South East Asian J. of Mathematics and Mathematical Sciences
Vol. 19, No. 1 (2023), pp. 241-256

DOI: 10.56827/SEAJMMS.2023.1901.20 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

GENERALIZED CONFORMAL CURVATURE TENSOR OF LP-SASAKIAN MANIFOLD

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(**Received:** *Mar. 06, 2022* **Accepted:** *Feb. 24, 2023* **Published:** *Apr. 30, 2023)*

Abstract: The object of the present paper is to generalize conformal curvature tensor of LP-Sasakian manifold with the help of a new generalized $(0, 2)$ symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [7]. Various geometric properties of the generalized conformal curvature tensor of LP-Sasakian manifold have been studied. It is shown that a generalized conformally ϕ-Symmetric LP-Sasakian manifold is an η-Einstein manifold.

Keywords and Phrases: LP-Sasakian manifold, conformal curvature tensor, generalized conformal curvature tensor, Einstein manifold.

2020 Mathematics Subject Classification: 53C25, 53D15.

1. Introduction

In 1960, a Japanese Mathematician S. Sasaki introduced the notion of Sasakian structure [13]. A contact metric manifold is said to be Sasakian if it is normal. In some respects, Sasakian manifolds may be viewed as an odd dimensional analogous of Kâhler manifolds. T. Adati and K. Matsumoto [1] defined para-Sasakian and special para-Sasakian manifolds which are considered as special cases of an almost paracontact manifold introduced by I. Sato and K. Matsumoto [14].

The notation of Lorentizian para-Sasakian manifold was introduced by K. Matsumoto [9] in the year 1989.Later in 1992, the same notation was defined by Mihai and Rosca [10] independently. Since 1989, various geometric properties of LP-Sasakian manifolds have been explored by several researchers such as ([5], [8], [12], [16], [17]) and others.

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ was introduced by Mantica and Suh [7] in the year 2012 and they studied various geometric properties of it on Riemannian manifold. A new tensor $\mathcal{Z}$ defined as

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y) \tag{1.1}$$

is called $\mathcal{Z}-$tensor, where ψ is an arbitrary scalar function.

Due to the generalization of projective curvature tensor with the help of this new tensor $\mathcal{Z}$ defined in equation (1.1) on LP-Sasakian manifold our study become more significant.

In this paper, we consider the generalized conformal curvature tensor of LP-Sasakian manifold and study some geometric properties of it.The organization of the article is as follows:

After preliminaries on LP-Sasakian manifold in section 2, we introduce and briefly describe the generalized conformal curvature tensor of LP-Sasakian manifold in section 3 and studied some geometric properties of it. In section 4, we prove that a generalized conformal semi-symmetric LP-Sasakian manifold is of constant curvature. In section 5, we show that a generalized conformal Ricci semi-symmetric LP-Sasakian manifold is either Einstein manifold or $\psi = -(n-1)$. In the last section,we prove that a generalized conformally ϕ-symmetric LP-Sasakian manifold is an $\eta-$ Einstein manifold.

2. Preliminaries

An n-dimensional differentiable manifold M^n is said to be a Lorentzian para-Sasakian manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g that satisfies ([8], [9]),

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X),\ \nabla_X \xi = \phi X, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = \eta(Y)X + g(X,Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g. It can easily be seen that in an LP-Sasakian manifold M^n following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0,\ rank(\phi) = n-1. \tag{2.6}$$

Further, on an LP-Sasakian manifold following relations also hold:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.7}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(\xi,X)\xi = X + \eta(X)\xi, \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.11}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

for any vector fields X, Y, Z, where $R(X,Y)Z$ is the Riemannian curvature tensor and S is the Ricci tensor.
In [3] Blaga have given an example of LP-Sasakian manifold.

Example. [3] We consider the manifold
M=$\{(x,y,z) \in R^3,$: (x,y,z) are the standred coordinates in $R^3\}$.
Let the system of vector field

$$E_1 = e^z\frac{\partial}{\partial y}, E_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}), E_3 = e^z\frac{\partial}{\partial z},$$

are linearly independent at each point of the manifold.
Define the Lorentzian metric g by

$$g(E_1,E_1) = 1,$$

$$g(E_2,E_2) = 1,$$

$$g(E_3,E_3) = -1,$$

and

$$g(E_1,E_2) = g(E_2,E_3) = g(E_3,E_1) = 0.$$

Taking $\xi = E_3$. Let η be the 1-form defined by

$$\eta(Z) = g(Z,E_3)$$

for any vector field Z. Let the $(1,1)$ tensor field ϕ be defined by

$$\phi(E_1) = -E_1, \;\; \phi(E_2) = -E_2, \;\; \phi(E_3) = 0.$$

Using the linearty of ϕ and g, we have

$$\eta(E_3) = g(E_3, E_3) = -1,$$

$$\phi^2 Z = Z + \eta(Z)E_3,$$

$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any vector fields $Z, W \in TM$.
Then the structure (ϕ, E_3, η, g) defines the Lorentzian paracontact structure.
Let ∇ be the Levi-Civita connection for Lorentzian metric g, then by Koszul's formula, we have

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1,$$

$$\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = -E_3, \quad \nabla_{E_2} E_3 = -E_2,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

From these calculations, it can be easily seen that $M(\phi, E_3, \eta, g)$ is an LP-Sasakian manifold.

3. Generalized Conformal Curvature Tensor

In this section, we give brief account of generalized conformal curvature tensor of LP-Sasakian manifold and study various geometric properties of it.

The conformal curvature tensor is given as [6]

$$\begin{aligned} C(U,V,X,Y) =& R(U,V,X,Y) - \frac{1}{2(n-1)}[S(V,X)g(U,Y) \\ & - S(U,X)g(V,Y) + S(U,Y)g(V,X) - S(V,Y)g(U,X)] \\ & + \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \end{aligned} \tag{3.1}$$

where $R(U,V,X,Y)$ is the Riemannian curvature tensor and S is the Ricci tensor of the manifold. The conformal curvature tensor $C(U,V,X,Y)$ is skew-symmetric in the first two slots, skew-symmetric in the last two slots and symmetric in the pair of slots [6].
From equation (3.1), we have

$$\begin{aligned} C(U,V)X =& R(U,V)X - \frac{1}{2(n-1)}[S(V,X)U - S(U,X)V \\ & + g(V,X)QU - g(U,X)QV] \\ & + \frac{r}{(n-1)(n-2)}[g(V,X)U - g(U,X)V]. \end{aligned} \tag{3.2}$$

Covariant derivative of the conformal curvature tensor is given as

$$\begin{aligned}(\nabla_W C)(U,V)X = (\nabla_W R)(U,V)X - \frac{1}{2(n-1)}[(\nabla_W S)(V,X)U \\ - (\nabla_W S)(U,X)V + (\nabla_W Q)(U)g(V,X) - (\nabla_W Q)(V)g(U,X)] \\ + \frac{dr(W)}{(n-1)(n-2)}[g(V,X)U - g(U,X)V].\end{aligned} \tag{3.3}$$

Divergence of the conformal curvature tensor is given as

$$\begin{aligned}(divC)(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X) \\ &- \frac{1}{2(n-1)}\{g(V,X)dr(U) - g(U,X)dr(V)\}].\end{aligned} \tag{3.4}$$

In view of equation (1.1), equation (3.1) takes the form

$$\begin{aligned}C(U,V,X,Y) = R(U,V,X,Y) - \frac{1}{2(n-1)}[\mathcal{Z}(V,X)g(U,Y) \\ - \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)] \\ + \frac{r}{(n-1)(n-2)}[g(V,X)U - g(U,X)V] \\ + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{3.5}$$

Define

$$\begin{aligned}C^*(U,V,X,Y) = R(U,V,X,Y) - \frac{1}{n-1}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y) \\ + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)] \\ + \frac{r}{(n-1)(n-2)}[g(V,X)U - g(U,X)V].\end{aligned} \tag{3.6}$$

By using the equation (3.6), equation (3.5) reduces to

$$C(U,V,X,Y) = C^*(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],$$

which gives

$$C^*(U,V,X,Y) = C(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]. \tag{3.7}$$

The curvature tensor $C^*(U,V,X,Y)$ defined in equation (3.6) is called generalized conformal curvature tensor of LP-Sasakian manifold.
If $\psi = 0$, then from equation (3.7), we obtain

$$C^*(U,V,X,Y) = C(U,V,X,Y). \tag{3.8}$$

Thus we can state as follows

Theorem 3.1. *If the scalar function ψ vanishes on LP-Sasakian manifold, then the conformal curvature tensor and the generalized conformal curvature tensor coincide.*

From equation (3.7), we have

$$C^*(V,U,X,Y) = C(V,U,X,Y) - \frac{2\psi}{(n-2)}[g(U,X)g(V,Y) - g(V,X)g(U,Y)]. \tag{3.9}$$

Now adding equations (3.7) and (3.9) with the fact that
$C(U,V,X,Y) + C(V,U,X,Y) = 0$, we get

$$C^*(U,V,X,Y) + C^*(V,U,X,Y) = 0, \tag{3.10}$$

which shows that the generalized conformal curvature tensor is skew-symmetric in first two slots.
Again from equation (3.7), we have

$$C^*(U,V,Y,X) = C(U,V,Y,X) - \frac{2\psi}{(n-2)}[g(V,Y)g(U,X) - g(U,Y)g(V,X)]. \tag{3.11}$$

Now adding equations (3.7) and (3.11) with the fact that
$C(U,V,Y,X) + C(U,V,X,Y) = 0$, we have

$$C^*(U,V,X,Y) + C^*(U,V,Y,X) = 0, \tag{3.12}$$

which shows that the generalized conformal curvature tensor is skew-symmetric with respect to last two slots.
Again interchanging pair of slots in equation (3.7), we obtain

$$C^*(X,Y,U,V) = V(X,Y,U,V) - \frac{2\psi}{(n-2)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)]. \tag{3.13}$$

Now subtracting equations (3.13) with (3.7) with the fact that
$M(X,Y,U,V) - M(U,V,X,Y) = 0$, we get

$$M^*(U,V,X,Y) - M^*(X,Y,U,V) = 0, \tag{3.14}$$

which shows that the generalized conformal curvature tensor is symmetric in the pair of slots.
Thus we can state as follows

Theorem 3.2. *A generalized conformal curvature tensor of LP-Sasakian manifold is*
(*i*) *skew-symmetric in the first two slots,*
(*ii*) *skew-symmetric in the last two slots,*
(*iii*) *symmetric in the pair of slots.*

Now, writing two more equations by the cyclic permutations of U, V and X of equation (3.7), we obtain

$$C^*(V,X,U,Y) = C(V,X,U,Y) - \frac{2\psi}{(n-2)}[g(X,U)g(V,Y) - g(V,U)g(X,Y)] \quad (3.15)$$

and

$$C^*(X,U,V,Y) = C(X,U,V,Y) - \frac{2\psi}{(n-2)}[g(U,V)g(X,Y) - g(X,V)g(U,Y)]. \quad (3.16)$$

Adding equations (3.7), (3.15) and (3.16), with the fact that
$C(U,V,X,Y) + C(V,X,U,Y) + C(X,U,V,Y) = 0$, we obtain

$$C^*(U,V,X,Y) + C^*(V,X,U,Y) + C^*(X,U,V,Y) = 0, \quad (3.17)$$

which shows that the generalized conformal curvature tensor satisfies Bianchi's first identity.
Thus we can state as follows

Theorem 3.3. *A generalized conformal curvature tensor of LP-Sasakian manifold satisfies Bianchi's first identity.*

From equation (3.7), we have

$$C^*(U,V)X = C(U,V)X - \frac{2\psi}{(n-2)}[g(V,X)U - g(U,X)V]. \quad (3.18)$$

Putting $U = \xi$ in equation (3.18) and using equations (3.2) and (2.4), we obtain

$$C^*(\xi,V)X = R(\xi,V)X - \frac{1}{2(n-1)}[S(V,X)\xi - S(\xi,X)V + Q(\xi)g(V,X) - Q(V)g(\xi,X)] - \frac{r}{(n-1)(n-2)}[g(V,X)\xi - g(\xi,X)V] - \frac{2\psi}{(n-2)}[g(V,X)\xi - \eta(X)V]. \quad (3.19)$$

Now using equations (2.8) and (2.11) in equation (3.19), we get

$$C^*(\xi,V)X =[\frac{-n+1+r-2\psi(n-1)}{(n-1)(n-2)}]g(V,X)\xi + [\frac{-n+1+r-2\psi(n-1)}{(n-1)(n-2)}]\eta(X)V - \frac{1}{(n-2)}S(V,X)\xi + \frac{1}{(n-2)}Q(V)\eta(X). \tag{3.20}$$

Again putting $X=\xi$ in equation (3.18) and using equation (3.2),
we obtain

$$C^*(U,V)\xi =R(U,V)\xi - \frac{1}{n-2}[S(V,\xi)U - S(U,\xi)V + Q(U)\eta(V) - Q(V)\eta(U)] - \frac{r-2\psi(n-1)}{(n-1)(n-2)}[\eta(V)U-\eta(U)V], \tag{3.21}$$

which on using equations (2.10) and (2.11), gives

$$C^*(U,V)\xi =\frac{-n+r+1-2\psi(n-1)}{(n-1)(n-2)}[\eta(V)U-\eta(U)V]- \frac{1}{(n-2)}[Q(U)\eta(V) - Q(V)\eta(U)]. \tag{3.22}$$

Taking inner product of equation (3.18) with respect to ξ, we have

$$\eta(C^*(U,V)Y) = \eta(C(U,V)Y) - \frac{2\psi}{(n-2)}[g(V,Y)\eta(U) - g(U,X)\eta(V)]. \tag{3.23}$$

Using equations (3.3) and (2.10) in equation (3.23), we have

$$\eta(C^*(U,V)Y) = [\frac{(n-1)(n-2)+r-2\psi(n-1)}{(n-1)(n-2)}][g(V,X)\eta(U) - g(U,X)\eta(V)] - \frac{1}{(n-2)}[S(V,X)\eta(U) - S(U,X)\eta(V) + g(V,X)\eta(QU) - g(U,X)\eta(QV)]. \tag{3.24}$$

Thus we can state as follows

Theorem 3.4. *A generalized conformal curvature tensor of LP-Sasakian manifold*

satisfies the following-
(i)

$$C^*(\xi,V)X = [\frac{-n+1+r-2\psi(n-1)}{(n-1)(n-2)}]g(V,X)\xi + [\frac{-n+1+r-2\psi(n-1)}{(n-1)(n-2)}]\eta(X)V - \frac{1}{(n-2)}S(V,X)\xi + \frac{1}{(n-2)}Q(V)\eta(X),$$

(ii)

$$C^*(U,V)\xi = \frac{-n+r+1-2\psi(n-1)}{(n-1)(n-2)}[\eta(V)U - \eta(U)V] - \frac{1}{(n-2)}[Q(U)\eta(V) - Q(V)\eta(U)],$$

(iii)

$$\eta(C^*(U,V)Y) = [\frac{(n-1)(n-2)+r-2\psi(n-1)}{(n-1)(n-2)}][g(V,X)\eta(U) - g(U,X)\eta(V)] - \frac{1}{(n-2)}[S(V,X)\eta(U) - S(U,X)\eta(V) + g(V,X)\eta(QU) - g(U,X)\eta(QV)].$$

4. Generalized Conformal Curvature Tensor of LP-Sasakian Manifold satisfying $R(\xi,X).C^*=0$

In this section, we define a generalized conformally semi-symmetric LP-Sasakian manifold and proved some important results on it.

Definition 4.1. *A Riemannian manifold is said to be semi-symmetric if it satisfies the condition [15]*

$$R(X,Y).R = 0, \tag{4.1}$$

where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of manifold.

Analogous to definition (4.1), we define

Definition 4.2. *An LP-Sasakian manifold is said to be generalized conformally semi-symmetric if it satisfies the condition*

$$R(X,Y).C^* = 0, \tag{4.2}$$

where C^* is the generalized conformal curvature tensor of LP-Sasakian manifold and $R(X,Y)$ is considered as the derivation condition at each point of LP-Sasakian manifold.

Consider $(R(\xi, X).C^*)(U, V)Y = 0$ for any vector fields X, U, V, Y, where C^* is the generalized conformal curvature tensor of LP-Sasakian manifold, then we have

$$\begin{aligned} R(\xi, X).C^*(U, V)Y - C^*(R(\xi, X)U, V)Y \\ - C^*(U, R(\xi, X)V)Y - C^*(U, V)R(\xi, X)Y = 0. \end{aligned} \tag{4.3}$$

In view of equation (2.8), above equation gives

$$\begin{aligned} &\eta(C^*(U, V)Y)X - C^*(U, V, Y, X)\xi - \eta(U)C^*(X, V)Y \\ &+ g(X, U)C^*(\xi, V)Y - \eta(V)C^*(U, X)Y + g(X, V)C^*(U, \xi)Y \\ &- \eta(Y)C^*(U, V)X + g(X, Y)C^*(U, V)\xi = 0. \end{aligned} \tag{4.4}$$

Taking inner product of above equation with ξ, we get

$$\begin{aligned} &\eta(C^*(U, V)Y)\eta(X) - C^*(U, V, Y, X)\eta(\xi) - \eta(U)\eta(C^*(X, V)Y) \\ &+g(X, U)\eta(C^*(\xi, V)Y) - \eta(V)\eta(C^*(U, X)Y) + g(X, V)\eta(C^*(U, \xi)Y) \\ &- \eta(Y)\eta(C^*(U, V)X) + g(X, Y)\eta(C^*(U, V)\xi) = 0. \end{aligned} \tag{4.5}$$

Using equations (2.1), (3.7) and (3.24) in equation (4.5), we have

$$\begin{aligned} R(U, V, Y, X) = &\frac{1}{n-2}g(V, Y)g(X, U) - g(U, Y)g(X, V) \\ &+ \frac{1}{(n-2)}[S(U, X)g(V, Y) - S(V, X)g(U, Y)] \\ &- \frac{n-1}{n-2}[g(X, U)\eta(V) - g(X, V)\eta(U)]\eta(Y) \\ &+ \frac{1}{(n-2)}[S(U, X)\eta(V) - S(V, X)\eta(U)]\eta(Y). \end{aligned} \tag{4.6}$$

Let $\{e_i : \ i = 1, 2, 3, ..., n\}$ be orthonormal basis vectors. Putting $U = X = e_i$ in above equation and taking summation over $i, 1 \leq i \leq n$, we get

$$S(V, Y) = \frac{r-n+1}{n-1}g(V, Y) + \frac{-n(n-1)+r}{n-1}\eta(V)\eta(Y), \tag{4.7}$$

which is of the form

$$S(V, Y) = ag(V, Y) + b\eta(V)\eta(Y), \tag{4.8}$$

where $a = \frac{r-n+1}{n-1}$ and $b = \frac{-n(n-1)+r}{n-1}$.
Thus we can state as follow

Theorem 4.1. *A generalized conformally symmetric LP-Sasakian manifold is an η-Einstine manifold.*

5. Generalized Conformally Ricci Semi-symmetric LP-Sasakian Manifold

In this section, we define a generalized conformally Ricci semi-symmetric LP-Sasakian manifold and proved that a generalized conformally Ricci semi-symmetric LP-Sasakian manifold is either an Einstein manifold or the scalar function $\psi = -(n-1)$.

Definition 5.1. *A Riemannian manifold M^n is said to be Ricci semi-symmetric [2] if the condition*

$$R(X,Y).S = 0, \tag{5.1}$$

hold for all vector field X, Y.
Analogous to definition (5.1), we define

Definition 5.2. *An LP-Sasakian manifold M^n is said to be a generalized conformally Ricci semi-symmetric if the condition*

$$C^*(X,Y).S = 0, \tag{5.2}$$

hold for all vector fields X, Y, where C^* is the generalized conformal curvature tensor of LP-Sasakian manifold.

Consider LP-Sasakian manifold is generalized conformally Ricci semi-symmetric, *i.e.*

$$(C^*(\xi,X).S)(U,V) = 0,$$

which gives

$$S(C^*(\xi,X)U,V) + S(U,C^*(\xi,X)V) = 0. \tag{5.3}$$

Using equation (3.20) in above equation, we have

$$\begin{aligned} \alpha(n-1)[g(X,U)\eta(V) + g(X,V)\eta(U)] \\ - \alpha[S(X,U)\eta(V) + S(X,V)\eta(U)] \\ - \beta(n-1)[S(X,U)\eta(V) + S(X,V)\eta(U)] \\ + \beta[S^2(V,X)\eta(U) + S(U,QX)] = 0, \end{aligned} \tag{5.4}$$

where $\alpha = [\frac{-1}{(n-2)} + \frac{r}{(n-1)(n-2)} - \frac{2\psi}{(n-2)}]$ and $\beta = \frac{1}{n-2}$.
Putting $U = \xi$ in equation (5.4) and using equations (2.1), (2.4) and (2.11), we obtain

$$\begin{aligned} S^2(X,V) = -(n-1)^2\eta(X)\eta(V) - \{r-(n-1)(2\psi+1)\}g(X,V) \\ + \frac{\{r-(n-1)(2\psi-n+2)\}}{n-1}S(X,V) + (n-1)^2\eta(X). \end{aligned} \tag{5.5}$$

Thus we can state as follows-

Theorem 5.1. *On a generalized conformally Ricci semi-symmetric [4] LP-Sasakian manifold*
$S^2(X,V) = -(n-1)^2\eta(X)\eta(V) - \{r-(n-1)(2\psi+1)\}g(X,V) + \frac{\{r-(n-1)(2\psi-n+2)\}}{n-1}$
$S(X,V) + (n-1)^2\eta(X)$ *holds.*

6. Generalized Conformally ϕ-Symmetric LP-Sasakian Manifold

In this section, we define a generalized conformally locally ϕ-symmetric LP-Sasakian manifold and a generalized conformally ϕ-symmetric LP-Sasakian manifold and obtained an intresting result on it.

Definition 6.1. *A Riemannian manifold M^n is said to be locally ϕ-symmetric [18] if*

$$\phi^2((\nabla_W R)(X,Y)U) = 0, \tag{6.1}$$

for all vector fields X, Y, U, W orthogonal to ξ.

Definition 6.2. *A Riemannian manifold is said to be ϕ-symmetric [18] if*

$$\phi^2((\nabla_W R)(X,Y)U) = 0, \tag{6.2}$$

for arbitrary vector fields X, Y, U, W.
These notions were introduced by Takahashi for Sasakian manifold [18].
Analogous to these definitions, we define

Definition 6.3. *An LP-Sasakian manifold is said to be generalized conformally locally ϕ-symmetric if*

$$\phi^2(\nabla_W C^*(X,Y)U) = 0, \tag{6.3}$$

for all vector fields X, Y, U, W orthogonal to ξ.

Definition 6.4. *An LP-Sasakian manifold is said to be generalized conformally ϕ-symmetric if*

$$\phi^2((\nabla_W C^*)(X,Y)U) = 0, \tag{6.4}$$

for arbitrary vector fields X, Y, U, W.

From equations (3.3) and (3.18), we have

$$\begin{aligned}(\nabla_W C^*)(X,Y)U =& (\nabla_W R)(X,Y)U \\ &- \frac{1}{n-1}[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y] \\ &- \frac{2dr(W)}{(n-2)}[g(Y,U)X - g(X,U)Y.\end{aligned} \tag{6.5}$$

Assume that the manifold is a generalized conformally ϕ-symmetric, then from equations (6.4) and (6.5), we get

$$\begin{aligned}(\nabla_W R)(X,Y)U - \frac{1}{n-2}[(\nabla_W S)(Y,U)X - (\nabla_W S)(X,U)Y + (\nabla_W Q)(X)g(Y,U)\\ - (\nabla_W Q)(Y)g(X,U)] - \frac{dr(W)}{(n-1)}[g(Y,U)X - g(X,U)Y]\\ = -\eta((\nabla_W R)(X,Y)U + \frac{1}{n-2}[(\nabla_W S)(Y,U)\eta(X) - (\nabla_W S)(X,U)\eta(Y)\\ + (\nabla_W S)(X,\xi)g(Y,U) - (\nabla_W S)(Y,\xi)g(X,U)]\xi\\ + \frac{dr(W)}{(n-1)}[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\xi.\end{aligned} \tag{6.6}$$

Taking inner product of above equation with V, we obtain

$$\begin{aligned}g((\nabla_W R)(X,Y)U,V) - \frac{1}{(n-2)}[(\nabla_W S)(Y,U)g(X,V) - (\nabla_W S)(X,U)g(Y,V)\\ + (\nabla_W S)(X,V)g(Y,U) - (\nabla_W S)(Y,V)g(X,U)]\\ - \frac{2dr(W)}{(n-2)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)]\\ = -\eta((\nabla_W R)(X,Y)U)\eta(V)\\ + \frac{1}{n-2}[(\nabla_W S)(Y,U)\eta(X) - (\nabla_W S)(X,U)\eta(Y)\\ + (\nabla_W S)(X,\xi)g(Y,U) - (\nabla_W S)(Y,\xi)g(X,U)]\eta(V)\\ + \frac{2dr(W)}{n-2}[g(Y,U)\eta(X) - g(X,U)\eta(Y)]\eta(V).\end{aligned} \tag{6.7}$$

Putting $X = V = e_i$ in equation (6.7) and taking summation over i, we have

$$\begin{aligned}-dr(W)g(Y,U) - \frac{1}{2}(\nabla_W S)(Y,U) - \frac{1}{(2n)}[dr(W)g(Y,U) - (\nabla_W S)(Y,U)]\\ = -\eta((\nabla_W R)(e_i,Y)U)\eta(e_i) + \frac{1}{2(n-1)}[(\nabla_W S)(Y,U)\\ - (\nabla_W S)(e_i,U)\eta(Y)\eta(e_i) + (\nabla_W S)(e_i,\xi)g(Y,U)\eta(e_i)\\ - (\nabla_W S)(Y,\xi)\eta(U)] + \frac{dr(W)}{(n-1)}[g(Y,U) - \eta(Y)]\eta(U)].\end{aligned} \tag{6.8}$$

Putting $U = \xi$ in above equation and using equations (2.1) and (2.4), we get

$$-\eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) - \frac{n+2}{(n-1)}dr(W)\eta(Y) + \frac{n-2}{2(n-1)}(\nabla_W S)(Y,\xi)\eta(Y) = 0. \tag{6.9}$$

Now,

$$\eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi) \tag{6.10}$$

and

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) =& g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ & - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \end{aligned} \tag{6.11}$$

Since e_i is an orthonormal basis, so $\nabla_W e_i = 0$ and using equation (6.10), we get

$$(\nabla_W S)(Y, \xi) = -(n+1)dr(W)\eta(Y). \tag{6.12}$$

Putting $Y = \xi$ in above equation, we get $dr(W) = 0$, which shows that r is constant. From equation (6.12), We have

$$(\nabla_W S)(Y, \xi) = 0.$$

This shows that

$$S(Y, W) = ag(Y, W) + b\eta(Y)\eta(W), \tag{6.13}$$

where $a = -1$ and $b = -n$, which shows that M^n is an η-Einstein Manifold. Thus we can state as follows

Theorem 6.1. *A generalized conformally ϕ-symmetric LP-Sasakian Manifold is an η-Einstein Manifold.*

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