Tôhoku Math. Journ.
25 (1973), 129-137.

GENERALIZED CENTRAL SPHERES AND THE NOTION OF SPHERES IN RIEMANNIAN GEOMETRY

Dedicated to Professor Shigeo Sasaki on his 60th birthday

KATSUMI NOMIZU

(Received November 20, 1971)

In a euclidean space E^{n+1} an n-plane or an n-sphere of radius r may be characterized as an umbilical hypersurface with mean curvature equal to 0 or $1/r$. A similar characterization is possible for an n-plane or an n-sphere in a euclidean space E^{n+p} where $p > 1$, as shown by E. Cartan [1], p. 231. Indeed, it is possible to determine all umbilical submanifolds of dimension n in an $(n+p)$-dimensional space form $\tilde{M}$, which can be regarded as "n-planes" or "n-spheres" according to whether the mean curvature is 0 or not.

In an arbitrary Riemannian manifold $\tilde{M}$ of dimension $n+p$, a natural analogue of an n-plane is an n-dimensional totally geodesic submanifold (equivalently, umbilical submanifold with zero mean curvature). In terms of a geometric notion of the development of curves, Cartan [1], p. 116, characterizes such n-planes in $\tilde{M}$ as follows. Let M be an n-dimensional submanifold of $\tilde{M}$. For every point x of M and for every curve τ in M starting at x, the development τ^* of τ into the euclidean tangent space $T_x(\tilde{M})$ lies in the euclidean subspace $T_x(M)$ if and only if M is totally geodesic in $\tilde{M}$.

The purpose of the present paper is to show that a natural analogue of an n-sphere in an arbitrary Riemannian manifold M is an n-dimensional *umbilical submanifold with non-zero parallel mean curvature vector* by characterizing such a submanifold as follows: for every point x of M and for every curve τ in M starting at x, the development τ^* lies in an n-sphere in $T_x(\tilde{M})$. The situation can be further clarified by introducing a generalization of central sphere defined in [5], which is also a generalization of the notion of osculating circle for a space curve. Namely, for an n-dimensional submanifold M with non-zero mean curvature in an arbitrary Riemannian manifold $\tilde{M}$, we associate to each point x of M a certain n-sphere $S^n(x)$ in $T_x(\tilde{M})$ which we call the *central n-sphere* at x. For every curve τ in M from x to y, the affine parallel displace-

Work supported by an NSF grant GP 28419.

ment along τ (with respect to the affine connection in $\tilde{M}$) maps $S^n(x)$ upon $S^n(y)$ if and only if M is an "n-sphere" in $\tilde{M}$. This fact (in the case of codimension 1) is quite similar to the result on umbilical hypersurfaces in a space with normal conformal connection due to S. Sasaki [4]. It is perhaps possible to relate these two results in a direct way.

Our main results are stated as Theorems 1, 2 and 3.

Finally, we remark that it is proved in [3] that if a Riemannian manifold $\tilde{M}$ admits sufficiently many n-spheres for some n, $2 \leqq n < \dim \tilde{M}$, then $\tilde{M}$ is a space form.

1. Preliminaries. We shall summarize the notations and facts which we need in this paper.

Let M be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold $\tilde{M}$. The Riemannian connections of $\tilde{M}$ and M are denoted by $\tilde{\nabla}$ and ∇, respectively, whereas the normal connection (in the normal bundle of M in $\tilde{M}$) is denoted by $\nabla^{\perp}$. The second fundamental form α is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) ,$$

where X and Y are vector fields tangent to M. For any vector field ξ normal to M, the tensor field A_ξ of type (1, 1) on M is given by

$$\tilde{\nabla}_X \xi = - A_\xi(X) + \nabla^{\perp}_X \xi ,$$

where X is a vector field tangent to M. We have

$$g(\alpha(X, Y), \xi) = g(A_\xi X, Y)$$

for X and Y tangent to M and ξ normal to M, where g is the Riemannian metric on $\tilde{M}$. For the detail, see [2], Vol. II, Chap. 7.

The mean curvature vector field η of M is defined by the relation

$$\text{trace } A_\xi / n = g(\xi, \eta)$$

for all ξ normal to M. We say that η is parallel (with respect to the normal connection) if $\nabla^{\perp}_X \eta = 0$ for every X tangent to M.

We say that M is umbilical in $\tilde{M}$ if

$$\alpha(X, Y) = g(X, Y)\eta$$

for all X and Y tangent to M. Equivalently, M is umbilical in $\tilde{M}$ if

$$A_\xi = g(\xi, \eta)I$$

for all ξ normal to M, where I is the identity transformation.

It is known that if $\tilde{M}$ is a space form (a Riemannian manifold of constant sectional curvature), then an umbilical submanifold M of $\tilde{M}$ has parallel mean curvature vector.

We now recall the notion of development of a curve. Let $\tilde{M}$ be a Riemannian manifold, and let τ be a curve from x to y. In addition to the linear parallel displacement along τ, we consider the affine parallel displacement $\tilde{\tau}$ along τ which is an affine transformation of the affine tangent space $T_x(\tilde{M})$ at x onto the affine tangent space $T_y(\tilde{M})$ at y. By parametrizing τ by x_t so that $x_0 = x$ and $x_1 = y$, we denote by τ_0^t and $\tilde{\tau}_0^t$ the linear and affine parallel displacements along the curve τ (in the reversed direction) from x_t to x_0. When the point x_t is considered as the origin of the affine tangent space $T_{x_t}(M)$, $\tilde{\tau}_0^t(x_t)$, $0 \leqq t \leqq 1$, is a curve in the affine space $T_x(\tilde{M})$, which is called the development τ^* of τ into $T_x(\tilde{M})$. For the detail, see [2], Vol. I, p. 131. Proposition 4.1 there shows, for a *smooth* curve $\tau = x_t$, $0 \leqq t \leqq 1$, how we can obtain the development τ^*: Set

$$Y_t = \tau_0^t \dot{x}_t, \quad 0 \leqq t \leqq 1 ,$$

where $\dot{x}_t$ denotes the tangent vector of τ at x_t. Then the development τ^* of τ is a (unique) curve C_t, $0 \leqq t \leqq 1$, in the affine tangent space $T_x(\tilde{M})$ with $C_0 = x$ such that the tangent vector dC_t/dt is parallel to Y_t in $T_x(\tilde{M})$.

This process can be extended to the case of a piecewise smooth curve. For simplicity, consider a curve composed of two smooth curves $\tau = x_t$, $0 \leqq t \leqq a$, and $\mu = x_t$, $a \leqq t \leqq b$. Let $\tau^* = C_t$, $0 \leqq t \leqq a$, be the development τ in $T_x(\tilde{M})$. Let C_t, $a \leqq t \leqq b$, be a (unique) curve starting at the end point of τ^* such that its tangent vector dC_t/dt is parallel to $\tau_0^a \mu_a^t(\dot{x}_t)$ for each t, $a \leqq t \leqq b$. Then C_t, $0 \leqq t \leqq b$, is the development of the composed curve $\mu \cdot \tau$. This fact depends on the following. If τ is a curve (smooth or piecewise smooth) from x to y and if μ is a curve from y to z, then the affine parallel displacement along $\mu \cdot \tau$ is the composite of those along τ and μ. It also follows that if μ^* is the development of μ in $T_y(\tilde{M})$, then the development $(\mu \cdot \tau)^*$ in $T_x(M)$ is equal to the composite $\tilde{\tau}^{-1}(\mu^*) \cdot \tau^*$. We shall make use of these facts.

2. Main results. Let M be an n-dimensional submanifold in an $(n + p)$-dimensional Riemannian manifold $\tilde{M}$. For each point x of M, let η_x be the mean curvature vector and $H_x = \|\eta_x\|$ the mean curvature. If $H_x \neq 0$, we consider the n-dimensional sphere $S^n(x)$ with center at η_x/H_x^2 and of radius $1/H_x$ that lies in the euclidean subspace of dimension $n + 1$ of $T_x(\tilde{M})$ spanned by $T_x(M)$ and η_x. We shall call $S^n(x)$ the *central n-sphere* at x for the submanifold M.

REMARK. If the ambient space $\tilde{M}$ is a euclidean space E^{n+p}, then the affine tangent space $T_x(\tilde{M})$ can be naturally identified with E^{n+p} itself. Thus the central n-sphere $S^n(x)$ is indeed an n-sphere in E^{n+p}. We consider two special cases:

(1) If M is a surface in E^3 with non-zero mean curvature H_x, then the central sphere $S^2(x)$ is a sphere in E^3 with radius $1/H_x$ that is tangent to M at x.

(2) Let $M = x(s)$ be a curve in E^3 parametrized by arc length s with non-zero curvature $k(s)$. Considering M as a 1-dimensional submanifold, we find that the mean curvature vector is equal to ke_2, where e_2 is the principal normal vector. Thus the central 1-sphere at $x(s)$ is nothing but the osculating circle at this point.

We now assume that M has non-zero mean curvature at each point x and consider the following three properties:

(A) *For every x in M and for every curve τ in M starting at x, the development τ^* of τ into $T_x(\tilde{M})$ lies in the central n-spheres $S^n(x)$.*

(B) *For every curve τ in M from x to y, the affine parallel displacement $\tilde{\tau}$ maps $S^n(x)$ upon $S^n(y)$.*

(C) *M is umbilical and has parallel mean curvature vector.*

We now state our main results.

THEOREM 1. *Let M be a connected n-dimensional submanifold in an $(n + p)$-dimensional Riemannian manifold $\tilde{M}$ with non-vanishing mean curvature. Then conditions* (A), (B) *and* (C) *are equivalent.*

In the case of $\tilde{M} = E^{n+p}$, the central n-spheres are n-spheres in E^{n+p}. On the other hand, if τ is a curve in M from x to y, the development τ^* of τ into $T_x(\tilde{M}) = E^{n+p}$ is nothing but τ itself. Thus if M satisfies condition (A), every point y of M lies in the central n-sphere $S^n(x)$, and hence M is part of the n-sphere $S^n(x)$ in E^{n+p}. The converse is obvious. We may also paraphrase condition (B) by the statement that all central n-spheres $S^n(x), x \in M$, coincide. As for condition (C), note that an umbilical submanifold of E^{n+p} (more generally, of any space form) has parallel mean curvature vector, provided $\dim M \geqq 2$. For $\dim M = 1$, if $M = x(s)$ is a curve with non-vanishing curvature, then the assumption of parallel mean curvature implies that the curvature is constant and the torsion is 0, that is, M is (part of) a circle.

THEOREM 2. *Let M and $\tilde{M}$ be as in Theorem* 1. *Under condition* (C), *the development τ^* of a geodesic τ in M starting at x is a great circle of the central n-sphere $S^n(x)$.*

Finally, we consider a condition weaker than (A) which does not involve the mean curvature vector, namely,

(A_0) *At some point x of M, there is an n-sphere $\Sigma^n(x)$ in $T_x(\tilde{M})$ such that every curve τ in M starting at x is developed upon a curve on $\Sigma^n(x)$.*

We have

THEOREM 3. *Let M and $\tilde{M}$ be as in Theorem* 1. *If M satisfies condition* (A_0), *then M satisfies condition* (C), *hence* (A) *and* (B) *as well, and $\Sigma^n(x)$ is indeed the central n-sphere $S^n(x)$.*

3. **Proofs.** We shall proceed to prove (1) equivalence of (A) and (B); (2) implication (C) $\rightarrow$ (A); (3) Theorem 2; and, finally, (4) implication (A_0) $\rightarrow$ (C).

(1) Assume (B) and let τ be a curve from x to y. Then $\tilde{\tau}^{-1}(S^n(y)) \subset S^n(x)$. Thus the end point $\tilde{\tau}^{-1}(y)$ of the development τ^* of τ into $T_x(\tilde{M})$ lies in $S^n(x)$. Conversely, assume (A), and let τ be a curve from x to y. In order to show $\tilde{\tau}(S^n(x)) \subset S^n(y)$, it is sufficient to show that there exists a neighborhood U^* of x in $S^n(x)$ such that $\tilde{\tau}(U^*) \subset S^n(y)$. For this purpose we first consider a mapping f of a normal neighborhood V of x in M into $S^n(x)$: for any point $z \in V$, let $f(z)$ be the end point of the development μ^* of the geodesic μ in V from x to z. Since f is a differentiable mapping of V into $S^n(x)$ whose differential at x is the identity mapping, it follows that there is a neighborhood U of x in M such that $U^* = f(U)$ is a neighborhood of x in $S^n(x)$. In order to prove that $\tilde{\tau}(U^*) \subset S^n(y)$, let $z^* \in U^*$, $z^* = f(z)$, $z \in U$, and let μ be the geodesic in U from x to z. Then the development $(\mu\cdot\tau^{-1})^*$ of the composed curve $\mu\cdot\tau^{-1}$ lies in $S^n(y)$. Since $(\mu\cdot\tau^{-1})^* = \tilde{\tau}(\mu^*)\cdot(\tau^{-1})^*$, its end point $\tilde{\tau}(z^*)$ lies in $S^n(y)$.

(2) We now assume (C) and prove (A). Let $\tau = x_t$ be a curve in M with $x_0 = x$. Let $\xi_1, \xi_2, \cdots, \xi_p$ be an orthonormal basis in the normal space at x such that $\xi_1 = \eta_x/H_x$ (unit mean curvature vector). We displace $\xi_1, \cdots, \xi_p$ along τ with respect to the normal connection $\nabla^\perp$ to obtain $(\xi_1)_t, \cdots, (\xi_p)_t$, which form an orthonormal basis in the normal space at x_t for each t. Since the mean curvature vector η is parallel with respect to $\nabla^\perp$ by assumption, $(\xi_1)_t$ is the unit mean curvature vector at x_t (and, of course, H is a constant). Since M is umbilical, we have

$$A_{(\xi_1)_t} = HI \text{ and } A_{(\xi_i)_t} = 0 \text{ for } 2 \leqq i \leqq p$$

along τ.

We observe that each $(\xi_i)_t, 2 \leqq i \leqq p$, is parallel along τ with respect to the linear connection $\tilde{\nabla}$ in $\tilde{M}$. Indeed, we have

$$\tilde{\nabla}_{\vec{x}_t}(\xi_i)_t = -A_{(\xi_i)_t}(\vec{x}_t) + \nabla^\perp_{\vec{x}_t}(\xi_i)_t = 0$$

along t.

We set

$$\tilde{X}_t = \tau_0^t(\bar{x}_t) \text{ for each } t ,$$

and let $\tau^* = \tilde{x}_t$ be the development of τ into $T_x(\tilde{M})$ so that $d\tilde{x}_t/dt = \tilde{X}_t$. The relations

$$g(\tilde{X}_t, \xi_i) = g(\bar{x}_t, (\xi_i)_t) = 0 , \qquad 2 \leqq i \leqq p ,$$

show that τ^* lies in the euclidean subspace of dimension $n+1$ in $T_x(\tilde{M})$ spanned by $T_x(M)$ and ξ_1.

Define $(\tilde{\xi}_1)_t \in T_x(\tilde{M})$ by $(\tilde{\xi}_1)_t = \tau_0^t((\xi_1)_t)$ for each t. Since

$$g(\tilde{X}_t, (\tilde{\xi}_1)_t) = g(\bar{x}_t, (\xi_1)_t) = 0 ,$$

we see that $(\tilde{\xi}_1)_t$ is perpendicular to τ^* at $\tilde{x}_t$. Set

$$u_t = \tilde{x}_t + (1/H)(\tilde{\xi}_1)_t ,$$

which is a curve in $T_x(\tilde{M})$. We shall show that u_t is actually a single point, say, $u = x + (1/H)\xi_1$ and so

$$\|\tilde{x}_t - u\| = 1/H ,$$

which shows that τ^* lies on the hypersphere in $T_x(\tilde{M})$ with center u and of radius $1/H$. Thus τ^* lies on the central n-sphere $S^n(x)$.

To show that u_t is a single point we need

LEMMA. $d(\tilde{\xi}_1)_t/dt = -H\tilde{X}_t$.

By definition of $(\xi_1)_t$ and $(\tilde{\xi}_1)_t$ we have

$$(\tilde{\xi}_1)_{t+h} = \tau_0^t \tau_t^{t+h}(\xi_1)_{t+h}$$

and

$$(\tilde{\xi}_1)_t = \tau_0^t(\xi_1)_t .$$

By linearity of τ_0^t we have

$$[(\tilde{\xi}_1)_{t+h} - (\tilde{\xi}_1)_t]/h = \tau_0^t[\tau_t^{t+h}(\xi_1)_{t+h} - (\xi_1)_t]/h .$$

As $h \to 0$, we get $d(\tilde{\xi}_1)/dt$ from the left-hand side. The right-hand side gives

$$\begin{aligned} \tau_0^t(\tilde{\nabla}_{\bar{x}_t}(\xi_1)_t) &= \tau_0^t(-A_{(\xi_1)_t}\bar{x}_t) \\ &= -\tau_0^t(H\bar{x}_t) = -H\tilde{X}_t . \end{aligned}$$

This proves the lemma.

Now we use the lemma to obtain

$$\begin{aligned} du_t/dt &= d\tilde{x}/dt + (1/H)d(\tilde{\xi}_1)_t/dt \\ &= \tilde{X}_t + (1/H)(-H\tilde{X}_t) = 0 , \end{aligned}$$

which shows that u_t is a single point and completes the proof that (C) implies (A).

(3) We prove Theorem 2. Assume (C) and let $\tau = x_t$ be a geodesic in M such that $x_0 = x$. As before, let $\tilde{X}_t = \tau_0^t(\bar{x}_t)$ for each t. For the fixed value of t, $\tilde{X}_t$ is obtained as follows: let Y_s, $0 \leqq s \leqq t$, be a unique parallel family of tangent vectors along τ such that $Y_t = \bar{x}_t$. Then $\tilde{X}_t = Y_0$. Now choosing $(\xi_1)_t, \cdots, (\xi_p)_t$ along τ as before, we may write

$$Y_s = Z_s + \sum_{i=1}^{p} \varphi^i(s)(\xi_i)_s, \qquad 0 \leqq s \leqq t\,,$$

where Z_s is tangent to M at x_s. We find

$$\begin{aligned}\tilde{\nabla}_{\bar{x}_s} Y_s &= \tilde{\nabla}_{\bar{x}_s} Z_s + \sum_{i=1}^{p} (d\varphi^i/ds)(\xi_i)_s \\ &\qquad - \sum_{i=1}^{p} \varphi^i A_{(\xi_i)_s}(\bar{x}_s) + \sum_{i=1}^{p} \varphi^i \nabla^{\perp}_{\bar{x}_s}(\xi_i)_s \\ &= \nabla_{\bar{x}_s} Z_s + Hg(\bar{x}_s, Z_s)(\xi_1)_s \\ &\qquad + \sum_{i=1}^{p} (d\varphi^i/ds)(\xi_i)_s - H\varphi^1(s)\bar{x}_s\,,\end{aligned}$$

by virtue of $\alpha(\bar{x}_s, Z_s) = g(\bar{x}_s, Z_s)\eta_s$, $A_{(\xi_1)_s} = HI$, $A_{(\xi_i)_s} = 0$ for $2 \leqq i \leqq p$, and $\nabla^{\perp}_{\bar{x}_s}(\xi_i)_s = 0$ for $1 \leqq i \leqq p$. Thus the equation $\nabla_{\bar{x}_s} Y_s = 0$ is equivalent to a system of equations

$$\begin{aligned}\nabla_{\bar{x}_s} Z_s &= H\varphi^1(s)\bar{x}_s \\ d\varphi^1/ds &= -Hg(\bar{x}_s, Z_s) \\ d\varphi^i/ds &= 0,\ 2 \leqq i \leqq p\,,\end{aligned}$$

and the terminal condition $Y_t = \bar{x}_t$ is given by

$$Z_t = \bar{x}_t \text{ and } \varphi^i(t) = 0 \text{ for } 1 \leqq i \leqq p\,.$$

Since τ is a geodesic, that is, $\nabla_{\bar{x}_s}\bar{x}_s = 0$, we see that the unique solution is given by

$$\begin{aligned}Z_s &= \cos H(t-s)\bar{x}_s \\ \varphi^1(s) &= \sin H(t-s) \\ \varphi^i(s) &= 0 \text{ for } 2 \leqq i \leqq p\,.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\tilde{X}_t = Y_0 &= Z_0 + \varphi^1(0)(\xi_1)_0 \\ &= \cos(Ht)\bar{x}_0 + \sin(Ht)(\xi_1)_0\,,\end{aligned}$$

where $(\xi_1)_0$ is the unit mean curvature vector ξ_1 at x. And $\bar{x}_0$ is the initial (unit) tangent vector of the geodesic τ. Thus the development τ^* of τ

is given by

$$\tilde{x}_t = (x + \xi_1/H) + (\sin(Ht)\bar{x}_0 - \cos(Ht)\xi_1)/H ,$$

which is a great circle on the central n-sphere $S^n(x)$. We have thus proved Theorem 2.

(4) We now prove Theorem 3. Assume (A_0) and let u and r be the center and the radius of the given sphere $\Sigma^n(x)$. Let y be an arbitrary point of M. For any curve $\tau = x_t$ in M such that $x_0 = x$ and $x_1 = y$, its development $\tau^* = \tilde{x}_t$ lies on $\Sigma^n(x)$. For each t, we define

$$(\tilde{\xi}_1)_t = (u - \tilde{x}_t)/r \in T_x(\tilde{M}) .$$

Let $\xi_1 = (\tilde{\xi}_1)_0, \xi_2, \cdots, \xi_p$ be an orthonormal basis in the normal space to M at x. We define $(\xi_i)_t \in T_{x_t}(\tilde{M})$ along τ as follows:

$$\tau_0^t((\xi_1)_t) = (\tilde{\xi}_1)_t,\ \tau_0^t((\xi_i)_t) = \xi_i \text{ for } 2 \leqq i \leqq p .$$

We show that for each value, say, s, of t, $(\xi_i)_s$ is perpendicular to M at x_s, where $1 \leqq i \leqq p$. Indeed, if we alter the curve τ after x_s so that it goes out of x_s in the direction of a tangent vector $Y \in T_{x_s}(M)$ and call the new curve τ', then its development τ'^* still lies on $\Sigma^n(x)$. Hence $\tau_0^s(Y)$ is perpendicular to $(\tilde{\xi}_1)_s$, as well as to $\xi_2, \cdots, \xi_p$. Thus Y is perpendicular to $(\xi_1)_s, (\xi_2)_s, \cdots, (\xi_p)_s$. Since Y is an arbitrary tangent vector to M at x_s, this proves our assertion.

Now, by definition of $(\tilde{\xi}_1)_t$, we have

$$d(\tilde{\xi}_1)_t/dt = -\tilde{X}_t/r = -(1/r)\tau_0^t(\bar{x}_t) ,$$

where $\tilde{X}_t = d\tilde{x}_t/dt$. From the argument for the preceding lemma we have

$$d(\tilde{\xi}_1)_t/dt = \tau_0^t(\tilde{\nabla}_{\bar{x}_t}(\xi_1)_t) .$$

These two equations imply

$$\tilde{\nabla}_{\bar{x}_t}(\xi_1)_t = -(1/r)\bar{x}_t ,$$

that is,

$$\nabla^{\perp}_{\bar{x}_t}(\xi_1)_t = 0 \text{ and } A_{(\xi_1)_t}(\bar{x}_t) = (1/r)\bar{x}_t .$$

The second equation is valid at each point x_t of τ if $\bar{x}_t$ is replaced by any tangent vector $Y \in T_{x_t}(M)$, because the curve τ may be altered to a new curve τ' which goes out of x_t in the direction Y just as in the previous argument, whereas $A_{(\xi_1)_t}$ depends only on $(\xi_1)t$ and is not affected by the alteration of τ. We have thus

$$\nabla^{\perp}_{\bar{x}_t}(\xi_1)_t = 0 \tag{1}$$

$$A_{(\xi_1)t} = (1/r)I . \tag{2}$$

For $2 \leqq i \leqq p$, $(\xi_i)_t$ is parallel along τ, that is,

$$\tilde{\nabla}_{\bar{x}_t}(\xi_i)_t = 0 ,$$

which implies

$$\nabla^{\perp}_{\bar{x}_t}(\xi_i)_t = 0 \text{ and } A_{(\xi_i)_t}(\bar{x}_t) = 0 .$$

Applying the previous argument, we see that the second equation is valid if $\bar{x}_t$ is replaced by any $Y \in T_{x_t}(M)$. Hence

$$(3) \qquad \nabla^{\perp}_{\bar{x}_t}(\xi_i)_t = 0, \qquad 2 \leqq i \leqq p$$

$$(4) \qquad A_{(\xi_i)_t} = 0, \qquad 2 \leqq i \leqq p .$$

From (1) and (3) it follows that $(\xi_i)_t$, $1 \leqq i \leqq p$, form an orthonormal basis in the normal space at x_t. From (2) and (4) we see that the mean curvature vector η is given by

$$(5) \qquad (\eta)_{x_t} = (1/r)(\xi_1)_t$$

and that for each point x_t

$$(6) \qquad A_{\xi} = g(\xi, \eta)I \text{ for every } \xi \text{ normal to } M \text{ at } x_t .$$

The relation (5) for $t = 0$ shows that $1/r = H_x = \|\eta_x\|$ and $\xi_1 = \eta_x/H_x$. Thus the given sphere $\Sigma^n(x)$ is indeed the central n-sphere $S^n(x)$.

The relation (6) for $t = 1$, namely, at the end point y of τ shows that y is umbilical. Since y is an arbitrary point of M, we conclude that every point of M is umbilical. It now remains to show that η is parallel with respect to $\nabla^{\perp}$. Let $y \in M$ and $Y \in T_y(M)$. Let μ be a curve starting at y in the direction of Y. By applying our argument to the curve $\mu \cdot \tau$, we see that (5) is valid at every point, namely, the mean curvature vector η is $1/r$ times $(\xi_1)_t$ which is parallel along the curve with respect to $\nabla^{\perp}$ by virtue of (1). In particular, $\nabla^{\perp}_Y \eta = 0$ at y. This completes the proof of Theorem 3.

BIBLIOGRAPHY

[1] E. CARTAN, Leçons sur la géométrie des espaces de Riemann, Gauthier-Villars, Paris, 1946.

[2] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry, Volumes I and II, Wiley-Interscience, New York, 1963 and 1969.

[3] D. S. LEUNG AND K. NOMIZU, The axiom of spheres in Riemannian geometry, J. Differential Geometry 5(1971), 487-489.

[4] S. SASAKI, Geometry of the conformal connexion, Sci. Rep. Tôhoku Imp. Univ. Ser. I, 29(1940), 219-267; also, Geometry of conformal connexions (in Japanese), Kawade Shobo, Tokyo, 1947.

[5] G. THOMSEN, Über konforme Geometrie I. Grundlagen der konformem Flächentheorie, Abh, Math. Seminar Hamburg 3(1923), 31-56.

DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY