

# GENERALIZED BRANCHING PROCESSES II: ASYMPTOTIC THEORY<sup>1</sup>

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## 1. Introduction

This paper continues the study of the process  $X(t)$  = total energy at  $t$ , defined in [9], the latter paper to be referred to from now on as I. All the notations and definitions of I carry over here. In addition some strong regularity conditions will be imposed on the underlying distributions  $G$ ,  $\Phi_j$ ,  $\{q_j\}$ . These will be essentially (a) a homogeneity restriction on  $\Phi_j$ ; (b) the requirement that  $1 - G$  and  $\Phi_j$  be sufficiently small, respectively, for large and small values of their arguments; (c) the existence of certain moments.

After some preliminaries in Section 2, it will be shown in Section 3 that the moments of the process satisfy renewal equations, and thence their asymptotic behavior is given. In Section 4 the convergence, to a random variable, of the energy  $X(t)$  divided by its mean is studied. Some regularity properties of the limit random variable are given in Section 5. Section 6 deals briefly with the total energy of all particles which have existed up to time  $t$ .

Since paper I will be frequently referred to, specific expressions or theorems from that paper will be identified by preceding them by a I, e.g., expression (I.2.1) or Theorem I.3.

The results and techniques of the present paper follow closely the development of Bellman and Harris [1] for the standard age-dependent process, and the purpose of the present paper is to extend the results of [1] to the generalized process. In certain instances it has been possible to reduce problems in this paper to ones which have already been dealt with in [1]. To save space and avoid repetitious analysis, this device has been adopted wherever practicable.

## 2. Preliminaries and assumptions

It will be assumed throughout that for any integer  $j$  and any constant  $c$

$$(A.1) \quad \Phi_j(cx_1, \dots, cx_j | cx_0) = \Phi_j(x_1, \dots, x_j | x_0).$$

If  $\Phi^{(n)}(x | x_0)$  is defined as in (I.3.8.1), then (A.1) implies  $\Phi^{(n)}(cx | cx_0) = \Phi^{(n)}(x | x_0)$ . Write  $\Phi^{(n)}(x | 1) \equiv \Phi^{(n)}(x)$ . It will be further assumed that

$$(A.2) \quad \int_0^\infty x^2 \Phi^{(2)}(dx) \text{ exists;}$$

also that

$$(A.3) \quad q_0 + q_1 < 1;$$

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and

$$(A.4) \quad \int_0^\infty t \, dG(t) < \infty.$$

In Sections 3 and 4 it is also assumed that  $G(t)$  is nonperiodic, i.e., not a step function with all its jumps at integral multiples of some constant.

The homogeneity condition (A.1) is essential in the subsequent analysis. The condition (A.2) will guarantee the existence of the required moments of the process. Note that by Lemma I.3, (A.2) is somewhat stronger than requiring the existence of the second moment of the total energy of the first generation offspring of a parent particle of unit energy. It also implies that  $\sum n^2 q_n < \infty$  (see I.3.8.1). The purpose of (A.3) is to avoid certain uninteresting degeneracies.

By Theorem I.1, (A.2) is thus sufficient to guarantee that (I.2.1) has a unique, bounded solution  $P(x, t | x_0)$ , which is a distribution function. By replacing  $x$  by  $cx$ , and  $x_i, i = 0, 1, \dots, j$ , by  $cx_i$  in (I.2.1), and using the homogeneity property (A.1), one can show that  $P(cx, t | cx_0)$  is also a solution of (I.2.1). Hence by the uniqueness of the solution, it follows that

$$(2.1) \quad P(cx, t | cx_0) = P(x, t | x_0)$$

for any constant  $c$ . From (2.1) it follows at once that if

$$\mu^{(n)}(t | x_0) = \int_0^\infty x^n P(dx, t | x_0)$$

exists, then

$$(2.2) \quad \mu^{(n)}(t | cx_0) = c^n \mu^{(n)}(t | x_0).$$

Now write  $P(x, t | 1) \equiv P(x, t), \Phi_j(x_1, \dots, x_j | 1) \equiv \Phi_j(x_1, \dots, x_j)$ , and let  $\hat{P}(s, t)$  be the characteristic function of  $P(x, t)$ . Then (I.2.1) and (I.2.2) become

$$(2.3) \quad \begin{aligned} P(x, t) &= [1 - G(t)]Z(x - 1) + q_0 G(t)Z(x) \\ &+ \sum_{j=1}^\infty q_j \int_0^t dG(y) \int_0^\infty \cdots \int_0^\infty \Phi_j(dx_1, \dots, dx_j) \\ &\quad \cdot \left[ P\left(\frac{x}{x_1}, t - y\right) * \cdots * P\left(\frac{x}{x_j}, t - y\right) \right] \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} \hat{P}(s, t) &= [1 - G(t)]e^{is} + q_0 G(t) \\ &+ \sum_{j=1}^\infty q_j \int_0^t dG(y) \int_0^\infty \cdots \int_0^\infty \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j \hat{P}(x_i s, t - y). \end{aligned}$$

By Theorem I.1, (2.3) and (2.4) have unique bounded solutions;  $P(x, t)$  is a distribution function, and  $\hat{P}(s, t)$  is its characteristic function. Furthermore,

writing  $\mu^{(n)}(t | 1) \equiv \mu^{(n)}(t)$ , the moment equation (I.3.9) becomes

$$(2.5) \quad \mu^{(n)}(t) = [1 - G(t)] + \sum c(n_1, \dots, n_j) \int_0^t dG(y) \prod_{i=1}^j \mu^{(n_i)}(t - y),$$

where

$$c(n_1, \dots, n_j) = (n_1, \dots, n_j) q_j \int_0^\infty \dots \int_0^\infty \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j x_i^{n_i}$$

and the summation in (2.5) is over  $j = 1, 2, \dots$ , and all  $n_1, \dots, n_j$  such that  $n_1 + \dots + n_j = n$ ,  $n_i \geq 0$ , for all  $i$ .

The existence and uniqueness properties of  $\mu^{(n)}(t)$  can be established directly from Theorem I.2.

**THEOREM 1.** *If  $\int_0^\infty x^n \Phi^{(n)}(dx)$  exists for  $n = 1, \dots, r$ , then*

$$\mu_n(t) = \int_0^\infty x^n P(dx, t), \quad n = 1, \dots, r,$$

*exist, are of exponential order, satisfy (2.5), and are the unique simultaneous solutions (for  $n = 1, \dots, r$ ) of (2.5) among the class of solutions of exponential order.*

*Proof.* Let  $N_k^{(n)}(x_0)$  be the function defined in (I.3.7) and (I.3.8). Then

$$N_k^{(n)}(x_0) = x_0^n \left[ \int_0^\infty x^n \Phi^{(n)}(dx) \right]^k.$$

Hence  $N_{k+1}^{(n)}(x_0)/N_k^{(n)}(x_0) = \int_0^\infty x^n \Phi^{(n)}(dx)$ , and the theorem then follows by hypothesis and Theorem I.2.

Let

$$(2.6) \quad c_n = \sum_{j=1}^\infty q_j \int_0^\infty \dots \int_0^\infty \Phi_j(dx_1, \dots, dx_j) (x_1^n + \dots + x_j^n) = \int_0^\infty x^n \Phi(dx),$$

where  $\Phi(x) \equiv \Phi(x | 1)$ , and  $\Phi(x | x_0)$  is defined in (I.3.3). If  $c_1 \geq 1$ , then the equation

$$(2.7) \quad c_1 \int_0^\infty e^{-st} dG(t) = 1$$

will have a nonnegative root  $s = \sigma$ . If  $c_1 < 1$ , then (2.7) may not have a root. In what follows, it will be seen that the asymptotic behavior of the moments  $\mu_n(t)$  will depend on the constants  $c_n$ , and several cases will have to be considered, depending on the values of the constants. In order to simplify the enumeration of hypotheses for these cases, it will be assumed, once and for all, that:

(A.5) If  $c_1 < 1$ , then  $G$  is such that  $\int_0^\infty e^{-st} dG(t)$  converges in a negative  $s$ -interval containing a root  $\sigma$  of (2.7). Furthermore  $1 - G(t) = O(e^{2\sigma t})$ .

As a final preliminary, some well known results in renewal theory which

will be extensively used in the next section will be summarized in two lemmas. The results contained in these lemmas can be extracted from Feller [3], and from similar summary lemmas in Bellman and Harris [1]. Further references are given in the latter paper.

LEMMA 1. If  $U(t)$  satisfies the equation

$$(2.8) \quad U(t) = bK(t) + a \int_0^t U(t-y) dH(y),$$

where  $|K(t)|$  is bounded,  $K(t) \rightarrow 1$  as  $t \rightarrow \infty$ , and  $H$  is a distribution function, then

(i) if  $a > 1$  and  $K$  is a distribution function, then

$$U(t) \sim \frac{be^{\sigma t}}{a\sigma} \left[ \int_0^\infty e^{-\sigma t} dK(t) \right] \left[ \int_0^\infty te^{-\sigma t} dH(t) \right]^{-1},$$

where  $\sigma$  is the positive root of

$$(2.9) \quad a \int_0^\infty e^{-\sigma t} dH(t) = 1;$$

(ii) if  $a = 1$ , then

$$U(t) \sim bt \left[ \int_0^\infty t dH(t) \right]^{-1};$$

(iii) if  $a < 1$ , then  $U(t) \rightarrow b/(1-a)$ .

LEMMA 2. If  $u(t)$  satisfies the equation

$$(2.10) \quad u(t) = bk(t) + a \int_0^t u(t-y) dH(y),$$

where  $k$  is a density function and  $H$  is a nonperiodic distribution function, then

(i) if  $a > 1$ , then

$$(2.11) \quad u(t) \sim \frac{be^{\sigma t}}{a} \left[ \int_0^\infty e^{-\sigma t} k(t) dt \right] \left[ \int_0^\infty te^{-\sigma t} dH(t) \right]^{-1};$$

(ii) if  $a = 1$ , then

$$u(t) \rightarrow b \left[ \int_0^\infty t dH(t) \right]^{-1};$$

(iii) if  $a < 1$  and if  $\int_0^\infty e^{-st} dH(t)$  converges in a negative  $s$ -interval containing a root  $\sigma$  of (2.9), and if  $\int_0^\infty e^{-\sigma t} k(t) dt$  converges, then (2.11) holds.

### 3. Asymptotic behavior of moments

THEOREM 2. Let  $c_1$  be defined as in (2.6),  $\sigma$  as defined in (2.7). (Write  $\mu_1(t) = \mu(t)$ .)

(i) If  $c_1 \neq 1$ , then

$$(3.1) \quad \mu(t) \sim e^{\sigma t} (c_1 - 1) c_1^{-2} \sigma^{-1} \left[ \int_0^\infty te^{-\sigma t} dG(t) \right]^{-1}.$$

(ii) If  $c_1 = 1$ , then  $\mu(t) \rightarrow 1$ .

*Proof.* (A.1), (A.2), and Theorem 1 imply the existence of  $\mu(t)$ , which by (2.5) therefore satisfies the equation

$$(3.2) \quad \mu(t) = [1 - G(t)] + c_1 \int_0^t \mu(t-y) dG(y).$$

Application of Lemma 2, and integration by parts of  $\int_0^\infty e^{-\sigma t} [1 - G(t)] dt$ , yields the theorem.

Let

$$(3.3) \quad c_{11} = 2 \sum_{j=1}^{\infty} q_j \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \sum_{1 \leq i < k \leq j} x_i x_k.$$

If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) > 1$ , then denote by  $\sigma_2$  the positive root of the equation

$$(3.4) \quad c_2 \int_0^\infty e^{-(\sigma+2\sigma_2)t} dG(t) = 1.$$

**THEOREM 3.** Let  $c_1, c_2, c_{11}, \sigma, \sigma_2$  be as defined in (2.6), (3.3), (2.7), (3.4).

(i) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) > 1$ , then

$$(3.5) \quad \mu_2(t) e^{-(\sigma_2+2\sigma)t} \rightarrow c_2^{-1} \left[ \int_0^\infty B(y) dy \right] \left[ \int_0^\infty y e^{-(\sigma_2+2\sigma)y} dG(y) \right]^{-1},$$

where

$$(3.6) \quad B(t) = e^{-(\sigma_2+2\sigma)t} \left[ 1 - G(t) + c_{11} \int_0^t \mu^2(t-y) dG(y) \right].$$

(ii) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) = 1$ , then

$$(3.7) \quad \mu_2(t) t^{-1} e^{-2\sigma t} \rightarrow c_{11} c_2^{-1} d^2 \int_0^\infty e^{-2\sigma y} dG(y) \left[ \int_0^\infty y e^{-2\sigma y} dG(y) \right]^{-1},$$

where

$$(3.8) \quad d = (c_1 - 1) c_1^{-2} \sigma^{-1} \left[ \int_0^\infty t e^{-\sigma t} dG(t) \right]^{-1} \quad \text{if } c_1 \neq 1, \\ = 1 \quad \text{if } c_1 = 1.$$

(iii) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) < 1$ , then

$$(3.9) \quad \mu_2(t) e^{-2\sigma t} \rightarrow c_{11} d^2 \int_0^\infty e^{-2\sigma y} dG(y) \left[ 1 - c_2 \int_0^\infty e^{-2\sigma t} dG(t) \right]^{-1}.$$

**Remark 1.** The integrals on the right sides of (3.7) and (3.9) are explicitly computable, and hence the limits in these expressions can be evaluated. This is however not the case for (3.5), since  $B(t)$  is not explicitly known. Thus  $\int_0^\infty B(t) dt$  is to be regarded as a finite but unknown constant.

**Remark 2.**  $\sigma_2 + 2\sigma$  ( $>$ ,  $=$ ,  $<$ ) 0 when  $c_2$  ( $>$ ,  $=$ ,  $<$ ) 1.

*Proof of Theorem 3.* (A.1), (A.2), and Theorem 1 imply the existence of  $\mu_2(t)$ , which by (2.5) therefore satisfies the equation

$$(3.10) \quad \mu_2(t) = 1 - G(t) + c_{11} \int_0^t \mu^2(t-y) dG(y) + c_2 \int_0^t \mu_2(t-y) dG(y).$$

(i) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) > 1$ , then

$$(3.11) \quad \mu_2(t) e^{-(\sigma_2+2\sigma)t} = B(t) + c_2 \int_0^t [\mu_2(t-y) e^{-(\sigma_2+2\sigma)(t-y)}] e^{-(\sigma_2+2\sigma)y} dG(y).$$

Theorem 2 and (A.5) imply that  $\int_0^\infty B(y) dy < \infty$ , and hence the result follows from Lemma 2(ii) and (3.11).

(ii) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) = 1$ , write

$$(3.12) \quad \begin{aligned} \mu_2(t) e^{-2\sigma t} &= [1 - G(t)] e^{-2\sigma t} + c_{11} \int_0^t [\mu(t-y) e^{-\sigma(t-y)}]^2 e^{-2\sigma y} dG(y) \\ &\quad + c_2 \int_0^t [\mu_2(t-y) e^{-2\sigma(t-y)}] e^{-2\sigma y} dG(y). \end{aligned}$$

By (A.5),  $[1 - G(t)] e^{-2\sigma t} \rightarrow 0$ . Using Theorem 2 one can see that as  $t \rightarrow \infty$

$$(3.13) \quad c_{11} \int_0^t [\mu(t-y) e^{-\sigma(t-y)}]^2 e^{-2\sigma y} dG(y) \rightarrow c_{11} d^2 \int_0^\infty e^{-2\sigma y} dG(y),$$

and hence Lemma 1(ii) and (3.12) imply (3.7).

(iii) If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) < 1$ , then again  $[1 - G(t)] e^{-2\sigma t} \rightarrow 0$ . By Theorem 2(i), (3.13), and Lemma 1(iii), one obtains (3.9).

*Example.* In the case of the binary age-dependent process treated in [1],  $c_1 = c_2 = c_{11} = 2$ . Hence  $\sigma > 0$ , and  $1 = 2 \int_0^\infty e^{-\sigma t} dG(t) > 2 \int_0^\infty e^{-2\sigma t} dG(t)$ . Thus Theorem 3 is always in case (iii), which with  $c_{11} = c_2 = 2$  is Lemma 7 of [1]. Similarly in the general age-dependent process  $c_1 = c_2$ , and hence  $1 = c_1 \int_0^\infty e^{-\sigma t} dG(t) > c_2 \int_0^\infty e^{-2\sigma t} dG(t)$ .

#### 4. Convergence of $X(t)/\mu(t)$

In this section we will study the mean square convergence of  $X(t)/\mu(t) = w(t)$  to a random variable. In the case of the binary branching process, where  $X(t)$  is the number of particles at  $t$ , such convergence was proved by Bellman and Harris [1]. Convergence in probability was proved by Levinson [7] for the general age-dependent process, without requiring  $\sum n^2 q_n < \infty$ , but subject to rather strong regularity conditions on  $G$ . He also assumed  $\sum n q_n = \nu > 1$ . (Recall that for this process  $c_1 = \nu$ .) Since

$$\mu(t) \rightarrow (\infty, 0, 1) \quad \text{if } \nu (>, <, =) 1,$$

one would expect limit theorems for  $X(t)$  to show different character for these cases. Most work for age-dependent branching processes has been for  $\nu > 1$  (see [5], [7]).

It will be seen below that the mean square convergence of  $w(t)$  does not,

in fact, depend on  $\nu > 1$ , but rather on  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) < 1$ . In Theorem 4 it will be shown that this inequality (together with some regularity conditions) is sufficient for mean square convergence. Conversely, if the inequality is violated, then it is clear from Theorems 2 and 3 that  $EW^2(t)$  diverges as  $t \rightarrow \infty$ , and hence in this situation  $\mu(t)$  is not the correct normalizing factor for  $X(t)$ .

If  $c_1 = c_2$ , as is the case in the age-dependent branching process, then the condition  $\nu < 1$  implies  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) > 1$ , as is indicated in the example at the end of Section 3. In the present process, however, one can easily construct examples for which  $c_1 < 1$  and  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) < 1$ . Such an example will be given after Theorem 4.

It will be necessary to consider the joint distribution of  $X(t_1)$  and  $X(t_2)$  given an initial particle of unit energy at time zero. Denote this distribution by  $P_2(x_1, x_2, t_1, t_2)$ . The law of total probability suggests that it satisfies the equation

$$\begin{aligned}
 P_2(x_1, x_2, t_1, t_2) = & [1 - G(t_2)]Z(x_1 - 1)Z(x_2 - 1) + q_0 G(t_1)Z(x_1)Z(x_2) \\
 & + q_0[G(t_2) - G(t_1)]Z(x_1 - 1)Z(x_2) \\
 & + Z(x_1 - 1) \sum_{j=1}^{\infty} q_j \int_{t_1}^{t_2} dG(y) \int \cdots \int \Phi_j(du_1, \dots, du_j) \\
 (4.1) \quad & \cdot P\left(\frac{x_2}{u_1}, t_2 - y\right) * \cdots * P\left(\frac{x_2}{u_j}, t_2 - y\right) \\
 & + \sum_{j=1}^{\infty} q_j \int_0^{t_1} dG(y) \int \cdots \int \Phi_j(du_1, \dots, du_j) \\
 & \cdot P_2\left(\frac{x_1}{u_1}, \frac{x_2}{u_1}, t_1 - y, t_2 - y\right) * \cdots * P_2\left(\frac{x_1}{u_j}, \frac{x_2}{u_j}, t_1 - y, t_2 - y\right).
 \end{aligned}$$

An argument analogous to that of Theorem I.1 shows that if  $\nu < \infty$ , then (4.1) has a unique bounded solution, that this solution is a distribution function, and that the bivariate characteristic function  $\hat{P}_2$  of  $P_2$  is the unique bounded solution of the equation

$$\begin{aligned}
 \hat{P}_2(s_1, s_2, t_1, t_2) = & [1 - G(t_2)]e^{is_1 + is_2} + q_0 G(t_1) + q_0[G(t_2) - G(t_1)]e^{is_1} \\
 (4.2) \quad & + e^{is_1} \sum_{j=1}^{\infty} q_j \int_{t_1}^{t_2} dG(y) \int \cdots \int \Phi_j(du_1, \dots, du_j) \prod_{i=1}^j \hat{P}(u_i s_2, t_2 - y) \\
 & + \sum_{j=1}^{\infty} q_j \int_0^{t_1} dG(y) \int \cdots \int \Phi_j(du_1, \dots, du_j) \prod_{i=1}^j \hat{P}_2(u_i s_1, u_i s_2, t_1 - y, t_2 - y).
 \end{aligned}$$

By the methods of Theorem I.2, and using (4.2), one can then show that (A.2) implies that  $m(t_1, t_2) = \int_0^\infty \int_0^\infty x_1 x_2 P(dx_1, dx_2, t_1, t_2)$  exists, and is the unique exponential order solution of

$$\begin{aligned}
 (4.3) \quad m(t_1, t_2) &= [1 - G(t)] + c_1 \int_{t_1}^{t_2} \mu(t_2 - y) dG(y) \\
 &+ c_{11} \int_0^{t_1} \mu(t_1 - y) \mu(t_2 - y) dG(y) + c_2 \int_0^{t_1} m(t_1 - y, t_2 - y) dG(y).
 \end{aligned}$$

**THEOREM 4.** If  $c_2 \int_0^\infty e^{-2\sigma t} dG(t) < 1$ , then  $w(t)$  converges in mean square to a random variable  $w$ .

*Proof.* As in [1], set  $t_1 = t$ ,  $t_2 = t + h$  for any  $t, h > 0$ , and let  $m(t, t + h)e^{-\sigma(h+2t)} = u(t, h)$ . Then by (4.3)

$$\begin{aligned}
 (4.4) \quad u(t, h) &= e^{-\sigma(h+2t)}[1 + G(t + h)] + c_1 e^{-\sigma(h+2t)} \int_t^{t+h} \mu(t + h - y) dG(y) \\
 &+ c_{11} \int_0^t [\mu(t - y)e^{-\sigma(t-y)}][\mu(t + h - y)e^{-\sigma(t+h-y)}]e^{-2\sigma y} dG(y) \\
 &+ c_2 \int_0^t u(t - y, h)e^{-2\sigma y} dG(y).
 \end{aligned}$$

By (A.5),  $e^{-\sigma(h+2t)}[1 - G(t + h)] \rightarrow 0$  as  $t \rightarrow \infty$ . Also

$$c_1 e^{-\sigma(h+2t)} \int_t^{t+h} \mu(t + h - y) dG(y) \leq \text{constant} \cdot e^{-2\sigma t}[G(t + h) - G(t)] \rightarrow 0$$

as  $t \rightarrow \infty$ . Define  $d$  as in (3.8). By Theorem 2 the third term on the right side of (4.4) converges to  $c_{11} d^2 \int_0^\infty e^{-2\sigma y} dG(y)$ . Hence by hypothesis and Lemma 1(iii)

$$(4.5) \quad u(t, h) \rightarrow c_{11} d^2 \int_0^\infty e^{-2\sigma y} dG(y) \left[ 1 - c_2 \int_0^\infty e^{-2\sigma y} dG(y) \right]^{-1} = u$$

as  $t \rightarrow \infty$ . But by Theorem 3(iii),  $\mu_2(t)e^{-2\sigma t}$  converges to  $u$ . Hence

$$\begin{aligned}
 E[e^{-\sigma t} X(t) - e^{-\sigma(t+h)} X(t + h)]^2 &= \mu_2(t)e^{-2\sigma t} + \mu_2(t + h)e^{-2\sigma(t+h)} - 2u(t, h) \\
 &\rightarrow 0 \qquad \qquad \qquad \text{as } t \rightarrow \infty,
 \end{aligned}$$

and  $E[w(t) - w(t + h)]^2 \rightarrow 0$ .

*Example.* To show that the remarks preceding the above theorem are not void, we give a simple example satisfying (A.1)-(A.5) and

$$(i) \quad c_1 < 1; \quad (ii) \quad c_1 \int e^{-\sigma t} dG(t) = 1; \quad (iii) \quad c_2 \int e^{-2\sigma t} dG(t) < 1.$$

Take  $q_1 = q_2 = \frac{1}{2}$ ;  $G(t) = 0$  if  $t < 1$ ,  $= 1$  if  $t \geq 1$ ;  $\Phi_1(x) = 0$  if  $x < \frac{1}{2}$ ,  $= 1$



if  $x \geq \frac{1}{2}$ ;  $\Phi_2(x_1, x_2) = 1$  if  $x_1 \geq \frac{1}{2}$  and  $x_2 \geq \frac{1}{2}$ ,  $= 0$  otherwise. Then  $c_1 = \frac{3}{4}$ ,  $c_2 = \frac{3}{8}$ ; (ii) becomes  $\frac{3}{4}e^{-\sigma} = 1$ , and hence  $\frac{3}{8}e^{-2\sigma} < \frac{9}{16}e^{-2\sigma} = 1$ .

The special case when  $G$  is an exponential distribution has received much attention in the literature on cascades, and deserves special comment in the present process. The following property was pointed out to the author by T. E. Harris.

**THEOREM 5.** *If  $G(t)$  is exponential, then  $w(t)$  is a martingale.*

*Proof.* From (2.2) it follows that  $\mu(t | x)$  is a linear function of  $x$ . This implies that the mean total energy at  $t$  due to any number of independent particles with total energy  $x_0$  at time zero, is simply  $\mu(t | x_0)$ .

Define the random vector  $V(t) = (N(t), X_1(t), \dots, X_{N(t)}(t))$ , where  $N(t)$  is the number of particles at  $t$ , and  $X_i(t)$ ,  $i = 1, \dots, N(t)$ , is the energy of the  $i^{\text{th}}$  particle. If  $G(\cdot)$  is exponential, then the process defined by  $V(t)$  is a time-homogeneous Markov process. For a further discussion of such processes, see Moyal [8]. Let  $P_{V(t)}(\cdot)$  denote the distribution function of  $V(t)$ . For a formal definition of  $P_{V(t)}(\cdot)$  see [8]. Choose  $t_1 \leq \dots \leq t_n \leq t_{n+1}$ . Then using the linearity of  $\mu(t | \cdot)$  and the Markovian property of  $V$ ,

$$\begin{aligned} E[X(t_{n+1}) | X(t_1) = x_1, \dots, X(t_n) = x_n] \\ &= \int E[X(t_{n+1}) | X(t_1) = x_1, \dots, X(t_n) = x_n, V(t_n) = v] P_{V(t_n)}(dv) \\ &= \int E[X(t_{n+1}) | X(t_n) = x_n, V(t_n) = v] P_{V(t_n)}(dv) \\ &= \int \mu(t_{n+1} - t_n | x_n) P_{V(t_n)}(dv) = \mu(t_{n+1} - t_n | x_n) \\ &= x_n \mu(t_{n+1} - t_n). \end{aligned}$$

Hence

$$\begin{aligned} E[w(t_{n+1}) | w(t_1) = w_1, \dots, w(t_n) = w_n] \\ &= [\mu(t_{n+1})]^{-1} E[X(t_{n+1}) | X(t_1) = w_1 \mu(t_1), \dots, X(t_n) = w_n \mu(t_n)] \\ &= [\mu(t_{n+1})]^{-1} w_n \mu(t_n) \mu(t_{n+1} - t_n) = w_n, \end{aligned}$$

since  $\mu(t)$  is an exponential function in the present case.

**COROLLARY.** *If  $G(t)$  is exponential, then  $w(t)$  converges with probability one to a random variable.*

*Proof.* The corollary is a direct consequence of Theorem 5, the martingale convergence theorem (see Doob [2, p. 319]), and the fact that  $EW(t) \equiv 1$ .

In the next section some properties of the limit random variable will be studied.

### 5. The limit random variable

**THEOREM 6.** *If  $w(t)$  converges in probability to a random variable  $w$ , then the characteristic function of  $w$ , say  $f$ , satisfies the equation*

$$(5.1) \quad f(s) = q_0 + \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f(x_i e^{-\sigma y} s).$$

(Note that the hypothesis of Theorem 4 or 5 implies that of Theorem 6.)

*Proof.* Let  $f(s, t)$  denote the characteristic function of  $w(t)$ . Then  $f(s, t) = E \exp \{isX(t)/\mu(t)\} = \hat{P}(s/\mu(t), t)$ . Also

$$\hat{P}\left(x_i \frac{s}{\mu(t-y)} \cdot \frac{\mu(t-y)}{\mu(t)}, t-y\right) = f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right).$$

Hence, substituting  $s/\mu(t)$  for  $s$  in (2.4) yields

$$(5.2) \quad f(s, t) = [1 - G(t)]e^{is/\mu(t)} + q_0 G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y).$$

Clearly by hypothesis,  $f(s, t) \rightarrow f(s)$  as  $t \rightarrow \infty$ .

Turning to the right side of (5.2), and given any  $\varepsilon > 0$ , we may choose a  $T_1 < \infty$  and a  $K_1 < \infty$  such that

$$(5.3) \quad 1 - G(T_1) < \varepsilon/8$$

and

$$(5.4) \quad \sum q_j \int_{R'_j} \cdots \int \Phi_j(dx_1, \dots, dx_j) < \varepsilon/8,$$

where  $R_j = \{(x_1, \dots, x_j) : x_i \leq K_1, i = 1, \dots, j\}$ , and the prime denotes complementation. This is clearly possible since  $|f| \leq 1$  and  $\Phi_j$  and  $G$  are distributions. By Theorem 2,  $\mu(t-y)/\mu(t) \rightarrow e^{\sigma y}$  uniformly in every bounded  $y$ -interval, and hence there is a  $T_2 < \infty$  and a  $K_2 < \infty$  such that for all  $t > T_2$ ,  $y < T_1$ , one has  $\mu(t-y)/\mu(t) < K_2$ . But  $f(u, t) \rightarrow f(t)$  uniformly in any bounded  $u$ -interval, in particular for  $|u| \leq K_1 K_2 s$  (see e.g. Doob [2, p. 39]); and hence there is a  $T'_3$  such that if  $t > T'_3$  and  $|u_i| \leq K_1 K_2 s, i = 1, \dots, j$ , then  $|\prod_{i=1}^j f(u_i, t) - \prod_{i=1}^j f(u_i)| < \varepsilon/4$ . Choose  $T_3 > T_1 + T'_3$ . Finally, since  $f(u)$  is uniformly continuous in any closed interval, one may choose a  $\delta$  such that if  $|u_{i1} - u_{i2}| < \delta$  for  $i = 1, \dots, j$ , then

$$|\prod_{i=1}^j f(u_{i1}) - \prod_{i=1}^j f(u_{i2})| < \varepsilon/4,$$

and since  $\mu(t-y)/\mu(t) \rightarrow e^{-\sigma y}$  uniformly on  $0 \leq y \leq T_1$ , one may choose  $T_4$  so that for  $y \leq T_1$ , and  $t \geq T_4$  one has  $|\mu(t-y)/\mu(t) - e^{-\sigma y}| < \delta/sK_1$ .

Now take  $T \geq \max(T_2, T_3, T_4)$ . Then for  $t > T$

$$\begin{aligned}
& \left| \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) \right. \\
& \quad \left. - \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right| \\
& \leq \left| \sum_{j=1}^{\infty} q_j \int_{T_1}^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) \right| \\
(5.5) \quad & + \left| \sum_{j=1}^{\infty} q_j \int_{T_1}^{\infty} dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right| \\
& + \left| \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_{R'_j} \cdots \int \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) \right| \\
& + \left| \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_{R'_j} \cdots \int \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right| \\
& + \sum_{j=1}^{\infty} q_j \int_0^{T_1} dG(y) \int_0^{K_1} \cdots \int_0^{K_1} \Phi_j(dx_1, \dots, dx_j) \\
& \quad \cdot \left| \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) - \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right|.
\end{aligned}$$

The sum of the first four terms on the right side of (5.5) is  $\leq \varepsilon/2$ . To the last term add and subtract  $\prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s\right)$  under the absolute value sign. Then (5.5)  $\leq$

$$\begin{aligned}
& \varepsilon/2 + \sum_{j=1}^{\infty} q_j \int_0^{T_1} dG(y) \int_0^{K_1} \cdots \int_0^{K_1} \Phi_j(dx_1, \dots, dx_j) \\
& \quad \cdot \left| \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) - \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s\right) \right| \\
& \quad + \sum_{j=1}^{\infty} q_j \int_0^{T_1} dG(y) \int_0^{K_1} \cdots \int_0^{K_1} \Phi_j(dx_1, \dots, dx_j) \\
& \quad \cdot \left| \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s\right) - \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right|.
\end{aligned}$$

But now for all  $t > T$  and for  $y < T_1$ ,  $x_i \leq K_1$ , one has

$$x_i \frac{\mu(t-y)}{\mu(t)} s \leq K_1 K_2 s \quad \text{and} \quad \left| x_i \frac{\mu(t-y)}{\mu(t)} s - x_i e^{-\sigma y} s \right| < \delta.$$

Also  $y < T_1$  and  $t > T$  imply  $t-y > T_3 - T_1 > T'_3$ . Thus for all  $t > T$ ,  $y < T_1$ , one has

$$\left| \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s, t-y\right) - \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s\right) \right| < \frac{\varepsilon}{4},$$

and

$$\left| \prod_{i=1}^j f\left(x_i \frac{\mu(t-y)}{\mu(t)} s\right) - \prod_{i=1}^j f(x_i e^{-\sigma y} s) \right| < \frac{\varepsilon}{4}.$$

Hence (5.5)  $\leq \varepsilon/2 + \varepsilon/4 + \varepsilon/4$  for all  $t > T$ , which proves the theorem.

The next result is the generalization of Theorems 5 and 6 of [1]. It will be shown that  $f(t)$  is of exponential order as  $t \rightarrow \pm\infty$ , and thence that the distribution function of  $w$  is absolutely continuous. To state the conditions of the theorem we will first have to introduce some more notation. Let

$$A(j, k_j) = \{(y, x_1, \dots, x_j) : x_i e^{-\sigma y} \leq 1 \text{ for at least } k_j \text{ subscripts } i\},$$

$$B(j, k_j) = \{(y, x_1, \dots, x_j) : x_i e^{-\sigma y} \geq 1 \text{ for at least } k_j \text{ subscripts } i\},$$

$$u(j, k_j) = \int_{A(j, k_j)} \dots \int dG(y) \Phi_j(dx_1, \dots, dx_j),$$

$$v(j, k_j) = \int_{B(j, k_j)} \dots \int dG(y) \Phi_j(dx_1, \dots, dx_j).$$

$\alpha_j$  = the conditional expected total energy of the offspring of a particle of unit energy, given that there are  $j$  offspring

$$= \int_0^\infty \dots \int_0^\infty \Phi_j(dx_1, \dots, dx_j)(x_1 + \dots + x_j).$$

The statement and proof of the theorem will be substantially simplified if we assume that  $q_0 = 0$ , and we will hence do so. In some remarks following the proof of the theorem we shall indicate what modifications must be made to treat  $q_0 \neq 0$ .

**THEOREM 7.** Assume that

- (i) there is a  $\Delta > 0$  such that for all  $j = 1, 2, \dots$ , we have  $\Phi_j(x_1, \dots, x_j) = 0$  if  $x_i < \Delta$  for any  $i = 1, \dots, j$ ;
- (ii)  $q_1 \alpha_1 / c_1 < \Delta$ ;
- (iii) there is a sequence of integers  $\{k_j\}$ ,  $j = 1, 2, \dots$ , such that  $0 < k_j \leq j$  and  $\max[\sum_j k_j q_j u(j, k_j), \sum_j k_j q_j v(j, k_j)] > 1$ ;
- (iv)  $1 - G(t)$  is of exponential order;
- (v)  $q_0 = 0$ .

Then either  $w$  has an absolutely continuous distribution, or is identically equal to a constant with probability one.

*Remarks.* Since some of the conditions of the theorem may at first sight appear artificial, the following remarks might be in order.

Condition (i) is far from necessary, and is made because it greatly simplifies the analysis. A close inspection of the following proof will show that the difficulties created by the  $\Phi_j$ -distributions near zero are very similar to those

of the  $G$ -distribution for large values of the argument. (The proof would be further simplified if we assumed  $G(t) \equiv 1$  for sufficiently large  $t$ .) In the present case  $1 - G(t)$  has been assumed of exponential order as  $t \rightarrow \infty$ , and it seems fairly clear that if the kind of treatment of  $G$  carried out in [1, Lemma 2, p. 290], and below, were mirrored for  $\Phi_j$  near zero, then condition (ii) could be replaced by the weaker one that  $\Phi_j$  be of exponential order near zero.

Next note that if we had  $q_1 = 1$ , i.e., that all offspring were "only" children, then the theorem would be false. (Take for  $\Phi_1(x)$  a step function with unit mean.) What is in fact required is that not too large a proportion of the energy of a parent should be transmitted through an "only child". The precise formulation of this idea is condition (ii), namely that the conditional expected energy of the offspring given that there is exactly one of these, times the probability that there is one, divided by the unconditional mean energy transmitted by the parent, should be less than  $\Delta$ .

Condition (iii) is a technical one needed in the proof which will be satisfied in a great variety of circumstances. In the case of the age-dependent branching process it is equivalent to  $\sum j q_j > 1$ , and in more general circumstances it will always be satisfied if the  $\{q_j\}$  distribution is large enough in its upper tail. It thus re-enforces the kind of property already required in (ii).

Together, (ii) and (iii) assure us that a sufficiently large part of the transmitted energy is spread among a sufficiently large number of offspring particles.

Assumption (iv) has already been made throughout the paper for the case  $c_1 < 1$ , and is now needed for all  $c_1$ .

Note also that (by (5.1)) it would be easy to guarantee that  $w$  not be identically equal to a positive constant by such conditions as that  $G$  not be a step function of one step, or  $q_j \neq 1$  for any  $j$ , or  $\Phi_j$  not be a step function of one step, etc.

*Proof.* The main outline of the proof is as in [1], and certain steps will be treated by reducing them to situations in [1]. We shall show that

$$(5.6) \quad f(s) \rightarrow 0 \quad \text{as} \quad s \rightarrow \pm \infty;$$

thence that

$$(5.7) \quad f(s) = O(|s|^{-\beta}) \quad \text{as} \quad s \rightarrow \pm \infty$$

for some  $\beta > 0$ ; and finally that (5.7) implies

$$(5.8) \quad \int_{-\infty}^{\infty} |f'(it)| \, dt < \infty,$$

where  $f'$  is the derivative of  $f$ .

The conclusion of the theorem follows by an argument similar to that given in Theorem 4.1 of [4], and given in detail in [10].

It will first be shown that there exist an  $s'$  and a  $\delta$  such that

$$(5.9) \quad |f(s)| < 1 - \delta \quad \text{for all} \quad s > s'.$$

As in [1], suppose the contrary, and write

$$(5.10) \quad f(s) = 1 + isEw - (s^2/2)Ew^2 + o(s^2).$$

Either  $w = a$  constant with probability one, or  $Ew^2 > (Ew)^2$ . In the latter case (5.10) implies that there is a pair  $(\alpha, s_\alpha)$  such that

$$(5.11) \quad \begin{aligned} |f(s)| &< 1 - \alpha && \text{when } s = s_\alpha, \\ &< 1 && \text{when } 0 < s < s_\alpha. \end{aligned}$$

Let  $s_0$  and  $s_{00}$  be the first points to the left and right respectively of  $s_\alpha$ , such that

$$(5.12) \quad |f(s_0)| = |f(s_{00})| = 1 - \alpha.$$

The former exists since  $f(s)$  is a continuous function of  $s$  with  $|f(0)| = 1$  and  $|f(s_\alpha)| < 1 - \alpha$ . The latter exists by assumption of the nonexistence of  $(s', \delta)$  satisfying (5.9).

Suppose now that there is a sequence  $\{k_j\}$  as hypothesized in (iii) such that  $\sum_j k_j q_j u(j, k_j) > 1$ . Let  $T_1 = (1/\sigma) \log(s_{00}/s_0)$  if  $\sigma > 0$ ,  $T_1 = \infty$  if  $\sigma \leq 0$ . Let  $b_1 = (s_0/s_{00})$ , and  $C_j = \{(y, x_1, \dots, x_j) : x_i \geq b_1, i = 1, \dots, j; y \leq T_1\}$ . Then by (5.1) (and the assumption that  $q_0 = 0$ )

$$\begin{aligned} |f(s)| \leq \sum_{j=1}^{\infty} q_j \int \cdots \int_{\{C_j \cap A(j, k_j)\}} dG(y) \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j |f(x_i e^{-\sigma y} s)| \\ + \sum q_j [1 - P\{C_j \cap A(j, k_j)\}]. \end{aligned}$$

Setting  $s = s_{00}$  in (5.1) yields

$$(1 - \alpha) \leq \sum_{j=1}^{\infty} q_j P\{C_j \cap A(j, k_j)\} (1 - \alpha)^{k_j} + \sum_{j=1}^{\infty} q_j [1 - P\{C_j \cap A(j, k_j)\}],$$

or

$$(5.13) \quad 1 \geq \sum q_j P\{C_j \cap A(j, k_j)\} [(1 - (1 - \alpha)^{k_j})/\alpha].$$

If  $\alpha \rightarrow 0$ , then  $s_0 \rightarrow 0$  (continuity of the characteristic function at zero), and since  $s_{00}$  will not decrease,  $b_1 \rightarrow 0$ , and thus  $T_1 \rightarrow \infty$  (if it does not already equal  $\infty$ ). Therefore  $P\{C_j \cap A(j, k_j)\} \rightarrow u(j, k_j)$ , and the right side of (5.13) converges to  $\sum q_j u(j, k_j) k_j$ , which is  $> 1$  by hypothesis. Thus (5.13) is contradicted.

Suppose on the other hand that

$$\sum k_j q_j u(j, k_j) \leq 1 \quad \text{but} \quad \sum k_j q_j v(j, k_j) > 1.$$

In this case let  $T_2 = (1/\sigma) \log(s_0/s_{00})$  if  $\sigma < 0$ ,  $T_2 = \infty$  if  $\sigma \geq 0$ ; and let  $b_2 = (s_{00}/s_0)$ . Then setting  $s = s_0$  in (5.1) and arguing as above but with  $A(j, k_j)$  replaced by  $B(j, k_j)$  and  $u(j, k_j)$  by  $v(j, k_j)$ , we are again led to a contradiction like (5.13). Hence the pair  $(s', \delta)$  in (5.9) exists.

We shall now prove (5.6) for the case  $\sigma \geq 0$ . Take  $n_1, t_1$  so large that for a given  $\varepsilon > 0$  we have  $1 - G(t_1) < \varepsilon$ ,  $\sum_{j=n_1}^{\infty} q_j < \varepsilon$ . Take  $s$  so large that

$se^{-\sigma t_1} \Delta \geq s'$ . Then

$$|f(s)| < (1 - \alpha)q_1 G(t_1) + 3\varepsilon \\ + (1 - \alpha) \sum_{j=2}^{n_1} q_j \int_0^{t_1} dG(y) \int_{\Delta}^{\infty} \cdots \int_{\Delta}^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^{j-1} |f(x_i e^{-\sigma y} s)|.$$

Let  $M(s) = \sup \{|f(t)| : t \geq s\}$ . Then

$$(5.14) \quad M(s) < (1 - \alpha)q_1 G(t_1) + 3\varepsilon + (1 - \alpha)(1 - q_1)G(t_1)M(\Delta e^{-\sigma t_1} s).$$

Let  $a_1 = \max [\Delta^{-1} e^{\sigma t_1}, 2]$ . Then (5.14) implies

$$(5.15) \quad M(a_1^n s) < \frac{(1 - \alpha)q_1 + 3\varepsilon}{1 - (1 - \alpha)(1 - q_1)} + [(1 - \alpha)(1 - q_1)]^n M(s).$$

Since  $q_1/[1 - (1 - \alpha)(1 - q_1)] < 1$ , it is clear that one may choose  $\varepsilon$  so that  $[(1 - \alpha)q_1 + 3\varepsilon]/[1 - (1 - \alpha)(1 - q_1)] < (1 - \alpha)b_1$ , where  $b_1 < 1$ . Then it follows from (5.15) that there is an  $s_1$  such that for all  $s \geq s_1$ ,  $M(s) < (1 - \alpha)b_2$ , where  $b_2 < 1$ . Repeating this argument with  $(1 - \alpha)$  replaced by  $(1 - \alpha)b_2$ , then repeating it similarly  $n$ -fold, implies that there is an  $s_n$  such that  $M(s) < (1 - \alpha)b_2^n$  for all  $s \geq s_n$ . This proves (5.6) for  $\sigma \geq 0$ .

We shall next prove (5.7) for the case  $\sigma \geq 0$ . By (5.1) and hypothesis (i) of the theorem we see that

$$|f(s)| \leq 1 - G(A) \\ + \sum_{j=1}^{\infty} q_j \int_0^A dG(y) \int_{\Delta}^{\infty} \cdots \int_{\Delta}^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j |f(x_i e^{-\sigma y} s)|$$

for any  $A < \infty$ . In particular set  $A = (1/2\sigma) \log(s/\Delta^2)$ ,  $s > 1$ ; and let  $b(s) = 1 - G(A)$ . Then  $y \leq A$  implies  $e^{-\sigma y} \geq \sqrt{s}/\Delta$ , and hence

$$(5.16) \quad M(s) \leq \sum_{j=1}^{\infty} q_j M^j(\sqrt{s}) + bs \\ \leq q_1 M(\sqrt{s}) + (1 - q_1) M^2(\sqrt{s}) + b(s).$$

This expression is the same as that obtained by Bellman and Harris [1] for the age-dependent branching process. The binary case is identical to the right side of (5.16) with  $q_1 = 0$ , and is expression (11), [1, p. 290]. The not necessarily binary case is discussed in Section 9 of the same paper. The conclusion is (5.7).

Turning to the case  $\sigma < 0$  we shall prove (5.7) directly without going through the intermediate step (5.6). By (5.1)

$$M(t) \leq \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j M(x_i e^{-\sigma y} t) \\ \leq \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_{\Delta}^{\infty} \cdots \int_{\Delta}^{\infty} \Phi_j(dx_1, \dots, dx_j) \prod_{i=1}^j M(x_i t) \\ \leq \sum q_j M^j(\Delta t).$$

But this is a stronger inequality than (5.16), and hence implies (5.7).

Turning to the proof of (5.8) let  $M_1(t) = \sup \{ |f'(s)| : s \geq t \}$ , where  $f'$  is the derivative of  $f$ . Then

$$(5.17) \quad M_1(t) \leq \sum_{j=1}^{\infty} q_j \int_0^{\infty} e^{-\sigma y} dG(y) \int_{\Delta}^{\infty} \cdots \int_{\Delta}^{\infty} \Phi_j(dx_1, \dots, dx_j) \\ \cdot \left( \sum_{i=1}^j x_i \right) M_1(\Delta e^{-\sigma y} t) M^{j-1}(\Delta e^{-\sigma y} t).$$

Let  $B(T) = \int_0^T M_1(t) dt$ , and set  $te^{-\sigma y} = u$ . Then

$$(5.18) \quad B(T) \leq \sum_{j=1}^{\infty} q_j \alpha_j \int_0^{\infty} e^{-\sigma y} dG(y) \int_0^{Te^{-\sigma y}} M_1(\Delta u) M^{j-1}(\Delta u) du.$$

Suppose first that  $\sigma > 0$ . Then

$$B(T) \leq \sum_{j=1}^{\infty} q_j \alpha_j \int_0^{\infty} e^{-\sigma y} dG(y) \int_0^T M_1(\Delta u) M^{j-1}(\Delta u) du \\ \leq \frac{1}{\Delta} \sum_{j=1}^{\infty} q_j \alpha_j \int_0^{\infty} e^{-\sigma y} dG(y) \int_0^T M_1(u) M^{j-1}(u) du,$$

and hence, if we recall that  $\int_0^{\infty} e^{-\sigma y} dG(y) = 1/c_1$ ,

$$\left(1 - \frac{q_1 \alpha_1}{\Delta}\right) B(T) \leq \left(\frac{1}{\Delta c_1} \sum_{j=2}^{\infty} q_j \alpha_j\right) \int_0^{\infty} dG(y) \int_0^T M_1(u) M(u) du.$$

By (5.7) and hypothesis (ii) of the theorem it follows that there is a constant  $K < \infty$  such that

$$(5.19) \quad B(T) \leq K \int_0^T \frac{M_1(u) du}{(1+u)^{\beta}}.$$

This is of exactly the same form as (31), [1, p. 292], and the ensuing argument of the latter paper implies our result.

Now suppose that  $\sigma \leq 0$ . Then by (5.17)

$$M_1(t) \leq \sum_{j=1}^{\infty} q_j \int_0^{\infty} e^{-\sigma y} dG(y) \int_{\Delta}^{\infty} \cdots \int_{\Delta}^{\infty} \Phi_j(dx_1, \dots, dx_j) \\ \cdot \left( \sum_{i=1}^j x_i \right) M_1(\Delta t) M^{j-1}(\Delta t).$$

Defining  $B(T)$  as before, we get

$$B(T) \leq \frac{1}{c_1} \sum_{j=1}^{\infty} q_j \alpha_j \int_0^T M^{j-1}(\Delta t) M_1(\Delta t) dt,$$

and this again implies (5.19), proving the theorem.

*Remark.* It remains to consider the case when  $q_0 > 0$ . Define the function (as suggested in [1])

$$f^*(s) = \frac{f[(1-Q)s] - Q}{1-Q},$$



where  $Q$  is the unique nonnegative root, less than one, of  $s = \sum_{j=0}^{\infty} q_j s^j$ . Then

$$(5.20) \quad f(s) = Q + (1 - Q)f^*\left(\frac{s}{1 - Q}\right),$$

and substituting this into (5.1) yields

$$(5.21) \quad \begin{aligned} & Q + (1 - Q)f^*\left(\frac{s}{1 - Q}\right) \\ &= q_0 + \sum_{j=1}^{\infty} q_j \int_0^{\infty} dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \dots, dx_j) \\ & \quad \cdot \prod_{i=1}^j \left\{ Q + (1 - Q)f^*\left(\frac{x_i e^{-\sigma y} s}{1 - Q}\right) \right\}. \end{aligned}$$

If the product on the right side of (5.21) is multiplied out, then it is seen that it yields a term  $\sum_{j=1}^{\infty} q_j Q^j$ , which when added to  $q_0$ , cancels the  $Q$  on the left side of (5.21). We are thus left with an equation similar in form to (5.1) with  $q_0 = 0$ , but with the products  $\prod_{i=1}^j f(x_i e^{-\sigma y} s)$  of (5.1) replaced by linear combinations of the form  $\sum_{i=1}^j a_i \prod_{\{K_i\}} f^*(x_i e^{-\sigma y} s)$ , where the coefficients  $a_i$  are functions of  $Q$ , and the sets  $\{K_i\}$  over which the products are taken are appropriate subsets of the integers  $\{1, \dots, j\}$ . The argument of Theorem 7 can now be pushed through as before, provided that hypotheses (ii) and (iii) are suitably modified to correspond to the more complicated version of (5.1).

The conclusion will then be that  $f^*(s)$  is the characteristic function of a random variable which either equals a constant with probability one, or has an absolutely continuous distribution. In the latter case it follows from (5.20) that  $w$  itself has a distribution which is absolutely continuous except for a jump  $Q$  at zero. Thus  $P\{w = 0\} = Q$ , as was the case in the regular age-dependent branching process (see [1]).

Note that if  $q_0 = 0$ , then  $Q = 0$ .

## 6. The total energy of the process

In I we introduced the random variable  $Y(t)$ , namely the total energy of all particles which have existed up to time  $t$ . It is possible to carry out a development for  $Y(t)$  analogous to that of this paper for  $X(t)$ . Letting  $R(y, t | x_0)$  denote the distribution of  $Y(t)$  with an initial particle of energy  $x_0$ , and using Theorem I.4 and assumptions (A.1) and (A.2), we can show that  $R(cy, t | cx_0) = R(y, t | x_0)$ . Hence the characteristic function of  $R$ , say  $\hat{R}$ , satisfies  $\hat{R}(s, t | cx_0) = \hat{R}(cs, t | x_0)$ , and hence writing  $\hat{R}(s, t | 1) = \hat{R}(s, t)$ , (I.5.2) becomes

$$(6.1) \quad \begin{aligned} \hat{R}(s, t) &= [1 - (1 - q_0)G(t)]e^{is} \\ &+ e^{is} \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \dots, dx_j) \prod \hat{R}(x_i s, t - y). \end{aligned}$$

From (6.1) one may obtain equations for the moments of  $Y(t)$ . These are again renewal equations, and hence their asymptotic behavior may be determined. The convergence of a suitably normalized version of  $Y(t)$  to a random variable, the derivation of an equation satisfied by the characteristic function of this random variable, and regularity properties of the distribution of the limit random variable, may all be developed as for  $X(t)$ .

We will not go through this repetitious procedure, and will content ourselves with looking at the results for the mean  $\eta(t) = \frac{1}{i} \frac{\partial}{\partial s} \hat{R}(0, t)$ , and pointing out their similarity to known results for discrete branching processes. From (6.1) it follows that

$$(6.2) \quad \eta(t) = 1 + c_1 \int_0^t \eta(t-y) dG(y).$$

Then by Lemma 1, there follows at once from (6.2)

THEOREM 8. (i) If  $c_1 > 1$ , then

$$\eta(t) \sim \frac{e^{\sigma t}}{c_1 \sigma} \left[ \int_0^\infty t e^{-\sigma t} dG(t) \right]^{-1}.$$

(ii) If  $c_1 = 1$ , then

$$\eta(t) \sim t \left[ \int_0^\infty t dG(t) \right]^{-1}.$$

(iii) If  $c_1 < 1$ , then  $\eta(t) \rightarrow (1 - c_1)^{-1}$ .

In the discrete branching process, with  $\nu = \sum n q_n$ , let  $N$  be the mean total number of particles that ever exist in the course of the process until extinction (see Kemeny and Snell [6, p. 83]; Harris [4]). Then  $N$  is finite if and only if  $c_1 < 1$ . In the latter case  $N = (1 - \nu)^{-1}$ . This is obviously a special and less precise form of Theorem 8, with  $c_1 = \nu$  and  $G(t) = Z(t - 1)$ . In the continuous-parameter age-dependent process, which also leads to a specialization of Theorem 8, such questions have been studied only for binary processes, in which the case  $c_1 \leq 1$  does not arise. Hence the result  $\eta(t) \rightarrow (1 - \nu)^{-1}$  has not been explicitly stated before.

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