



# Generalized commutative quaternions of the Fibonacci type

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## Abstract

Quaternions are a four-dimensional hypercomplex number system discovered by Hamilton in 1843 and next intensively applied in mathematics, modern physics, computer graphics and other fields. After the discovery of quaternions, modified quaternions were also defined in such a way that commutative property in multiplication is possible. That number system called as commutative quaternions is intensively studied and used for example in signal processing. In this paper we define generalized commutative quaternions and next based on them we define and explore Fibonacci type generalized commutative quaternions.

**Keywords** Quaternions · Generalized quaternions · Fibonacci numbers · Horadam numbers

**Mathematics Subject Classification** 11B37 · 11B39

## 1 Introduction

One of the problems in applying quaternions is their noncommutative structure. In other words, a different order of multiplication of quaternions results in different quaternions. That structure makes hard to conduct applications among other to engineering problems, see for example [22]. Modified quaternions were proposed by Serge in [26], so that commutative property in multiplication is possible. Quaternions and commutative quaternions belong to the class of hypercomplex

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numbers. Let  $N \geq 0$  be an integer. Hypercomplex numbers are defined by the expression (see [5, 21, 27])

$$\mathbf{x} = \sum_{i=0}^{N-1} x_i e_i$$

where  $x_i \in \mathbb{R}$  are called components and  $e_i$  are called units. The element  $e_0$  is the scalar or the real element and it may be identified with the real number 1. For  $i = 1, \dots, N - 1$  we have  $e_i \notin \mathbb{R}$ .

Let  $\mathbf{x} = \sum_{i=0}^{N-1} x_i e_i$  and  $\mathbf{y} = \sum_{i=0}^{N-1} y_i e_i$  be any two hypercomplex numbers. We define equality, addition and multiplication by scalar in the following way:

$$\mathbf{x} = \mathbf{y} \text{ only if } x_i = y_i, i = 0, 1, \dots, N - 1 \text{ (equality)}$$

$$\mathbf{x} + \mathbf{y} = \sum_{i=0}^{N-1} (x_i + y_i) e_i \text{ (addition)}$$

$$s \cdot \mathbf{x} = \sum_{i=0}^{N-1} (s \cdot x_i) e_i \text{ (multiplication by scalar } s \in \mathbb{R}\text{)}.$$

The zero element is the element with all components equal to zero. From these definitions it follows that the addition satisfies commutative and associative properties and the zero element exists, see [6].

Important in defining the multiplication of hypercomplex numbers is that the product of hypercomplex numbers is still a hypercomplex number.

For real and complex numbers the product operation satisfies the following properties: distributive with respect to the sum, associative, commutative, does not have divisors of zero. It can be shown (see [25]) that, only the real and complex numbers can satisfy these four properties all together. As far as other systems are concerned, if they satisfy the first two properties they can satisfy just one of the last two.

Analogously as for quaternions the class (family) of hypercomplex numbers can be partitioned into two subclasses (subfamilies) non-commutative hypercomplex system and commutative hypercomplex system.

A generalized noncommutative quaternion  $\mathbf{x}$  is a vector in 4-dimensional vector-space of the form  $\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$ , where quaternionic units  $e_1, e_2, e_3$  satisfy the equalities

$$e_1^2 = -\alpha, \quad e_2^2 = -\beta, \quad e_3^2 = -\alpha\beta,$$

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = \beta e_1 \quad \text{and} \quad e_3 e_1 = -e_1 e_3 = \alpha e_2,$$

where  $\alpha, \beta \in \mathbb{R}$ . The class of all generalized noncommutative quaternions we denote by  $\mathbb{H}_{\alpha\beta}^n$ . For special values of  $\alpha$  and  $\beta$  we obtain well-known subclasses of non-commutative quaternions

- $\alpha = \beta = 1$  – real quaternions,
- $\alpha = 1, \beta = -1$  – split quaternions,
- $\alpha = 1, \beta = 0$  – semi-quaternions,

$\alpha = -1, \beta = 0$  – split semiquaternions,  
 $\alpha = \beta = 0$  – 1/4 quaternions.

Algebraic properties of generalized quaternions were described in [16]. Motivated by above we introduce generalized commutative quaternions as follows.

A generalized commutative quaternion  $\mathbf{x}$  is a vector of the form  $\mathbf{x} = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ , where quaternionic units  $e_1, e_2, e_3$  satisfy the equalities

$$e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha\beta, \quad (1)$$

$$e_1e_2 = e_2e_1 = e_3, \quad e_2e_3 = e_3e_2 = \beta e_1 \quad \text{and} \quad e_3e_1 = e_1e_3 = \alpha e_2, \quad (2)$$

where  $\alpha, \beta \in \mathbb{R}$ . The family of all generalized commutative quaternions we denote by  $\mathbb{H}_{\alpha\beta}^c$ . For special  $\alpha$  and  $\beta$  there are the following subclasses of commutative quaternions

$\alpha < 0, \beta = 1$  – elliptic quaternions, in particular canonical (Segre's) quaternions if  $\alpha = -1$ ,  
 $\alpha = 0, \beta = 1$  – parabolic quaternions,  
 $\alpha > 0, \beta = 1$  – hyperbolic quaternions, in particular canonical or bihyperbolic numbers if  $\alpha = 1$ ,  
 $\alpha = -1, \beta = 1$  – tessarines or complex hyperbolic numbers,  
 $\alpha = -1, \beta = -1$  – bicomplex numbers,  
 $\alpha = 1, \beta = -1$  – hyperbolic complex numbers.

Note that the above special cases are well-known and studied in the literature. It suffices to mention that in [6] Catoni et al. studied algebraic properties of three types of commutative quaternions i.e. elliptic, parabolic and hyperbolic quaternions. Although the theory of commutative quaternions is becoming important in recent years and has interesting applications in distinct areas of mathematics and modern physics, the study of commutative quaternions is focus mainly on elliptic quaternions, see for example [17–20]. Tessarines were introduced by Coclé in 1848 and next that theory was developed in the sequences of articles [7–10]. Bicomplex numbers were discovered by Serge, see [26] and recently studied by Rochon in [24]. Hyperbolic complex numbers also named as complex numbers with hyperbolic coefficients were investigated in [1]. Bihyperbolic numbers are one of the extension of hyperbolic numbers, see [4, 23].

In recent years the theory of hypercomplex numbers is complemented by their connections with Fibonacci sequences, for more details see the book [27].

Generalized noncommutative Fibonacci quaternions were introduced by Horadam in [15]. In [2, 11–13] the authors presented some properties of the generalized noncommutative Fibonacci quaternions.

In this paper we define generalized commutative quaternions of the Fibonacci type.

## 2 Generalized commutative quaternions of the Fibonacci type

Let  $n \geq 0$  be an integer. The  $n$ th Fibonacci number  $F_n$  is a number defined recursively by the second-order linear recurrence relation  $F_n = F_{n-1} + F_{n-2}$ , for  $n \geq 2$  with  $F_0 = 0$ ,  $F_1 = 1$ .

The Fibonacci numbers are generalized in different ways, see for example the list in [3] and [27]. Consider the well-known generalization of Fibonacci numbers given by Horadam in [14].

Let  $p, q, n$  be integers. For  $n \geq 0$  Horadam defined the numbers  $W_n = W_n(W_0, W_1; p, q)$  by the recursive equation

$$W_n = p \cdot W_{n-1} - q \cdot W_{n-2}, \quad (3)$$

for  $n \geq 2$  with fixed real numbers  $W_0, W_1$ .

For the historical reasons these numbers were later called the Horadam numbers.

Let  $t_1, t_2$  be the two distinct real roots of the equation

$$t^2 - pt + q = 0.$$

Then, the Binet type formula for the Horadam numbers has the form

$$W_n = At_1^n + Bt_2^n, \quad (4)$$

where

$$\begin{aligned} t_1 &= \frac{p - \sqrt{p^2 - 4q}}{2}, \\ t_2 &= \frac{p + \sqrt{p^2 - 4q}}{2}, \\ A &= \frac{W_1 - W_0 t_2}{t_1 - t_2}, \\ B &= \frac{W_0 t_1 - W_1}{t_1 - t_2}. \end{aligned} \quad (5)$$

For special  $W_0, W_1, p, q$  the equation (3) defines selected numbers of the Fibonacci type:

the Fibonacci numbers  $F_n = W_n(0, 1; 1, -1)$ , the Lucas numbers  $L_n = W_n(2, 1; 1, -1)$ , the Pell numbers  $P_n = W_n(0, 1; 2, -1)$ , the Pell-Lucas numbers  $Q_n = W_n(2, 2; 2, -1)$ , the Jacobsthal numbers  $J_n = W_n(0, 1; 1, -2)$ , the Jacobsthal-Lucas numbers  $j_n = W_n(2, 1; 1, -2)$ , the balancing numbers  $B_n = W_n(0, 1; 6, 1)$ , the Lucas-balancing numbers  $C_n = W_n(1, 3; 6, 1)$  etc.

Let  $n \geq 0$  be an integer. The  $n$ th generalized commutative Horadam quaternion  $gc\mathcal{H}_n$  is defined as

$$gc\mathcal{H}_n = W_n + W_{n+1}e_1 + W_{n+2}e_2 + W_{n+3}e_3, \quad (6)$$

where  $W_n$  is the  $n$ th Horadam number and  $e_1, e_2, e_3$  are units satisfy (1) and (2).

**Theorem 1** (Binet type formula for generalized commutative Horadam quaternions) Let  $n \geq 0$  be an integer. Then

$$gc\mathcal{H}_n = At_1^n(1 + t_1e_1 + t_1^2e_2 + t_1^3e_3) + Bt_2^n(1 + t_2e_1 + t_2^2e_2 + t_2^3e_3), \tag{7}$$

where  $t_1, t_2, A$  and  $B$  are given by (5).

**Proof** Using (6) and (4) we have

$$\begin{aligned} gc\mathcal{H}_n &= W_n + W_{n+1}e_1 + W_{n+2}e_2 + W_{n+3}e_3 \\ &= At_1^n + Bt_2^n + (At_1^{n+1} + Bt_2^{n+1})e_1 + (At_1^{n+2} + Bt_2^{n+2})e_2 \\ &\quad + (At_1^{n+3} + Bt_2^{n+3})e_3 \\ &= At_1^n(1 + t_1e_1 + t_1^2e_2 + t_1^3e_3) + Bt_2^n(1 + t_2e_1 + t_2^2e_2 + t_2^3e_3) \end{aligned}$$

and the result follows. □

For simplicity of notation let

$$\begin{aligned} \hat{t}_1 &= 1 + t_1e_1 + t_1^2e_2 + t_1^3e_3, \\ \hat{t}_2 &= 1 + t_2e_1 + t_2^2e_2 + t_2^3e_3. \end{aligned} \tag{8}$$

Then we can write (7) as

$$gc\mathcal{H}_n = At_1^n\hat{t}_1 + Bt_2^n\hat{t}_2.$$

**Theorem 2** (Catalan type identity for generalized commutative Horadam quaternions) Let  $n \geq 0, r \geq 0$  be integers such that  $n \geq r$ . Then

$$\begin{aligned} gc\mathcal{H}_{n-r} \cdot gc\mathcal{H}_{n+r} - (gc\mathcal{H}_n)^2 \\ = ABt_1^n t_2^n \left( \left( \frac{t_1}{t_2} \right)^r + \left( \frac{t_2}{t_1} \right)^r - 2 \right) \hat{t}_1 \hat{t}_2, \end{aligned}$$

where  $t_1, t_2, A, B$  and  $\hat{t}_1, \hat{t}_2$  are given by (5) and (8), respectively.

**Proof** By formula (7) we have

$$\begin{aligned} gc\mathcal{H}_{n+r} \cdot gc\mathcal{H}_{n-r} - (gc\mathcal{H}_n)^2 \\ = (At_1^{n+r}\hat{t}_1 + Bt_2^{n+r}\hat{t}_2) \cdot (At_1^{n-r}\hat{t}_1 + Bt_2^{n-r}\hat{t}_2) - (At_1^n\hat{t}_1 + Bt_2^n\hat{t}_2)^2 \\ = At_1^{n+r}\hat{t}_1 At_1^{n-r}\hat{t}_1 + At_1^{n+r}\hat{t}_1 Bt_2^{n-r}\hat{t}_2 + Bt_2^{n+r}\hat{t}_2 At_1^{n-r}\hat{t}_1 + Bt_2^{n+r}\hat{t}_2 Bt_2^{n-r}\hat{t}_2 \\ \quad - At_1^n\hat{t}_1 At_1^n\hat{t}_1 - At_1^n\hat{t}_1 Bt_2^n\hat{t}_2 - Bt_2^n\hat{t}_2 At_1^n\hat{t}_1 - Bt_2^n\hat{t}_2 Bt_2^n\hat{t}_2 \\ = At_1^{n+r}\hat{t}_1 Bt_2^{n-r}\hat{t}_2 + Bt_2^{n+r}\hat{t}_2 At_1^{n-r}\hat{t}_1 - At_1^n\hat{t}_1 Bt_2^n\hat{t}_2 - Bt_2^n\hat{t}_2 At_1^n\hat{t}_1 \\ = ABt_1^n t_2^n \left( \left( \frac{t_1}{t_2} \right)^r + \left( \frac{t_2}{t_1} \right)^r - 2 \right) \hat{t}_1 \hat{t}_2, \end{aligned}$$

which ends the proof. □

Note that for  $r = 1$  we get the Cassini type identity for generalized commutative Horadam quaternions.

**Corollary 3** (Cassini type identity for generalized commutative Horadam quaternions) *Let  $n \geq 1$  be an integer. Then*

$$\begin{aligned} &gc\mathcal{H}_{n+1} \cdot gc\mathcal{H}_{n-1} - (gc\mathcal{H}_n)^2 \\ &= ABt_1^n t_2^n \left( \frac{t_1}{t_2} + \frac{t_2}{t_1} - 2 \right) \hat{t}_1 \hat{t}_2, \end{aligned}$$

where  $t_1, t_2, A, B$  and  $\hat{t}_1, \hat{t}_2$  are given by (5) and (8), respectively.

**Theorem 4** (d’Ocagne type identity for generalized commutative Horadam quaternions) *Let  $m \geq 0, n \geq 0$  be integers such that  $m \geq n$ . Then*

$$\begin{aligned} &gc\mathcal{H}_m \cdot gc\mathcal{H}_{n+1} - gc\mathcal{H}_{m+1} \cdot gc\mathcal{H}_n \\ &= ABt_1^n t_2^n (t_1 - t_2) (t_2^{m-n} - t_1^{m-n}) \hat{t}_1 \hat{t}_2, \end{aligned}$$

where  $t_1, t_2, A, B$  and  $\hat{t}_1, \hat{t}_2$  are given by (5) and (8), respectively.

**Proof** By formula (7) we have

$$\begin{aligned} &gc\mathcal{H}_m \cdot gc\mathcal{H}_{n+1} - gc\mathcal{H}_{m+1} \cdot gc\mathcal{H}_n \\ &= (At_1^m \hat{t}_1 + Bt_2^m \hat{t}_2) (At_1^{n+1} \hat{t}_1 + Bt_2^{n+1} \hat{t}_2) \\ &\quad - (At_1^{m+1} \hat{t}_1 + Bt_2^{m+1} \hat{t}_2) (At_1^n \hat{t}_1 + Bt_2^n \hat{t}_2) \\ &= At_1^m \hat{t}_1 At_1^{n+1} \hat{t}_1 + Bt_2^m \hat{t}_2 At_1^{n+1} \hat{t}_1 + At_1^m \hat{t}_1 Bt_2^{n+1} \hat{t}_2 + Bt_2^m \hat{t}_2 Bt_2^{n+1} \hat{t}_2 \\ &\quad - At_1^{m+1} \hat{t}_1 At_1^n \hat{t}_1 - Bt_2^{m+1} \hat{t}_2 At_1^n \hat{t}_1 - At_1^{m+1} \hat{t}_1 Bt_2^n \hat{t}_2 - Bt_2^{m+1} \hat{t}_2 Bt_2^n \hat{t}_2 \\ &= Bt_2^m \hat{t}_2 At_1^{n+1} \hat{t}_1 + At_1^m \hat{t}_1 Bt_2^{n+1} \hat{t}_2 - Bt_2^{m+1} \hat{t}_2 At_1^n \hat{t}_1 - At_1^{m+1} \hat{t}_1 Bt_2^n \hat{t}_2 \\ &= ABt_1^n t_2^n (t_1 t_2^{m-n} + t_1^{m-n} t_2 - t_2 t_2^{m-n} - t_1 t_1^{m-n}) \hat{t}_1 \hat{t}_2 \\ &= ABt_1^n t_2^n (t_2^{m-n} (t_1 - t_2) - t_1^{m-n} (t_1 - t_2)) \hat{t}_1 \hat{t}_2 \\ &= ABt_1^n t_2^n (t_1 - t_2) (t_2^{m-n} - t_1^{m-n}) \hat{t}_1 \hat{t}_2, \end{aligned}$$

which ends the proof. □

Moreover, by simple calculations we obtain

$$\begin{aligned} &t_1 + t_2 = p, \quad t_1 t_2 = q, \\ &t_1^2 + t_2^2 = (t_1 + t_2)^2 - 2t_1 t_2 = p^2 - 2q, \\ &t_1^3 + t_2^3 = (t_1 + t_2)^3 - 3t_1 t_2 (t_1 + t_2) = p^3 - 3pq \end{aligned}$$

and

$$\begin{aligned}
 \hat{t}_1 \hat{t}_2 &= \hat{t}_2 \hat{t}_1 = (1 + t_1 e_1 + t_1^2 e_2 + t_1^3 e_3)(1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3) \\
 &= 1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3 + t_1 e_1 + t_1 e_1 t_2 e_1 + t_1 e_1 t_2^2 e_2 + t_1 e_1 t_2^3 e_3 + t_1^2 e_2 \\
 &\quad + t_1^2 e_2 t_2 e_1 + t_1^2 e_2 t_2^2 e_2 + t_1^2 e_2 t_2^3 e_3 + t_1^3 e_3 + t_1^3 e_3 t_2 e_1 + t_1^3 e_3 t_2^2 e_2 + t_1^3 e_3 t_2^3 e_3 \\
 &= 1 + t_2 e_1 + t_2^2 e_2 + t_2^3 e_3 + t_1 e_1 + t_1 t_2 \alpha + t_1 t_2^2 e_3 + t_1 t_2^3 \alpha e_2 + t_1^2 e_2 \\
 &\quad + t_1^2 t_2 e_3 + t_1^2 t_2^2 \beta + t_1^2 t_2^3 \beta e_1 + t_1^3 e_3 + t_1^3 t_2 \alpha e_2 + t_1^3 t_2^2 \beta e_1 + t_1^3 t_2^3 \alpha \beta \\
 &= 1 + t_1 t_2 \alpha + t_1^2 t_2^2 \beta + t_1^3 t_2^3 \alpha \beta + (t_1 + t_2 + t_1^2 t_2^2 \beta + t_1^3 t_2^3 \beta) e_1 \\
 &\quad + (t_1^2 + t_2^2 + t_1 t_2^3 \alpha + t_1^3 t_2 \alpha) e_2 + (t_1^3 + t_2^3 + t_1 t_2^2 + t_1^2 t_2) e_3 \\
 &= 1 + q\alpha + q^2\beta + q^3\alpha\beta + (p + pq^2\beta) e_1 + (p^2 - 2q)(1 + q\alpha) e_2 + (p^3 - 2pq) e_3.
 \end{aligned}$$

Substituting the appropriate values of  $p, q, W_0, W_1$  etc. and using the above calculations we can obtain Binet formulas, Catalan, Cassini and d’Ocagne identities for the mentioned numbers of the Fibonacci type.

Now we give matrix representation of commutative Horadam quaternions. Let  $gc\mathcal{H}(n) = \begin{bmatrix} gc\mathcal{H}_{n+1} & gc\mathcal{H}_n \\ gc\mathcal{H}_n & gc\mathcal{H}_{n-1} \end{bmatrix}$  be a matrix with entries being generalized commutative Horadam quaternions.

**Theorem 5** *Let  $n \geq 1$  be an integer. Then*

$$\begin{bmatrix} gc\mathcal{H}_{n+1} & gc\mathcal{H}_n \\ gc\mathcal{H}_n & gc\mathcal{H}_{n-1} \end{bmatrix} = \begin{bmatrix} gc\mathcal{H}_2 & gc\mathcal{H}_1 \\ gc\mathcal{H}_1 & gc\mathcal{H}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix}^{n-1}. \tag{9}$$

**Proof** If  $n = 1$ , then by simple calculations the result immediately follows. Assume that the equality (9) is true for all integers  $1, 2, \dots, n$ . We shall prove that the equality is true for integer  $n + 1$ . Using induction hypothesis we have

$$\begin{aligned}
 &\begin{bmatrix} gc\mathcal{H}_2 & gc\mathcal{H}_1 \\ gc\mathcal{H}_1 & gc\mathcal{H}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} = \begin{bmatrix} gc\mathcal{H}_{n+1} & gc\mathcal{H}_n \\ gc\mathcal{H}_n & gc\mathcal{H}_{n-1} \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} \\
 &= \begin{bmatrix} p \cdot gc\mathcal{H}_{n+1} - q \cdot gc\mathcal{H}_n & gc\mathcal{H}_{n+1} \\ p \cdot gc\mathcal{H}_n - q \cdot gc\mathcal{H}_{n-1} & gc\mathcal{H}_n \end{bmatrix} = \begin{bmatrix} gc\mathcal{H}_{n+2} & gc\mathcal{H}_{n+1} \\ gc\mathcal{H}_{n+1} & gc\mathcal{H}_n \end{bmatrix},
 \end{aligned}$$

which ends the proof. □

Properties of commutative Horadam quaternions allow the use determinant properties. Moreover

$$\begin{bmatrix} gc\mathcal{H}_2 & gc\mathcal{H}_1 \\ gc\mathcal{H}_1 & gc\mathcal{H}_0 \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} = \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix}^T \cdot \begin{bmatrix} gc\mathcal{H}_2 & gc\mathcal{H}_1 \\ gc\mathcal{H}_1 & gc\mathcal{H}_0 \end{bmatrix}$$

and

$$\det \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix} = \det \begin{bmatrix} p & 1 \\ -q & 0 \end{bmatrix}^T = q.$$

Consequently by algebraic operations and matrix algebra some interesting properties of commutative Horadam quaternions can be expected.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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## References

1. Akar, M., Yüce, S., Şahin, S.: On the dual hyperbolic numbers and the complex hyperbolic numbers. *J. Comput. Sci. Comput. Math.* **8**(1), 1–6 (2018). <https://doi.org/10.20967/jcscm.2018.01.001>
2. Akyiğit, M., Kosal, H.H., Tosun, M.: Fibonacci generalized quaternions. *Adv. Appl. Clifford Algebr.* **24**, 631–641 (2014). <https://doi.org/10.1007/s00006-014-0458-0>
3. Bednarz, U., Włoch, I., Wołowiec-Musiał, M.: Total graph interpretation of the numbers of the Fibonacci type. *J. Appl. Math.* **2015**, 7 (2015). <https://doi.org/10.1155/2015/837917>
4. Bilgin, M., Ersoy, S.: Algebraic properties of bihyperbolic numbers. *Adv. Appl. Clifford Algebr.* **30**, 13 (2020). <https://doi.org/10.1007/s00006-019-1036-2>
5. Cartan, E.: *Oeuvres Complètes*. Gauthier Villars, Paris (1953)
6. Catoni, F., Boccaletti, D., Cannata, R., Catoni, V., Nichelatti, E., Zampetti, P.: *The Mathematics of Minkowski Space-Time*. Birkhäuser Verlag AG, Basel Boston Berlin (2008)
7. Cockle, J.: On a new imaginary in algebra. *London Edinburgh Dublin Philos. Mag. J. Sci.* **34**, 37–47 (1849). <https://doi.org/10.1080/14786444908646169>
8. Cockle, J.: On certain functions resembling quaternions, and on a new imaginary in algebra. *London Edinburgh Dublin Philos. Mag. J. Sci.* **33**, 435–439 (1848). <https://doi.org/10.1080/14786444808646139>
9. Cockle, J.: On impossible equations, on impossible quantities, and on tessarines. *London Edinburgh Dublin Philos. Mag. J. Sci.* **37**, 281–283 (1850). <https://doi.org/10.1080/14786445008646598>
10. Cockle, J.: On the symbols of algebra, and on the theory of tesarines. *London Edinburgh Dublin Philos. Mag. J. Sci.* **34**, 406–410 (1849). <https://doi.org/10.1080/14786444908646257>
11. Flaut, C.: A Clifford algebra associated to generalized Fibonacci quaternions. *Adv. Differ. Equ.* **2014**, 279 (2014). <https://doi.org/10.1186/1687-1847-2014-279>
12. Flaut, C., Savin, D.: Quaternion algebras and generalized Fibonacci-Lucas quaternions. *Adv. Appl. Clifford Algebr.* **25**, 853–862 (2015). <https://doi.org/10.1007/s00006-015-0542-0>
13. Flaut, C., Shpakivskiy, V.: On generalized Fibonacci quaternions and Fibonacci-Narayana quaternions. *Adv. Appl. Clifford Algebr.* **23**, 673–688 (2013). <https://doi.org/10.1007/s00006-013-0388-2>
14. Horadam, A.F.: Basic properties of a certain generalized sequence of numbers. *Fibonacci Quart.* **3**(3), 161–176 (1965)



15. Horadam, A.F.: Complex Fibonacci Numbers and Fibonacci Quaternions. *American Mathematical Monthly* **70**, 289–291 (1963). <https://doi.org/10.2307/2313129>
16. Jafari, M., Yayli, Y.: Generalized quaternions and their algebraic properties. *Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat.* **64**(1), 15–27 (2015)
17. Kösal, H.H., Akyığıt, M., Tosun, M.: Consimilarity of commutative quaternion matrices. *Miskolc Math. Notes* **16**, 965–977 (2015). <https://doi.org/10.18514/MMN.2015.1421>
18. Kösal, H.H., Tosun, M.: Commutative quaternion matrices. *Adv. Appl. Clifford Algebr.* **24**, 769–779 (2014). <https://doi.org/10.1007/s00006-014-0449-1>
19. Kösal, H.H., Tosun, M.: Some equivalence relations and results over the commutative quaternions and their matrices. *Analele Universitatii “Ovidius” Constanta - Seria Matematica* **25**(3), 125–142 (2017). <https://doi.org/10.1515/auom-2017-0040>
20. Kösal, H.H., Tosun, M.: Universal similarity factorization equalities for commutative quaternions and their matrices. *Linear Multilinear Algebr.* **67**, 926–938 (2019). <https://doi.org/10.1080/03081087.2018.1439878>
21. Lie, S., Scheffers, M.G.: *Vorlesungen über continuerliche Gruppen*, Kap. 21. Teubner, Leipzig (1893)
22. Pei, S.-C., Chang, J.-H., Ding, J.-J.: Commutative reduced biquaternions and their Fourier transform for signal and image processing applications. *IEEE Trans. Signal Process.* **52**(7), 2012–2031 (2004). <https://doi.org/10.1109/TSP.2004.828901>
23. Pogorui, A.A., Rodríguez-Dagnino, R.M., Rodríguez-Said, R.D.: On the set of zeros of bihyperbolic polynomials. *Complex Var. Elliptic Equ.* **53**(7), 685–690 (2008). <https://doi.org/10.1080/17476930801973014>
24. Rochon, D., Shapiro, M.: On algebraic properties of bicomplex and hyperbolic numbers. *Anal. Univ. Oradea. Fasc. Math.* **11**, 71–110 (2004)
25. Scorza, G.: *Corpi numerici ed Algebre*. (Numerical corps and algebras), Principato, Messina (1921)
26. Segre, C.: *Le Rappresentazioni Reali delle Forme Complesse a Gli Enti Iperalgebrici*. *Mathematische Annalen* **40**, 413–467 (1892). <https://doi.org/10.1007/BF01443559>
27. Szynal-Liana, A., Włoch, I.: *Hypercomplex numbers of the Fibonacci type*. Oficyna Wydawnicza Politechniki Rzeszowskiej, Rzeszów (2019)

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