$, then $G(T_2^*) \subset G(T_1^*)$. Analogously, if $S_1, S_2 : Y^* \to X^*$ and $G(S_1) \subset G(S_2)$, then $G(^*S_2) \subset G(^*S_1)$.

For further details concerning adjoint and pre-adjoint relations (as well as the proofs of the statements given here without proof) see Section 2 of Coddington, Dijksma [9] (cf. also Arens [1] and Brown [5]).

2.3. Lemma. *Let F be a linear topological space and $1 \leqq p < \infty$. Then $H : W_m^{n,p} \to F$ is a linear continuous operator if and only if there exist linear continuous operators $U_j : C_m \to F$ and $V : L_m^p \to F$ such that*

$$Hy = \sum_{j=0}^{n-1} U_j(y^{(j)}) + V(y^{(n)}) \quad \text{for each} \quad y \in W_m^{n,p}. \tag{2,15}$$

Proof. Let U_j and V be given. Since for any $y \in W_m^{n,p}$ and $j = 0, 1, \ldots, n-1$

$$y^{(j)}(t) = \sum_{i=0}^{n-j-1} \frac{(t-a)^i}{i!} y^{(j+i)}(a) + \tag{2,16}$$

$$+ \int_a^t \left(\int_a^{t_1} \left(\ldots \left(\int_a^{t_{n-j-1}} y^{(n)} \, dt_{n-j} \right) dt_{n-j-1} \right) \ldots \right) dt_1 \quad \text{on} \quad [a, b],$$

it follows from the Hölder inequality that

$$\|y^{(j)}\|_\infty \leqq \sum_{i=0}^{n-j-1} \frac{(b-a)^i}{i!} |y^{(j+i)}(a)| + \left(\sum_{i=0}^{n-j} \frac{(b-a)^i}{i!} \right) \|y^{(n)}\|_p \leqq e^{(b-a)} \|y\|_{n,p}.$$

Setting $v(t) = y^{(n)}(t)$ and $u_j(t) = (t-a)^j/j!\, y^{(j)}(a)$ $(j = 0, 1, \ldots, n-1)$ and defining the operators $\tilde{U}_j$ and $\tilde{V}$ by

$$\tilde{U}_j : y \in W_m^{n,p} \to U_j(u_j) \in F \quad (j = 0, 1, \ldots, n-1),$$
$$\tilde{V} : y \in W_m^{n,p} \to V(v) \in F,$$

we can write $H = \sum_{j=0}^{n-1} \tilde{U}_j + \tilde{V}$. Since these $\tilde{U}_j$ and $\tilde{V}$ are continuous, H is continuous.

On the other hand, let a linear operator $H : W_m^{n,p} \to F$ be continuous. Define for $j = 0, 1, \ldots, n-1$

$$U_j : u \in C_m \to H\left(\frac{(t-a)^j}{j!} u(a)\right) \in F \quad \text{and} \quad V : v \in L_m^p \to Hw \in F$$

where

$$w(t) = \int_a^t \left(\int_a^{t_1} \left(\ldots \left(\int_a^{t_{n-1}} v \, \mathrm{d}t_n\right) \mathrm{d}t_{n-1}\right) \ldots\right) \mathrm{d}t_1 \quad \text{on} \quad [a, b].$$

It is clear that U_j and V are continuous. Now (2,15) follows from (2,16).

2.4. If F is a Banach space, then U_j $(j = 0, 1, \ldots, n-1)$ and V can be represented by abstract integrals. For example, U_j are integrable with respect to a certain F valued countably additive regular measures defined on the Borel sets of $[a, b]$ (see [10], p. 318, 492 for details). A similar representation theory exists for V in terms of an abstract generalization of the Riesz representation theorem (cf. Pettis [20]). Such an approach would generalize earlier work in which the boundary conditions were represented by systems of ordinary Stieltjes integrals, and has been exploited by Hönig in the monograph [12] dealing with the construction of Green's functions for problems with abstract boundary conditions (cf. [13]).

Since in all calculations of this paper the boundary conditions appear only in the form $[Hy, \varphi]$, $\varphi \in F^*$, it will be sufficient for our purposes to make use of the following obvious relation valid for p, $1 \leqq p < \infty$.

(2,17)
$$[Hy, \varphi]_F = [y, H^*\varphi]_W =$$
$$= \sum_{j=0}^{n-1} ((H^*\varphi)^{(j)}(a))^* \, y^{(j)}(a) + \int_a^b ((H^*\varphi)^{(n)})^* \, y^{(n)} \, \mathrm{d}t$$
$$\text{for all} \quad y \in W_m^{n,p} \quad \text{and} \quad \varphi \in F^*,$$

where the function

(2,18)
$$(H^*\varphi)(t) = \sum_{j=0}^{n-1} (H^*\varphi)^{(j)}(a) \frac{(t-a)^j}{j!} +$$
$$+ \int_a^t \left(\int_a^{t_1} \left(\ldots \left(\int_a^{t_{n-1}} (H^*\varphi)^{(n)} \, \mathrm{d}t_n\right) \mathrm{d}t_{n-1}\right) \ldots\right) \mathrm{d}t_1 \quad \text{on} \quad [a, b]$$

belongs to $W_m^{n,q}$ with $1/p + 1/q = 1$ if $p > 1$ and $q = \infty$ if $p = 1$.

Further, since by Lemma 2.3

(2,19)
$$[Hy, \varphi]_F = \sum_{j=0}^{n-1} [U_j(y^{(j)}), \varphi]_F + [V(y^{(n)}), \varphi]_F$$
$$\text{for any} \quad y \in W_m^{n,p} \quad \text{and} \quad \varphi \in F^*,$$

we obtain for $y \in W_m^{n,p}$ and $\varphi \in F^*$

$$[U_j(y^{(j)}), \varphi]_F = ((H^*\varphi)^{(j)}(a))^* y^{(j)}(a) \quad (j = 0, 1, \ldots, n-1), \tag{2,20}$$

$$V^*\varphi = (H^*\varphi)^{(n)}.$$

If $1 < p \leqq \infty$, $F = (^*F)^*$ and $H = (^*H)^*$ for some locally convex topological vector space $^*F$ and a linear continuous operator $^*H : {}^*F \to W_m^{n,q}$ (where $1/p + 1/q = 1$ if $p < \infty$ and $q = 1$ if $p = \infty$; $^*H$ is the pre-adjoint of H), we have for every $\varphi \in {}^*F$ a function $^*H\varphi \in W_m^{n,q}$ such that

$$[\varphi, Hy]_{*F} = [^*H\varphi, y]_W = \sum_{j=0}^{n-1} (y^{(j)}(a))^* (^*H\varphi)^{(j)}(a) + \tag{2.21}$$

$$+ \int_a^b (y^{(n)})^* (^*H\varphi)^{(n)} \, dt \quad \text{for any} \quad y \in W_m^{n,p}.$$

The identities (2,17)–(2,21) will be used freely throughout the paper. In particular, we shall usually write $V^*\varphi$ for $(H^*\varphi)^{(n)}$ and similarly

$$^*V\varphi = (^*H\varphi)^{(n)} \quad \text{for} \quad \varphi \in {}^*F. \tag{2,22}$$

3. CONTINUOUS THEORY

Throughout this section we suppose

3.1. Assumptions. $1 \leqq p \leqq \infty$, A_0 and A are $k \times m$-matrix valued functions, $k \geqq m$, A_0 is esentially bounded on $[a, b]$, A is L^p-integrable on $[a, b]$ if $p < \infty$ and essentially bounded on $[a, b]$ if $p = \infty$. F is a locally convex topological vector space and $H : W_m^{1,p} \to F$ is a linear and continuous operator; q is a number such that $1/p + 1/q = 1$ if $1 < p < \infty$, $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$.

The symbols $(H^*\varphi)(a)$, $V^*\varphi$, $(^*H\varphi)(a)$ and $^*V\varphi$ are defined by (2,17)–(2,22).

3.2. Definition. Let $\mathscr{L} : W_m^{1,p} \to L_k^p \times F$ and $\mathscr{L}_0 : W_m^{1,p} \to L_k^p \times F \times C^{2m}$ be given by

$$\mathscr{L}y = \begin{pmatrix} \ell y \\ Hy \end{pmatrix} \quad \text{and} \quad \mathscr{L}_0 y = \begin{pmatrix} \ell y \\ Hy \\ y(a) \\ y(b) \end{pmatrix}, \quad \ell y = A_0 y' + Ay.$$

Using the Minkowski and Hölder inequalities we obtain the following

3.3. Lemma. *$\mathscr{L}$ and $\mathscr{L}_0$ are linear continuous operators.*

(In fact, for any $y \in W_m^{1,p}$ we have $\|\ell y\|_p \leqq (\|A_0\|_\infty + \varkappa\|A\|_p) \|y\|_{1,p}$ where $\varkappa = (b - a)^{1/q}$ if $p > 1$ and $\varkappa = 1$ if $p = 1$.)

It follows that the adjoints $\mathscr{L}^* : L^q_k \times F^* \to W^{1,q}_m$ and $\mathscr{L}^*_0 : L^q_k \times F^* \times C^{2m} \to \to W^{1,q}_m$ of $\mathscr{L}$ and $\mathscr{L}_0$ exist for $1 \leqq p < \infty$ and satisfy the Green relations

$$[\mathscr{L}y, (z, \varphi)] = [y, \mathscr{L}^*(z, \varphi)]_W \tag{3,1}$$

and

$$[\mathscr{L}_0 y, (z, \varphi, \alpha, \beta)] = [y, \mathscr{L}^*_0(z, \varphi, \alpha, \beta)]_W \tag{3,2}$$

for all $y \in W^{1,p}_m$, $z \in L^q_k$, $\varphi \in F^*$, α and $\beta \in C^m$. The left-hand pairings in (3,1) and (3,2) are on $(L^p_k \times F) \times (L^q_k \times F^*)$ and $(L^p_k \times F \times C^{2m}) \times L^q_k \times F^* \times C^{2m})$, respectively, while the right-hand ones are given by (2,6).

Our aim now will be to find the analytic form of $\mathscr{L}^*$ and $\mathscr{L}^*_0$.

3.4. Theorem. *Let $1 \leqq p < \infty$. Then the adjoint $\mathscr{L}^* : L^q_k \times F^* \to W^{1,q}_m$ is given by*

$$\mathscr{L}^*(z, \varphi)(t) = \mathscr{L}^*(z, \varphi)(a) + \int_a^t \left\{A^*_0 z + V^*\varphi + \int_s^b A^* z \, d\tau\right\} ds\,, \quad t \in [a, b]\,, \tag{3,3}$$

$$\mathscr{L}^*(z, \varphi)(a) = \int_a^b A^* z \, ds + (H^*\varphi)(a) \quad \textit{for} \quad z \in L^q_k \quad \textit{and} \quad \varphi \in F^*\,.$$

Proof. Writing (3,1) explicitly (see (2,3)–(2,6) and (2,17)), we have

$$\int_a^b z^* A_0 y' \, dt + \int_a^b z^* A y \, dt + ((H^*\varphi)(a))^* y(a) + \int_a^b (V^*\varphi)^* y' \, dt =$$
$$= \int_a^b (\mathscr{L}^*(z, \varphi))'^* y' \, dt + (\mathscr{L}^*(z, \varphi)(a))^* y(a)\,.$$

Integrating $\int_a^b z^* A y \, dt$ by parts and rearranging terms gives

$$\int_a^b \left\{z^* A_0 + (V^*\varphi)^* + \int_t^b z^* A \, ds - (\mathscr{L}^*(z, \varphi))'^*\right\} y' \, dt +$$
$$+ \left\{\int_a^b z^* A \, dt + ((H^*\varphi)(a))^* - (\mathscr{L}^*(z, \varphi)(a))^*\right\} y(a) = 0$$

for any $y \in W^{1,p}_m$, $z \in L^q_k$ and $\varphi \in F^*$. Since this represents the zero functional on $W^{1,p}_m$,

$$(\mathscr{L}^*(z, \varphi))' = A^*_0 z + (V^*\varphi) + \int_t^b A^* z \, ds\,,$$

$$\mathscr{L}^*(z, \varphi)(a) = \int_a^b A^* z \, ds + (H^*\varphi)(a)$$

for $z \in L_k^q$ and $\varphi \in F^*$. Integrating the first expression completes the proof.

3.5. Corollary. *If* $1 \leqq p < \infty$, *then*

(3,4)
$$\mathscr{L}_0^*(z, \varphi, \alpha, \beta)(t) = \mathscr{L}_0^*(z, \varphi, \alpha, \beta)(a) + \int_a^t \left\{A_0^* z + V^*\varphi + \int_s^b A^* z \, d\tau\right\} ds + \beta(t - a), \quad t \in [a, b],$$

$$\mathscr{L}_0^*(z, \varphi, \alpha, \beta)(a) = \int_a^b A^* z \, dt + (H^*\varphi)(a) + \beta + \alpha$$

for all $z \in L_k^q, \quad \varphi \in F^*, \quad \alpha, \beta \in C^m$.

Proof. Setting

$$H_0 : y \in W_m^{1,p} \to \begin{pmatrix} Hy \\ y(a) \\ y(b) \end{pmatrix} \in F \times C^{2m} := F_0,$$

we have for any $y \in W_m^{1,p}$, $\varphi \in F^*$, α and $\beta \in C^m$

$$[H_0 y, (\varphi, \alpha, \beta)] := [Hy, \varphi]_F + \alpha^* y(a) + \beta^* y(b) =$$
$$= \{((H^*\varphi)(a))^* + \alpha^* + \beta^*\} y(a) + \int_a^b \{(V^*\varphi)^* + \beta^*\} y' \, dt.$$

Replacing in (3,3) $V^*\varphi$ by $V^*\varphi + \beta$ and $(H^*\varphi)(a)$ by $(H^*\varphi)(a) + \alpha + \beta$ we obtain (3,4).

3.6. Remark. If, in addition to 3.1, also
(3,5) $1 < p \leqq \infty$, $F = ({}^*F)^*$ *and* $H = ({}^*H)^*$ *for some locally convex topological vector space* ${}^*F$ *and linear continuous operator* ${}^*H : {}^*F \to W_m^{1,q}$,
we denote for $z \in W_m^{1,q}$ and $\varphi \in {}^*F$

(3,6)
$${}^*\mathscr{L}(z, \varphi)(t) = {}^*\mathscr{L}(z, \varphi)(a) + \int_a^t \left\{A_0^* z + ({}^*V\varphi) + \int_s^b A^* z \, d\tau\right\} ds,$$
$$t \in [a, b],$$
$${}^*\mathscr{L}(z, \varphi)(a) = \int_a^b A^* z \, dt + ({}^*H\varphi)(a).$$

It is easy to check that the operator ${}^*\mathscr{L} : L_k^q \times {}^*F \to W_m^{1,q}$ defined by (3,6) fulfils $({}^*\mathscr{L})^* = \mathscr{L}$, i.e. ${}^*\mathscr{L}$ is the pre-adjoint of $\mathscr{L}$ and $\mathscr{L}$ is weakly*-closed in $W_m^{1,p} \times (L_k^p \times F)$.

Similarly, by an obvious modification of (3,4) we would obtain an analytical representation for the pre-adjoint $^*\mathscr{L}_0$ of $\mathscr{L}_0$.

At this point we introduce a new notation which will be convenient in the remainder of the paper.

3.7. Notation. Given $z \in L_k^q$ and $\psi \in L_m^q$ $(1 \leqq q \leqq \infty)$ such that $A_0^* z + \psi \in AC_m$, we denote

(3,7)
$$\ell^+(z, \psi) := -(A_0^* z + \psi)' + A^* z .$$

In terms of Notation 3.7 we have

3.8. Corollary. *Let* $1 \leqq p < \infty$, *then* $N(\mathscr{L}^*)$ *consists of all pairs* $(z, \varphi) \in L_k^q \times F^*$ *such that*

(3,8)
$$A_0^* z + \psi \in AC_m ,$$

(3,9)
$$\ell^+(z, \psi) = 0 \quad a.e. \ on \quad [a, b] ,$$

(3,10)
$$[A_0^* z + \psi](b) = 0 , \quad [A_0^* z + \psi](a) = (H^* \varphi)(a)$$

for some $\psi = V^* \varphi$ *a.e. on* $[a, b]$.

Proof. By Theorem 3.4, $\mathscr{L}^*(z, \varphi) = 0$ if and only if

(3,11)
$$\int_a^t \left\{ A_0^* z + V^* \varphi + \int_s^b A^* z \, d\tau \right\} ds = 0 \quad \text{on} \quad [a, b]$$

and

(3,12)
$$\int_a^b A^* z \, dt + (H^* \varphi)(a) = 0 .$$

Differentiating (3,11) we obtain

$$\eta(t) := [A_0^* z + V^* \varphi] + \int_t^b A^* z \, ds = 0 \quad \text{a.e. on} \quad [a, b] .$$

If $\psi(t) := (V^* \varphi)(t) - \eta(t)$ on $[a, b]$, then $\psi = V^* \varphi$ a.e. on $[a, b]$ and

(3,13)
$$[A_0^* z + \psi](t) + \int_t^b A^* z \, ds = 0 \quad \text{on} \quad [a, b] .$$

In particular, $A_0^* z + \psi \in AC_m$, $[A_0^* z + \psi](b) = 0$ and in virtue of (3,12), $[A_0^* z + \psi](a) = (H^* \varphi)(a)$. Finally, differentiating (3,13) gives (3,9).

3.9. Corollary. *If $1 \leqq p < \infty$, then $N(\mathscr{L}_0^*)$ consists of all $(z, \varphi, \alpha, \beta) \in L_k^q \times F^* \times$ $\times C^{2m}$ such that (3,8) and (3,9) hold for some $\psi = V^*\varphi$ a.e. on $[a, b]$ and*

$$\alpha = [A_0^* z + \psi](a) - (H^*\varphi)(a), \quad \beta = -[A_0^* z + \psi](b). \tag{3,14}$$

Proof. By Corollaries 3.5 and 3.8 $N(\mathscr{L}_0^*)$ is the set of all $(z, \varphi, \alpha, \beta) \in L_k^q \times F^* \times$ $\times C^{2m}$ for which there exists $\chi \in L_k^q$ such that $\chi = V^*\varphi + \beta$ a.e. on $[a, b]$,

$$A_0^* z + \chi \in AC_m, \tag{3,15}$$

$$\ell^+(z, \chi) = 0 \quad \text{a.e. on} \quad [a, b] \tag{3,16}$$

and

$$[A_0^* z + \chi](a) = (H^*\varphi)(a) + \alpha + \beta, \quad [A_0^* z + \chi](b) = 0. \tag{3,17}$$

If we put $\psi(t) := \chi(t) - \beta$, then $\psi = V^*\varphi$ a.e. on $[a, b]$ and the conditions (3,15), (3,16) and (3,17) reduce to (3,8), (3,9) and (3,14), respectively. This completes the proof.

3.10. Remark. The modification of assertions 3.8 and 3.9 in the case that (3,5) holds is obvious (cf. 3.6 and 3.7).

3.11. Notation. $\tilde{A}_0$ and $\tilde{A}$ are the $m \times m$-matrices formed by the first m rows of A_0 and A, respectively.

3.12. Theorem. *If $\tilde{A}_0$ is invertible a.e. on $[a, b]$ and $\tilde{A}_0^{-1}$ is essentially bounded on $[a, b]$, then $\mathscr{L}$ and $\mathscr{L}_0$ have closed ranges in $L_k^p \times F$ and $L_k^p \times F \times C^{2m}$, respectively.*

Proof. It is sufficient to prove that $R(\mathscr{L})$ is closed. Let $\tilde{A}_0^{-1}$ be essentially bounded on $[a, b]$. Then there exist linear bounded operators $\Phi : C^m \to W_m^{1,p}$ and $\Psi : L_m^p \to$ $\to W_m^{1,p}$ such that for every $f \in L_m^p$ and $c \in C^m$ the function

$$x = \Phi c + \Psi f$$

is the unique solution of the initial value problem

$$\tilde{A}_0 x' + \tilde{A} x = f \quad \text{a.e. on} \quad [a, b], \quad x(a) = c$$

in AC_m. (If X denotes the fundamental matrix solution of the corresponding homogeneous equation, then

$$(\Phi c)(t) = X(t)\, c, \quad (\Psi f)(t) = X(t) \int_a^t X^{-1} f \, \mathrm{d}s \quad \text{on} \quad [a, b].)$$

Let us denote $\tilde{F} := L^p_{k-m} \times F$. Write

$$A_0 = \begin{bmatrix} \tilde{A}_0 \\ B_0 \end{bmatrix}, \quad A = \begin{bmatrix} \tilde{A} \\ B \end{bmatrix} \quad \text{and} \quad \tilde{H} : y \in W^{1,p}_m \to \begin{bmatrix} B_0 y' + By \\ Hy \end{bmatrix} \in \tilde{F}.$$

Then

$$\mathscr{L} : y \in W^{1,p}_m \to \begin{bmatrix} \tilde{A}_0 y' + \tilde{A} y \\ \tilde{H} y \end{bmatrix} \in L^p_m \times \tilde{F}.$$

Let $g \in L^p_m$ and $\varrho \in \tilde{F}$. Then $(g, \varrho) \in R(L)$ if and only if there exists $c \in C^m$ such that

$$\tilde{H}(\Phi c) = \varrho - \tilde{H}(\Psi g) := T(g, \varrho). \tag{3,18}$$

Clearly, $T : L^p_m \times \tilde{F} \to \tilde{F}$ is linear and continuous. Define $\Theta : C^m \to \tilde{F}$ by $\Theta c = \tilde{H}(\Phi c)$, then Θ is also linear and continuous. Furthermore, in virtue of (3,18), $R(\mathscr{L}) = T^{-1} R(\Theta)$. Since the domain of Θ is C^m, its range is finite dimensional and thus closed. Because T is continuous, $R(\mathscr{L})$ must be closed too.

3.13. Remark. Theorem 3.12 is in the case $k = m$ due to Wexler [28]. Our approach is essentially the same.

As a direct consequence of Theorem 3.12 we have

3.14. Theorem (Fredholm Alternatives). *Under the assumptions of Theorem* 3.12 *we have*

(i) $R(\mathscr{L}) = {}^{\perp}N(\mathscr{L}^*)$, $N(\mathscr{L}^*) = R(\mathscr{L})^{\perp}$, $R(\mathscr{L}_0) = {}^{\perp}N(\mathscr{L}_0^*)$ *and* $N(\mathscr{L}_0^*) = R(\mathscr{L}_0)^{\perp}$ *if* $1 \leqq p < \infty$;

(ii) *if* (3,5) *holds, then* $R(\mathscr{L}) = N({}^*\mathscr{L})^{\perp}$, $N({}^*\mathscr{L}) = {}^{\perp}R(\mathscr{L})$, $R(\mathscr{L}_0) = N({}^*\mathscr{L}_0)^{\perp}$ *and* $N({}^*\mathscr{L}_0) = {}^{\perp}R(\mathscr{L}_0)$.

3.15. Remark. Furthermore, we have

$$R(\mathscr{L}^*) = N(\mathscr{L})^{\perp}, \quad N(\mathscr{L}) = {}^{\perp}R(\mathscr{L}^*), \quad N(\mathscr{L}_0) = \{0\} \quad \text{and}$$

$$R(\mathscr{L}_0^*) = N(\mathscr{L}_0)^{\perp} = W^{1,q}_m \quad \text{if} \quad 1 \leqq p < \infty$$

and

$$R({}^*\mathscr{L}) = {}^{\perp}N(\mathscr{L}), \quad N(\mathscr{L}) = R({}^*\mathscr{L})^{\perp}, \quad N(\mathscr{L}_0) = \{0\} \quad \text{and}$$

$$R({}^*\mathscr{L}_0) = W^{1,q}_m \quad \text{if} \quad 1 < p \leqq \infty \quad \text{and (3,5) holds}.$$

3.16. Remark. $\mathscr{L}$ being continuous, it is closed. Thus if $1 < p < \infty$ and F is reflexive, then $\mathscr{L}^{**} = \mathscr{L}$.

4. "UNBOUNDED" THEORY ($p \neq \infty$)

In this section we endow $W_m^{1,p}$ with the L^p-norm (2,1) so that it can be viewed as a dense subspace of L_m^p. The homogeneous system corresponding to (1,1) determines a pair of unbounded maximal and minimal operators L, L_0 having domain in L_m^p and range in L_k^p.

Throughout the section we assume that the hypotheses of Theorem 3.12 hold.

4.1. Assumptions. A_0, A, H, F and p, q satisfy 3.1. Furthermore, the $m \times m$-matrix $\tilde{A}_0$ formed by the first m rows of A_0 (cf. 3.11) is invertible a.e. on $[a, b]$ and $\tilde{A}_0^{-1}$ is essentially bounded on $[a, b]$. We shall restrict ourselves to the case $p \neq \infty$. (Nevertheless, the assertions 4.3 and 4.4 are true also for $p = \infty$.)

4.2. Definition. L is the operator $L_m^p \to L_k^p$ with the definition domain

$$D := \{y \in L_m^p : y \in W_m^{1,p} \text{ and } Hy = 0\} \tag{4,1}$$

and with values

$$Ly := \ell y \quad \text{for} \quad y \in D .$$

L_0 is the restriction of L to

$$D_0 := \{y \in D : y(a) = y(b) = 0\} . \tag{4,2}$$

Clearly

$$R(L) = \{f \in L_k^p : (f, 0) \in R(\mathscr{L})\} \quad \text{and} \quad R(L_0) = \{f \in L_k^p : (f, 0) \in R(\mathscr{L}_0)\} .$$

Since by Theorem 3.12 $R(\mathscr{L})$ and $R(\mathscr{L}_0)$ are closed, it follows that $R(L)$ and $R(L_0)$ are also closed.

4.3. Theorem. *Linear operators L and L_0 have closed ranges.*

The purpose of this section is to determine the adjoints L^*, L_0^* of L and L_0. In general, D and D_0 are not dense in L_m^p (cf. [7]) and hence L^*, L_0^* are relations in $L_m^q \times L_k^q$ characterized by their graphs

$$G(L^*) = \{(u, z) \in L_m^q \times L_k^q : [\ell y, z] = [y, u] \text{ for all } y \in D\} \tag{4,3}$$

and

$$G(L_0^*) = \{(u, z) \in L_m^q \times L_k^q : [\ell y, z] = [y, u] \text{ for all } y \in D_0\} , \tag{4,4}$$

respectively (cf. (2,9)).*)

---

*) Throughout this section $[\cdot, \cdot]$ means the pairing in $L_m^p \times L_m^q$ or $L_k^p \times L_k^q$ according to the circumstances.

In virtue of the Fredholm Alternatives for nondensely defined operators with closed range (cf. 2.2) we have

4.4. Theorem. *If* $1 \leqq p < \infty$, *then*

$$R(L) = {}^{\perp}N(L^*), \quad N(L^*) = R(L)^{\perp},$$
$$R(L_0) = {}^{\perp}N(L_0^*), \quad N(L_0^*) = R(L_0)^{\perp}.$$

4.5. Theorem. *The graph* $G(L^*)$ *of the adjoint relation* L^* *to* L *is the set of all couples* $(u, z) \in L_m^q \times L_k^q$ *for which there exists* $\psi \in L_m^q$ *such that*

$$A_0^* z + \psi \in AC_m, \tag{4,5}$$

$$u = \ell^+(z, \psi) := [A_0^* z + \psi]' + A^* z, \tag{4,6}$$

$$[A_0^* z + \psi](b) = 0 \tag{4,7}$$

and

$$[A_0^* z + \psi]^*(a)\, y(a) + \int_a^b \psi^* y' \,dt = 0 \quad \textit{for all} \quad y \in D. \tag{4,8}$$

Proof. a) Let $(u, z) \in G(L^*) \subset L_m^q \times L_k^q$. Then by the definition (4,3) we have for all $y \in D$

$$0 = [\ell y, z] - [y, u] = \int_a^b z^*(A_0 y' + Ay)\,dt - \int_a^b u^* y \,dt.$$

Integrating by parts we obtain further

$$\left(\int_a^b (A^* z - u)\,dt\right)^* y(a) + \int_a^b \left(A_0^* z + \int_t^b (A^* z - u)\,d\tau\right)^* y' \,dt = 0 \quad \text{for all} \quad y \in D.$$

Let $\psi \in L_m^q$ be such that

$$[A_0^* z + \psi](t) + \int_t^b (A^* z - u)\,d\tau = 0 \quad \text{for any} \quad t \in [a, b]. \tag{4,9}$$

Then

$$\alpha^* y(a) - \int_a^b \psi^* y' \,dt = 0 \quad \text{for all } y \in D \tag{4,10}$$

where

$$\alpha := \int_a^b (A^* z - u)\,dt \in C^m. \tag{4,11}$$

It follows immediately from (4,9) that $A_0^* z + \psi$ is absolutely continuous on $[a, b]$

and vanishes at $t = b$ (i.e. (4,5) and (4.7) hold). Moreover, differentiation of (4,9) yields (4,6). Inserting (4,6) and (4,7) into (4,11) we obtain

$$\alpha = \int_a^b [A_0^* z + \psi]' \, dt = -[A_0^* z + \psi](a).$$

Hence (4,10) reduces to (4,8). To summarize: if $(u, z) \in G(L^*)$, then there exists $\psi \in L_m^q$ such that (4,5)–(4,8) hold.

b) Let $(u, z) \in L_m^q \times L_k^q$ and let $\psi \in L_m^q$ be such that (4,5)–(4,8) hold. Then using the integration-by-parts formula we obtain for every $y \in D$

$$\int_a^b u^* y \, dt = -\int_a^b [A_0^* z + \psi]'^* \, y \, dt + \int_a^b z^* A y \, dt =$$

$$= -[A_0^* z + \psi]^* \, y \Big|_a^b + \int_a^b (z^* A_0 + \psi^*) \, y' \, dt + \int_a^b z^* A y \, dt =$$

$$= \int_a^b z^*(\ell y) \, dt - [A_0^* z + \psi]^* \, y \Big|_a^b + \int_a^b \psi^* y' \, dt = \int_a^b z^*(\ell y) \, dt .$$

Consequently $(u, z) \in G(L^*)$ and this completes the proof.

4.6. Corollary. *Let us denote by D_0' the set of the derivatives of all functions from D_0. Then $G(L_0^*)$ is the set of all couples $(u, z) \in L_m^q \times L_k^q$ for which there exists $\psi \in (D_0')^{\perp}$ such that (4,5) and (4,6) hold.*

Proof. By 4.5 $G(L_0^*)$ is the set of all couples $(u, z) \in L_m^q \times L_k^q$ for which there exists $\chi \in L_m^q$ such that

$$A_0^* z + \chi \in AC_m, \quad u = \ell^+(z, \chi), \quad [A_0^* z + \chi](b) = 0$$

and (since $y(a) = 0$ for all $y \in D_0$)

$$\int_a^b \chi^* y' \, dt = 0 \quad \text{for all} \quad y \in D_0 .$$

Since $\ell^+(z, \psi + \delta) = \ell^+(z, \psi)$ for all $z \in L_k^q$, $\psi \in L_m^q$ and $\delta \in C^m$, it is easy to verify that this happens if and only if there exists $\psi \in L_m^q$ such that (4,5) and (4,6) hold and

(4,12) $$\int_a^b \psi^* y' \, dt = 0 \quad \text{for all} \quad y \in D_0 \quad (\text{i.e. } \psi \in (D_0')^{\perp}).$$

4.7. Remark. Let us notice that in the proofs of Theorem 4.5 and of its corollary 4.6 we used neither the assumption about the essential boundedness of $\tilde{A}_0^{-1}$ nor the special forms (4,1), (4,2) of the definition domains D, D_0 of the operators L, L_0.

We needed only that $D \subset W_m^{1,p}$, $D_0 \subset W_m^{1,p}$ and $y(a) = y(b) = 0$ for any $y \in D_0$. Hence the characterization (4,5)–(4,8) of the adjoint L^* obtained in 4.5 is true also for operators L defined on an arbitrary linear subset D of $W_m^{1,p}$ and with values $A_0 y' + Ay$ where A_0 and A satisfy Assumptions 3.1. (A_0 need not contain an invertible submatrix $\tilde{A}_0$ as required in 4.1.) Analogously, if A_0, A fulfill 3.1 and $D_0 \subset W_m^{1,p}$ is such that $y(a) = y(b) = 0$ for each $y \in D_0$, then $(u, z) \in L_m^q \times L_k^q$ belongs to $G(L_0^*)$ if and only if there exists $\psi \in (D_0')^\perp$ such that (4,5) and (4,6) hold.

The following alternative characterization of L^* makes use of the special form (4,1) of the definition domain D of L.

4.8. Theorem. *$G(L^*)$ is the set of all $(u, z) \in L_m^q \times L_k^q$ for which there exist $\zeta \in W_m^{1,q}$ and its derivative $\zeta' \in L_m^q$ such that*

$$(4,13) \qquad A_0^* z + \zeta' \in AC_m \,,$$

$$(4,14) \qquad u = \ell^+(z, \zeta) \quad a.e.\ on \quad [a, b] \,,$$

$$(4,15) \qquad [A_0^* z + \zeta'](b) = 0 \,, \quad [A_0^* z + \zeta'](a) = \zeta(a) \,,$$

$$(4,16) \qquad \zeta \in \mathrm{cl}^*(R(H^*)) \quad (\text{the weak*-closure in } W_m^{1,q}) \,.$$

Proof. a) Let $(u, z) \in G(L^*)$. Then there exists $\psi \in L_m^q$ such that (4,5)–(4,8) hold. Let us put

$$(4,17) \qquad \zeta(a) = [A_0^* z + \psi](a) \,, \quad \zeta(t) = \zeta(a) + \int_a^t \psi \,\mathrm{d}\tau \quad \text{for} \quad t \in [a, b] \,.$$

Then the relations (4,13)–(4,15) follows directly from (4,5)–(4,7). Furthermore we have by (4,8) and (4,17)

$$\zeta^*(a)\, y(a) + \int_a^b \zeta' y' \,\mathrm{d}t = 0 \quad \text{for all} \quad y \in D \,.$$

It means that $\zeta \in D^\perp \subset W_m^{1,q}$ (in the sense of the pairing (2,6)). As $D = N(H)$ and $H : W_m^{1,p} \to F$ is continuous, we have $D^\perp = N(H)^\perp = \mathrm{cl}^*(R(H^*))$ (cf. 2.2), i.e. (4,16) holds.

b) On the other hand, if $(u, z) \in L_m^q \times L_k^q$, $\zeta \in W_m^{1,q}$ and $\zeta' \in L_m^q$ are such that (4,13)–(4,16) hold, then it is easy to verify that (u, z) and $\psi = \zeta'$ fulfil (4,5)–(4,8). This complete the proof.

4.9. Remark. Let us recall that (cf. 2.3) $Hy = Vy'$ for $y \in D_0$ where V is a linear continuous operator mapping L_m^p into F. Let us denote by W the operator

$$(4,18) \qquad W : w \in L_m^p \to \left(Vw \,, \int_a^b w \,\mathrm{d}\tau \right) \in F \times C^m =: \Phi \,.$$

Then $D_0' = N(W)$ and

(4,19) $$(D_0')^\perp = N(W)^\perp = \text{cl}^*(R(W^*)) \quad \text{(the weak*-closure in } L_m^q) .$$

Furthermore, as

$$[Ww, (\varphi, \alpha)]_\Phi = [Vw, \varphi]_F + \alpha^* \int_a^b w \, d\tau = \int_a^b [V^*\varphi + \alpha]^* \, w \, dt$$

for all $w \in L_m^p$, $\varphi \in F^*$ and $\alpha \in C^m$, the adjoint $W^* : \Phi^* = F^* \times C^m \to L_m^q$ of W is given by

$$W^* : (\varphi, \alpha) \in \Phi^* \to V^*\varphi + \alpha \in L_m^q .$$

Consequently

(4,20) $$R(W^*) = R(V^*) + C^m ,$$

the set of all $\eta \in L_m^q$ for which there exists an m-vector α such that the function $\eta - \alpha$ belongs to $R(V^*)$. (C^m stands here for the set of all m-vector valued functions constant on $[a, b]$.) By (4,19) and (4,20) we have

$$(D_0')^\perp = \text{cl}^*(R(V^*) + C^m) .$$

Since $R(V^*) + C^m \subset \text{cl}^*(R(V^*)) + C^m$ and $\text{cl}^*(R(V^*)) + C^m$ is weakly*-closed in L_m^q (cf. Rudin [21], 1.42), it follows that

$$(D_0')^\perp \subset \text{cl}^*(R(V^*)) + C^m .$$

On the other hand, from the definition of the weak*-closure we obtain easily

$$\text{cl}^*(R(V^*)) + C^m \subset \text{cl}^*(R(V^*) + C^m) = (D_0')^\perp .$$

Hence

(4,21) $$(D_0')^\perp = \text{cl}^*(R(V^*)) + C^m .$$

This enables us to prove the following theorem.

4.10. Theorem. *$G(L_0^*)$ is the set of all $(u, z) \in L_m^q \times L_k^q$ for which there exists $\psi \in \text{cl}^*(R(V^*))$ such that (4,5) and (4,6) hold.*

Proof. Let us denote by G_0^+ the set of all $(u, z) \in L_m^q \times L_k^q$ for which there exists $\psi \in \text{cl}^*(R(V^*))$ such that $A_0^*z + \psi \in AC_m$ and $u = \ell^+(z, \psi) = -[A_0^*z + \psi]' + A^*z$. Since $\text{cl}^*(R(V^*)) \subset (D_0')^\perp$ by (4,21), it follows immediately from 4.6 that $G_0^+ \subset G(L_0^*)$.

On the other hand, if $(u, z) \in G(L_0^*)$, then by 4.6 and (4,21) there exist $\psi \in \text{cl}^*(R(V^*))$ and $\alpha \in C^m$ such that

$$u = \ell^+(z, \psi + \alpha) = -[A_0^*z + \psi + \alpha]' + A^*z = \\ = -[A_0^*z + \psi]' + A^*z = \ell^+(z, \psi)$$

and

$$(4,22)\qquad A_0^* z + \psi + \alpha \in AC_m .$$

Obviously, (4,22) implies that $A_0^* z + \psi \in AC_m$. Hence $G(L_0^*) = G_0^+$ and this completes the proof.

4.11. Remark. Given a $\varphi \in F^*$, there exists $z \in L_k^q$ such that $A_0^* z + V^*\varphi \in AC_m$, $\ell^+(z, V^*\varphi) \in L_m^q$, $[A_0^* z + V^*\varphi](a) = (H^*\varphi)(a)$, $[A_0^* z + V^*\varphi](b) = 0$. In fact, put $z^* = (\zeta^*, 0)$ where

$$\zeta(t) = -\tilde{A}_0^{-1}(t)\left[(V^*\varphi)(t) + (H^*\varphi)(a)\frac{b-t}{b-a}\right].$$

4.12. Remark. If $1 < p < \infty$, then L_m^p is reflexive and $\overline{N} = \mathrm{cl}^*(N)$ for any $N \subset L_m^p$. Thus, in this case we may replace the weak*-closure $\mathrm{cl}^*(R(V^*))$ of $R(V^*)$ in the characterization of L_0^* given in 4.10 by the norm closure $\overline{R(V^*)}$. Analogously for L^*.

4.13. Remark. Notice that by 4.8 the definition domain $D(L^*)$ of the adjoint relation L^* to L is the set of all $z \in L_k^q$ for which there exist $\zeta \in \mathrm{cl}^*(R(H^*))$ and its derivative $\zeta' \in L_k^q$ such that $A_0^* z + \zeta' \in AC_m$, $\ell^+(z, \zeta') \in L_m^q$, $[A_0^* z \in \zeta'](a) = \zeta(a)$ and $[A_0^* z + \zeta'](b) = 0$. Similarly $D(L_0^*)$ is the set of all $z \in L_k^q$ for which there exists $\psi \in \mathrm{cl}^*(R(V^*))$ such that $A_0^* z + \psi \in AC_m$ and $\ell^+(z, \psi) \in L_m^q$.

On the basis of 3.8 one could suggest as possible adjoints to L, L_0 the relations L^+, L_0^+ given by

$$G(L_0^+) = \{(\ell^+(z, V^*\varphi), z) : (z, \varphi) \in D_0^+\}$$

and

$$G(L^+) = \{(\ell^+(z, V^*\varphi), z) : (z, \varphi) \in D^+\}$$

where

$$D_0^+ = \{(z, \varphi) \in L_k^q \times F^* : [A_0^* z + V^*\varphi] \in AC_m ,\quad \ell^+(z, V^*\varphi) \in L_m^q\}$$

and

$$D^+ = \{(z, \varphi) \in D_0^+ : [A_0^* z + V^*\varphi](a) = (H^*\varphi)(a),\ [A_0^* z + V^*\varphi](b) = 0\} .$$

It is easy to verify that the definition domain $D(L^+)$ of L^+ is the set of all $z \in L_k^q$ for which there exist $\zeta \in R(H^*)$ and its derivative $\zeta' \in R(V^*)$ such that $A_0^* z + \zeta' \in AC_m$, $\ell^+(z, \zeta') \in L_m^q$, $[A_0^* z + \zeta'](a) = \xi(a)$ and $[A_0^* z + \zeta'](b) = 0$. Consequently $L^* = L^+$ if and only if

$$(4,23)\qquad \mathrm{cl}^*(R(H^*)) = R(H^*) .$$

(Then also $\mathrm{cl}^*(R(V^*)) = R(V^*)$, cf. (2,18) and (2,20).) Similarly $L_0 = L_0^+$ if and only if

$$(4,24)\qquad \mathrm{cl}^*(R(V^*)) = R(V^*) .$$

In particular, if $\dim F < \infty$, then also $\dim R(H^*) < \infty$ and $\dim R(V^*) < \infty$ and (4,23) and (4,24) hold. We close the paper with a converse result.

4.14. Lemma. *Suppose* $1 < p < \infty$, $R(H) = F$ *and* $R(V^*) \subset L^\infty_m$. If $L^*_0 = L^+_0$, *then* F *is finite dimensional.*

Proof. Since $L^*_0 = L^+_0$, $\mathrm{cl}^*(R(V^*)) = \overline{R(V^*)} = R(V^*)$. By the Grothendick lemma (Rudin [21], 5.2) $\dim R(V^*) < \infty$. Since

$$R(H^*) \subset \left\{\int_a^t w \, \mathrm{d}\tau : w \in R(V^*)\right\} + C^m ,$$

$R(H^*)$ is also finite dimensional. Because $R(H) = F$, it is $N(H^*) = \{0\}$. Hence H^* is a one-to-one mapping of F^* onto $R(H^*)$ and this implies that $\dim F < \infty$. Therefore F^* is isometrically isomorphic with F^{**}. The local convexity of F implies that F^* separates points (Hahn-Banach Theorem). Thus the natural mapping $\varkappa : f \in F \to g = \varkappa(f) \in F^{**}$ defined by $g(\varphi) = [f, \varphi]_F$ for $\varphi \in F^*$ is one-to-one and consequently $\dim F < \infty$.

4.15. Corollary. *Suppose* $1 < p < \infty$, $R(V^*) \subset L^\infty_m$ *and let* F *be a Banach space. Then* $L^*_0 = L^+_0$ *if and only if* $\dim R(H) < \infty$.

Proof. Let $U : \gamma \in C^m \to Hc \in F$ where c stands for the constant function $c(t) \equiv \gamma$ on $[a, b]$. We have

$$V : u \in L^p_m \to H\left(\int_a^t u \, \mathrm{d}\tau\right) \in F \quad (\text{cf. } 2.3) .$$

Then $R(H) = R(U) + R(V)$ and since $\dim R(U) < \infty$, $R(H)$ is closed if and only if $R(V)$ is closed. The operator V being continuous, $R(V)$ is closed if and only if $R(V^*)$ is closed. Therefore $L^*_0 = L^+_0$ if and only if $\Phi := R(H)$ is a Banach space. Now, if $R(V^*) \subset L^\infty_m$ and $L^*_0 = L^+_0$, then Lemma 4.14 implies that $\dim \Phi = \dim R(H) < \infty$.

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