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Generalized connection graph method for synchronization in asymmetrical networks

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Abstract

We present a general framework for studying global complete synchronization in networks of dynamical systems with asymmetrical connections. We extend the connection graph stability method, originally developed for symmetrically coupled networks, to the general asymmetrical case. The principal new component of the method is the transformation of the directed connection graph into an undirected graph. In our method for symmetrically coupled networks we have to choose a path between each pair of nodes. The extension of the method to asymmetrical coupling consists in symmetrizing the graph and associating a weight to each path. This weight involves the "node unbalance" of the two nodes. This quantity is defined to be the difference between the sum of connection coefficients of the outgoing edges and the sum of the connection coefficients of the incoming edges to the node. The synchronization condition for this symmetrized-and-weighted network then also guarantees synchronization in the original asymmetrical network.

Keywords: Synchronization; Connection graph; Stability; Path length

1. Introduction

The increasing interest in synchronization in limit-cycle and chaotic dynamical systems [1–3] has led many researchers to consider the phenomenon of synchronization in large complex networks of coupled oscillators (see, e.g. [4,5] for a sampling of this large field).

Much of this research has been inspired by technological and biological examples, including coupled synchronized lasers [6,7], networks of computer clocks [8], and synchronized neuronal firing [9,10]. Networks of identical or slightly non-identical oscillators often synchronize, or else form synchronous patterns that depend on the symmetry of the underlying network [11,12].

The strongest form of synchrony in oscillator networks is complete synchronization when all oscillators do the same thing at the same time. An important problem in the study of complete synchrony is how the stability of a synchronized behavior, where the behavior could be a fixed point, a limit cycle or a chaotic attractor, is influenced by the network topology and kind of interaction. This problem was intensively studied for networks of biological oscillators [13–16], and more generally of limit-cycle and chaotic oscillators [17–19,21–44].

Most methods for determining stability of synchronization in linearly coupled networks of chaotic systems are based on the calculation of the eigenvalues of the connection matrix and a term depending mainly on the dynamics of the individual oscillators (see, e.g., [17–27]). Pecora and Carroll [22] developed a general approach to the local stability of complete synchronization for any linear coupling network architecture. This approach, called the Master Stability function, is based on the calculation of the maximum Lyapunov exponent for the least stable transversal mode of the synchronous manifold and the eigenvalues of the connection matrix. This powerful method is widely used in local stability studies of synchronization

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in complex oscillator networks [28–33]. Global stability results based on the calculation of the connection matrix eigenvalues were also derived for oscillator networks coupled via undirected [34,35] and directed graphs [36]. These studies show that both local and global stabilities of complete synchronization depend on the eigenvalues of the Laplacian connection matrix.

We have previously developed an alternate way to establish synchrony which does not depend on explicit knowledge of the spectrum of the connection matrix [40]. This connection graph method combines the Lyapunov function approach with graph theoretical reasoning. It guarantees complete synchronization from arbitrary initial conditions and not just local stability of the synchronization manifold. It is also applicable to time-dependent networks. This approach was originally developed for undirected graphs and applied to global synchronization in complex networks [41,42]. More recently, we showed that the method can be directly applied to asymmetrically coupled networks with node balance [43,44]. Node balance means that the sum of the coupling coefficients of all edges directed to a node equals the sum of the coupling coefficients of all the edges directed outward from the node. We proved that for node balanced networks it is sufficient to symmetrize all connections by replacing a unidirectional coupling with a bidirectional coupling of half the coupling strength. The bound for global synchronization in this undirected network then holds also for the original directed network.

In this paper we extend our approach to networks with arbitrary asymmetrical connections. The connection graph of such a network is directed and the coupling coefficient from node i to node j is in general different from the coupling coefficient for the reverse direction. The new ingredient of the method is the transformation of the directed connection graph into an undirected weighted graph. This is done by symmetrizing the graph and associating a weight to each edge of the undirected graph and to each path between any two nodes. This weight involves the "node unbalance" of the two nodes. This quantity is defined to be the difference between the sum of connection coefficients of the outgoing edges and the sum of the connection coefficients of the incoming edges to the node. As in the case of node-balanced networks, the synchronization criterion derived for this symmetrical network then guarantees synchronization in the asymmetrical directed network.

The layout of this paper is as follows. First, in Section 2, we state the problem under consideration. Then, in Section 3, we derive a graph-based criterion for global synchronization in asymmetrically coupled networks and formulate the main theorem of our method. In Section 4, we show how to apply the generalized connection graph method to several examples of concrete networks. We start with the simplest network of two unidirectionally coupled oscillators, then we continue with a star-configuration and a directed network with an irregular topology. In Section 5, a brief discussion of the obtained results is given.

2. Problem statement

2.1. Systems under study

We consider a network of n interacting nonlinear l-dimensional dynamical systems (oscillators). We assume that the individual oscillators are all identical, even though our results can be generalized to slightly non-identical systems. The composed dynamical system is described by the $n \times l$ ordinary differential equations

$$\dot{x}_i = F(x_i) + \sum_{k=1}^{n} d_{ik}(t) P x_k, \quad i = 1, \ldots, n, \tag{1}$$

where $x_i = (x_i^1, \ldots, x_i^l)$ is the l-vector containing the coordinates of the ith oscillator, the function $F : R^l \to R^l$ is nonlinear and capable of exhibiting periodic or chaotic solutions, and P is a projection operator that selects the components of x_i that are involved in the interaction between the individual oscillators. Without loss of generality, we consider a vector version of the coupling with the diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_l)$, where $p_\nu = 1, \nu = 1, 2, \ldots, s$ and $p_\nu = 0$ for $\nu = s+1, \ldots, l$. Note that all the results that are obtained in this paper are directly applicable to other possible cases where the projection matrix P is non-diagonal (see [44] for the proof of synchronization among oscillators with non-diagonal coupling).

The connection matrix D with entries d_{ik} is an $n \times n$ matrix with zero row-sums and nonnegative off-diagonal elements such that

$$\sum_{k=1}^{n} d_{ik} = 0 \quad \text{and} \quad d_{ii} = -\sum_{k=1; k \neq i}^{n} d_{ik}, \quad i = 1, \ldots, n.$$

This ensures that the coupling is of diffusive nature (on an arbitrary coupling graph) and any solution $x(t)$ for a single oscillator is also a solution of the coupled system (1). The connection matrix D is assumed to be *asymmetrical* without any further constraints. This is in contrast to our previous papers, where we required the symmetry of the connection matrix [40] or the zero column-sums property of an asymmetrical connection matrix [43]. The coupling matrix D is associated with the edge-weighted directed *connection graph* **D**, where to each individual system corresponds a node and for each pair of nodes i, j with $i \neq j$ and such that $d_{ij} > 0$, there is an edge directed from from j to i. The weight assigned to this edge is d_{ij}. The connection graph is assumed to be connected.

We admit an arbitrary time dependence in the coupling matrix even if t is not explicitly stated everywhere. All constraints and criteria for the coupling matrix are understood to hold for all times t.

2.2. Type of synchronization considered

In this paper, we concentrate on the strongest form of synchrony, namely, *global complete synchronization*. Complete synchronization is defined by the invariant hyperplane $M = \{x_1(t) = x_2(t) = \cdots = x_n(t)\}$. The manifold M has the dimension of a single oscillator, and is often called the synchronization manifold. Completely synchronous solutions of all types (multi-stable, periodic, and chaotic solutions) are constrained to this manifold.

Definition 2.1. Network (1) synchronizes completely, if

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{for } \forall i, j. \tag{2}$$

The system (1) exhibits global complete synchronization, if the condition (2) holds for any solution. Local complete synchronization arises in the system (1), if any solution of the system (1) that starts sufficiently close to the synchronization manifold synchronizes completely.

A central goal of our study is the ability to predict when the completely synchronous state is globally stable. In particular, we want to derive upper bounds for global complete synchronization in the network (1). In what follows we present the stability analysis of synchronization in oscillator networks with an arbitrary directed connection graph, under the constraint that the graph allows synchronization of all the nodes. Indeed, synchrony in directly coupled networks is only possible if there is at least one node which directly or indirectly influences all the others [36]. In terms of the connection graph, this amounts to the existence of a uniformly directed tree involving all the vertices. A star-coupled network where secondary nodes drive the hub is a counter example, where such a tree does not exist and synchronization is impossible.

2.3. Hypothesis made

One can formally divide oscillator networks (1) into two classes of coupled systems with different synchronization behavior. Networks from the first class *globally* synchronize when the coupling is made sufficiently large and remain synchronized up to an infinitely large value of coupling. Of course, global synchronization in such networks is always preceded by local synchronization, typically arising at a lower coupling strength. There may be multiple local synchronization thresholds, each associated with an unstable periodic orbit [23].

This large class of networks contains a majority of known limit-cycle and chaotic systems. Examples include coupled Lorenz systems [40], Chua circuits [20], Hindmarsh–Rose and Hodgkin–Huxley-type neuron models [42], coupled driven chaotic pendula [44], Duffing oscillators, etc.

The second, narrower class corresponds to the arrays in which increasing the coupling between systems destabilizes the locally stable synchronous state. This phenomenon is characterized by the absence of global synchronization and is called a short-wavelength bifurcation [18]. The standard examples used in applications of the master stability function are x-coupled Rössler oscillators and electronic circuits modeled after the Rössler equations, where the quadratic nonlinearity is replaced by a piecewise linear function [24]. These unexpected desynchronization transitions in x-coupled Rössler systems can be explained by a singularity of the function defining the coordinates of the coupled system's equilibria. This leads to the appearance of equilibria, outside the synchronization manifold, that are associated with desynchronous states [45].

Evidently, the connection graph method to prove global synchronization, that we extend in this paper, is only applicable to the first class of networks admitting global complete synchronization. It is difficult to determine a priori what class the network (1) with a chosen type of oscillator belongs to. However, if we prove global complete synchronization in the simplest network (1) of two oscillators of the given type, then we will be able to conclude that a larger network (1) containing that oscillator is also globally synchronizable and belongs to the first class of networks. Therefore, to solve the problem of global synchronization in a network of oscillators, given their individual dynamics and coupling structure, one should always start from the question of whether or not two oscillators of the given type are capable of global synchronization. This amounts to imposing the following constraint;

Hypothesis 1 (*Sufficient Condition*). Global synchronization in the network (1) of two unidirectionally coupled oscillators with coupling strength d_{12} is globally stable, provided that d_{12} exceeds the threshold a.

We will reformulate this hypothesis in a more technical form in Section 3.

The existence of the threshold a is the principal requirement of our method such that Hypothesis 1 has to be proven for each particular situation (for the concrete individual node's dynamics and the projection matrix P). The proof involves the construction of a Lyapunov function along with the assumption of the eventual dissipativeness of the coupled system (1) [40]. For the chaotic systems from the first class, listed above [20, 40,42,44], the critical value of coupling a can be expressed explicitly through the parameters of the individual oscillator.

3. Graph-based criterion for network synchronization

3.1. Stability system for the difference variables

Since we are interested in complete synchronization, we introduce the difference variables $X_{ij} = x_j - x_i$ for any i and j. Similarly to our previous works [40,43], we can write the stability system for the difference variables

$$\dot{X}_{ij} = F(x_j) - F(x_i) + \sum_{k=1}^{n} \{d_{jk} P X_{jk} - d_{ik} P X_{ik}\}, \quad i, j = 1, \ldots, n. \tag{3}$$

The function difference $F(x_j) - F(x_i)$ can be rewritten in a compact vector form

$$F(x_j) - F(x_i) = \left[\int_0^1 DF(\beta x_j + (1-\beta)x_i)\mathrm{d}\beta\right] X_{ij},$$

where DF is an $l \times l$ Jacobi matrix of F. Hence, we obtain

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1-\beta)x_i)\mathrm{d}\beta\right] X_{ij} + \sum_{k=1}^{n} \{d_{jk} P X_{jk} - d_{ik} P X_{ik}\}, \quad i, j = 1, \ldots, n. \tag{4}$$

We will prove global complete synchronization in the system (1) by showing that the equilibrium $O = \{X_{ij} = 0, i, j = 1, \dots, n\}$ can be made globally asymptotically stable by increasing coupling. The first term in Eq. (4) is unstable and defines the divergence of trajectories within the individual dynamical systems. The second term $\sum_{k=1}^{n}\{d_{jk}PX_{jk} - d_{ik}PX_{ik}\}$ representing the coupling structure favors the stability and can overcome the unstable term, provided that the coupling is strong enough.

We will derive the stability conditions in two steps.

I. We first study global stability of synchronization in the simplest network (1) composed of two unidirectionally coupled systems (cf. Hypothesis 1). In this case, the stability system for the difference variables (4) is reduced to the system

$$\dot{X}_{12} = \left[\int_0^1 DF(\beta x_2 + (1-\beta)x_1)\mathrm{d}\beta\right] X_{12} - d_{12}PX_{12}. \quad (5)$$

We need to show that the origin of the difference equation system (5) is globally asymptotically stable. This can be done by applying the Lyapunov function method. We choose the Lyapunov function of the form

$$W_{12} = \frac{1}{2}\, X_{12}^T \cdot H \cdot X_{12}, \quad (6)$$

where $H = \mathrm{diag}(h_1, h_2, \dots, h_s, H_1), h_1 = 1, \dots, h_s = 1$, and the $(l-s)\times(l-s)$ matrix H_1 is positive definite.

To ensure the global stability of the origin, the derivative $\dot{W}_{12}$ with respect to the system (5)

$$\dot{W}_{12} = X_{12}^T H\left[\int_0^1 DF(\beta x_2 + (1-\beta)x_1)\mathrm{d}\beta - d_{12}P\right] X_{12}, \quad X_{ij} \neq 0 \quad (7)$$

has to be negative. This is true under the condition of Hypothesis 1.

II. To study global stability of synchronization in the network (1) with arbitrary network topologies, we construct the Lyapunov function for the system of the difference variables (4)

$$V = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}^T \cdot H \cdot X_{ij}, \quad (8)$$

where H is the matrix in (6).

The corresponding time derivative has the form

$$\dot{V} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{W}_{ij} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}^T aPX_{ij} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\{d_{jk}X_{ji}^T HPX_{jk} + d_{ik}X_{ik}^T HPX_{ij}\}, \quad (9)$$

where $W_{ij} = \frac{1}{2}\, X_{ij}^T \cdot H \cdot X_{ij}$ and a is the synchronization threshold in the two-oscillator network with the stability system (5).

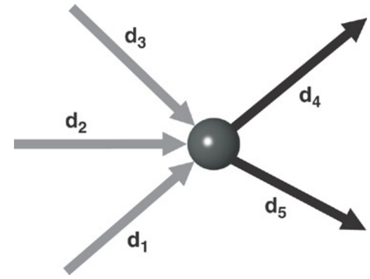

Fig. 1. "Node unbalance": The graph representation of the ith-column sum of the connection matrix D. Here, the node unbalance $D_i^c = (d_5 + d_4) - (d_3 + d_2 + d_1)$.

After some algebraic manipulations (see [43] for the details of this passage), we obtain the following inequality

$$\dot{V} \leq -\sum_{\nu=1}^{s}\sum_{j=1, i>i}^{n} h\left(n\frac{d_{ij}+d_{ji}}{2} - a\right) X_{ij}^{\nu\,2} + \sum_{\nu=1}^{s}\sum_{j=1, i>i}^{n} h\frac{D_i^c + D_j^c}{2} X_{ij}^{\nu\,2}, \quad (10)$$

where $D_i^c = \sum_{k=1}^{n} d_{ki}$ and $D_j^c = \sum_{k=1}^{n} d_{kj}$ are the ith and jth column sums of the connection matrix D, respectively.

To facilitate cross-paper reading, it is worth noticing that ε_{ij} and μ_{ij} in [43] stand for $\frac{d_{ij}+d_{ji}}{2}$ and $\frac{D_i^c+D_j^c}{2}$, respectively.

In terms of graphs, the column sum $D_i^c = \sum_{k=1}^{n} d_{ki} = \sum_{k\neq i} d_{ki} + d_{ii} = \sum_{k\neq i} d_{ki} - \sum_{k\neq i} d_{ik}$ amounts to the difference between the sum of the coupling coefficients of all edges directed outward from node i and the sum of the coupling coefficients of all the edges directed to node i (Fig. 1). We call this quantity the "node unbalance".

In order to establish synchronization, we have to prove that the right hand side (RHS) of the inequality (10) is a negative quadratic form. This is equivalent to the following inequality between quadratic forms

$$\sum_{\nu=1}^{s}\sum_{j=1, i>i}^{n}\left(\frac{d_{ij}+d_{ji}}{2}\right) X_{ij}^{\nu\,2} > \frac{a}{n}\sum_{\nu=1}^{s}\sum_{j=1, i>i}^{n} \times\left(1 + \frac{D_i^c + D_j^c}{2a}\right) X_{ij}^{\nu\,2}. \quad (11)$$

Here, the coupling coefficients $\frac{d_{ij}+d_{ji}}{2}$ define the edges on the symmetrized connection graph obtained by replacing an edge directed from node i to node j and another edge in the reverse direction by an undirected edge with this mean coupling coefficient. Consequently, the difference variables X_{ij} on the left hand side (LHS) of the inequality (11) correspond to pairs of nodes directly connected by an edge on the symmetrized graph. At the same time, the right hand side (RHS) contains the difference variables between any pair of nodes. The pivotal ingredient of the connection graph method for symmetrically coupled networks [40] is to express all difference variables $X_{ij}, i, j = 1, \dots, n$ on the RHS through the connection graph variables X_{ij} on the LHS of the inequality (11). This is done by establishing a bound on the total length of all chosen paths passing through an edge on the connection graph. In contrast to the symmetrical case, these path lengths will be weighted due to the presence of nonunit factors $\left(1+\frac{D_i^c+D_j^c}{2a}\right)$ on the RHS of the inequality (11).

The terms $\frac{a}{n}\frac{D_i^c+D_j^c}{2a}$ on the RHS of the inequality (11) are associated with the sum node unbalance of nodes i and j. If the term $\frac{a}{n}\frac{D_i^c+D_j^c}{2a}$ is negative for a given i and j, then it is favorable for lowering the inequality (11).

Therefore, we could have assigned all negative terms $\frac{a}{n}\frac{D_i^c+D_j^c}{2a}$ to the LHS (we discuss this possibility in Remark 2 after formulating the main theorem). However, it turns out that it is more advantageous, in general, to incorporate the term $\frac{a}{n}\frac{D_i^c+D_j^c}{2a}$ into the quadratic form on the LHS only if it is negative and if i and j are linked directly by an edge k of the symmetrized graph. We then denote $-\frac{1}{n}\frac{D_i^c+D_j^c}{2}$ by D_k. That is, we preserve the structure of the symmetrized graph, but make the graph weighted: some its edges have stronger coupling $\frac{d_{ij}+d_{ji}}{2}+D_k$.

If the term $\frac{1}{n}\frac{D_i^c+D_j^c}{2}$ is negative, but there is no edge linking node i and node j we leave it on the RHS of the inequality (11) along the terms $\frac{1}{n}\frac{D_i^c+D_j^c}{2}$ that are positive. Note that $1+\frac{D_i^c+D_j^c}{2a}$ may become negative. In this case we simply set it to 0 so that the RHS remains a positive quadratic form.

The redistribution of the terms $\frac{D_i^c+D_j^c}{2}X_{ij}^{\nu\,2}$ amounts to the following:

- To each edge of the symmetrized connection graph, we associate the quantity D_k defined by

$$D_k=\begin{cases}\left|\dfrac{D_i^c+D_j^c}{2n}\right|, & \text{if } D_i^c+D_j^c<0;\\ \quad\text{and } k \text{ links } i \text{ and } j & \\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

- For any pair of nodes (i,j), we choose a path P_{ij} and associate to each path P_{ij} its "length" $L(P_{ij})$ defined by

$$L(P_{ij})=\begin{cases}|P_{ij}|, & \text{if } D_i^c+D_j^c<0;\\ \quad\text{and there is a link } k \text{ between } i \text{ and } j & \\ |P_{ij}|\,\chi\left(1+\dfrac{D_{ij}}{a}\right), & \text{otherwise}\end{cases} \tag{13}$$

where $D_{ij}=\frac{D_i^c+D_j^c}{2}$ and $|P_{ij}|$ is the number of edges in P_{ij}. The function

$$\chi(x)=\begin{cases}x, & \text{if } x\ge 0\\ 0, & \text{if } x<0.\end{cases}$$

Thus, the inequality (11) becomes

$$\sum_{\nu=1}^{s}\sum_{j=1,i>i}^{n}(d_k+D_k)\,X_{ij}^{\nu\,2}>\frac{a}{n}\sum_{\nu=1}^{s}\sum_{j=1,i>i}^{n}\left(1+\frac{D_{ij}}{a}\right)X_{ij}^{\nu\,2}, \tag{14}$$

where $d_k=\frac{d_{ij}+d_{ji}}{2}$.

Exactly as in [40], we now replace the connection graph variables $X_{ij}^{\nu\,2}$, $j=1, i>i$ on the LHS of the inequality (14) by $Y_k^{\nu\,2}$, $k=1,\ldots,m$, where m is the number of edges of the symmetrized-and-weighted graph and apply the Cauchy–Schwarz inequality. This leads to the main theorem of this paper.

3.2. Connection graph method for arbitrary asymmetrical coupling

Theorem 1 (*Sufficient Conditions*). *Under Hypothesis 1, global complete synchronization is achieved in the network (1) with an arbitrary coupling graph* $\mathbf{D}$ *if for all k*

$$d_k+D_k>\frac{a}{n}b_k,\quad \text{where } b_k=\sum_{j>i;\ k\in P_{ij}}^{n}L(P_{ij}) \tag{15}$$

is the sum of the "lengths" $L(P_{ij})$ of all chosen paths P_{ij} which pass through a given edge k that belongs to the symmetrized undirected graph. This weighted path length $L(P_{ij})$ is defined in Eq. (13) as follows

$$L(P_{ij})=\begin{cases}|P_{ij}|, & \text{if } D_i^c+D_j^c<0;\\ \quad\text{and there is a link } k \text{ between } i \text{ and } j & \\ |P_{ij}|\,\chi(1+\dfrac{D_i^c+D_j^c}{2a})=|P_{ij}|\,\chi & \\ \quad\times\left(1+\dfrac{D_{ij}}{a}\right); & \text{otherwise,}\end{cases}$$

where the function χ is the identity for positive and 0 for negative arguments.

The mean coupling coefficient $d_k=\frac{d_{ij}+d_{ji}}{2}$ defines an edge k on the undirected symmetrized graph. An extra coupling strength $D_k=|\frac{D_i^c+D_j^c}{2n}|$ is added to the edges of the symmetrized connection graph for which the mean node unbalance $D_i^c+D_j^c$ is negative.

Remark 1. In the case where the directed connection graph is not a uniformly directed tree involving all nodes and complete synchronization of all the nodes is impossible, the condition for synchronization is simply impossible to satisfy.

Remark 2. The assignment of the different terms $\frac{D_i^c+D_j^c}{2}$ to the quadratic forms on the left and on the right sides of the inequality (14) is somewhat arbitrary. Another possibility is to assign all negative terms to the LHS. This implies that for those terms that do not have a direct link between i and j an additional edge with connection coefficient $\frac{D_i^c+D_j^c}{2n}$ has to be added between nodes i and j to the symmetrized connection graph. The final criterion (15) in Theorem 1 has to be evaluated with respect to this *augmented* symmetrized connection graph. This leads to a different set of inequalities that may be advantageous in certain examples.

Theorem 1 directly leads to the following method to establish our sufficient condition for global complete synchronization.

Step 1. Determine the "node unbalance" for each node $D_i^c = \sum_{j=1}^{n} d_{ji}$.

Step 2. Symmetrize the connection graph by replacing the edge directed from node i to node j by an undirected edge with half the coupling coefficient $d_{ij}/2$. In the case where there is an edge directed from node i to node j and another edge in the reverse direction, the pair of directed edges is replaced by an undirected edge with mean coupling coefficient $d_k = \frac{d_{ij}+d_{ji}}{2}$.

Step 3. Choose a path P_{ij} between each pair of nodes. Usually, the shortest path is chosen. Sometimes, however, a different choice of paths can lead to lower bounds [42].

Step 4. For each path P_{ij} determine the mean node unbalance of the endnodes i and j.

Identify paths of length 1, i.e. edges of the symmetrized graph, with negative mean node unbalance $D_i^c + D_j^c$. For these edges, calculate and add extra strength $D_k = |\frac{D_i^c+D_j^c}{2n}|$ to the symmetrized coupling d_k.

For all other paths P_{ij}, namely, paths of length 1 with nonnegative mean node unbalance and any paths composed of at least two edges, calculate the quantities $D_{ij} = \frac{D_i^c+D_j^c}{2}$ and $1 + \frac{D_{ij}}{a}$. Associate weight $1 + \frac{D_{ij}}{a}$ to the path length of P_{ij} if $1 + \frac{D_{ij}}{a} > 0$, and zero weight, otherwise.

Step 5. For each edge k of the symmetrized-and-weighted connection graph determine the inequality

$$d_k + D_k > \frac{a}{n} b_k, \quad \text{where } b_k = \sum_{j>i;\ k \in P_{ij}}^{n} L(P_{ij}).$$

Step 6. Combine the inequalities either to describe the set of common values for all connection coefficients that guarantee global complete synchronization or to describe in general the set of connection coefficient vectors that guarantee synchronization if we allow for coefficients that vary from link to link. Finally, the bound for global synchronization in the symmetrized-and-weighted network holds also for the original asymmetrical network.

Let us show how to apply the general method to three examples of concrete asymmetrical networks.

4. Examples: Application of the method

To find an upper bound for the synchronization threshold in concrete networks, we should follow the steps of the above study.

4.1. Two unidirectionally coupled oscillators

Consider the simplest directed network with $n = 2$ and coupling strength d (Fig. 2(a)).

Step 1. Determine the node unbalance for node 1 and 2 : $D_1^c = -d$ and $D_2^c = d$.

Step 2. Symmetrize the graph as shown in Fig. 2(b): $d_1 = \frac{d+0}{2} = \frac{d}{2}$.

Fig. 2. Simplest directed network and its symmetrized analog. The directed link is replaced by the undirected edge with half the coupling strength. Here, the mean node unbalance, $\frac{D_1^c+D_2^c}{2} = 0$, so that the symmetrize-and-weight operation amounts to symmetrization. The path length P_{12} also remains unweighted.

Step 3. Choose a path between each pair of nodes. Here, the graph has only one branch.

Steps 4. For each path determine the mean node unbalance of the endnodes. Here, this quantity is equal to $0 : D_1^c + D_2^c = d - d = 0$. Therefore, $D_k \equiv D_1 = \frac{0}{2 \cdot 2} = 0$ and $D_{ij} \equiv D_{12} = 0$.

Steps 5–6. For the edge 1 determine the inequality: $\frac{d}{2} + 0 > \frac{a}{2}|P_{12}|$. The path length $|P_{12}| = 1$ such that the final inequality becomes $d > a$.

Recall that by Hypothesis 1, a is an upper bound for synchronization in this network such that our method gives the correct synchronization bound.

4.2. Star-coupling

This is a well-known coupling scheme where the network has a central hub (this node is marked as the first one) and all other nodes are linked to this node. The coupling coefficient d_{out} of the edges directed outward from the hub differs from the coupling coefficient d_{in} of the edges directed to the hub (see Fig. 3).

The criterion of Theorem 1 is applied to this network as follows.

Step 1. Calculate the node unbalance for each node: $D_1^c = (n-1)(d_{\text{out}} - d_{\text{in}})$ and $D_i^c = (d_{\text{in}} - d_{\text{out}}), i = 2, \ldots, n$.

Step 2. Symmetrize the graph: $d_k = \frac{d_{\text{out}}+d_{\text{in}}}{2}, k = 1, \ldots, n-1$.

Step 3. Choose a path between each pair of nodes:

$$P_{1j} : 1 \leftrightarrow j, \quad j = 2, \ldots, n$$
$$P_{ij} : i \leftrightarrow 1 \leftrightarrow j, \quad i, j = 2, \ldots, n.$$

Step 4. For each path P_{ij} determine $\frac{D_i^c+D_j^c}{2}$ for nodes i and j and its place in the inequalities (15)

$$P_{1j} : \frac{D_1^c + D_j^c}{2} = \frac{n-2}{2}(d_{\text{out}} - d_{\text{in}}), \quad j = 2, \ldots, n$$
$$P_{ij} : \frac{D_i^c + D_j^c}{2} = d_{\text{in}} - d_{\text{out}}, \quad i, j = 2, \ldots, n.$$

For $d_{\text{in}} > d_{\text{out}}$, the term associated with P_{1j} is negative and there is an edge linking nodes 1 and j. Therefore, this term transforms into $D_k = \left|\frac{n-2}{2n}(d_{\text{out}} - d_{\text{in}})\right|$. For all other paths P_{ij}, the terms $\frac{D_i^c+D_j^c}{2} = d_{\text{in}} - d_{\text{out}}$ are positive and become $D_{ij} = d_{\text{in}} - d_{\text{out}}$.

For $d_{\text{in}} < d_{\text{out}}$, the term associated with P_{1j} is positive and therefore becomes $D_{1j} = \frac{n-2}{2}(d_{\text{out}} - d_{\text{in}})$. In all other cases,

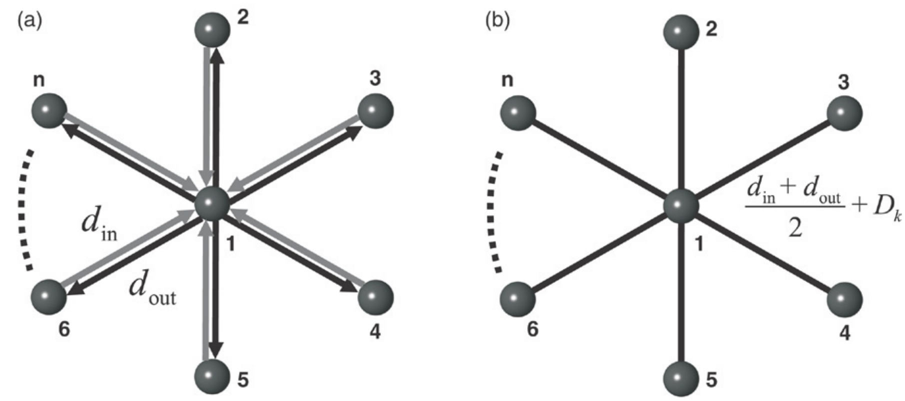

Fig. 3. (a) Star-network with asymmetrical connections: d_{out} and d_{in} are different. (b) Symmetrized-and-weighted analog of (a) with bidirectional connections with strength $\frac{d_{\text{out}}+d_{\text{in}}}{2} + D_k$, $k = 1, \ldots, n-1$. For $d_{\text{in}} > d_{\text{out}}$, $D_k = \frac{n-2}{2}(d_{\text{in}} - d_{\text{out}})$, and for $d_{\text{out}} > d_{\text{in}}$, $D_k = 0$. In general, for both cases, the length of the chosen path P_{ij} is weighted.

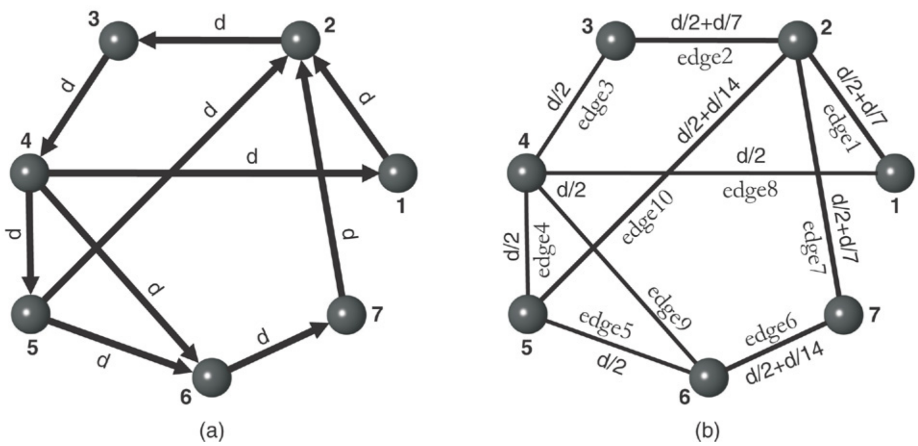

Fig. 4. (a) Unidirectionally coupled network with uniform coupling d. (b) Symmetrized analog of (a) with weighted bidirectional connections. Arrows indicate the direction of coupling along an edge; edges without arrows are coupled bidirectionally. The width of the links may be thought of as the coupling strength.

the terms $\frac{D_i^c + D_j^c}{2} = d_{\text{in}} - d_{\text{out}}$ are negative but there is no direct link between i and j for $i, j = 2, \ldots, n$. Hence, these terms also become $D_{ij} = d_{\text{in}} - d_{\text{out}}$.

Step 5. For each edge of the graph determine the inequality (15). Here, all edges are equivalent.

Case $d_{\text{in}} > d_{\text{out}}$:

$$\frac{d_{\text{out}} + d_{\text{in}}}{2} + \frac{n-2}{2}(d_{\text{in}} - d_{\text{out}}) > \frac{a}{n}\left[1 + 2(n-2)\chi\left(1 + \frac{d_{\text{in}} - d_{\text{out}}}{a}\right)\right].$$

Case $d_{\text{out}} > d_{\text{in}}$:

$$\frac{d_{\text{out}} + d_{\text{in}}}{2} > \frac{a}{n}\left[1 \cdot \chi\left(1 + \frac{n-2}{2a}(d_{\text{out}} - d_{\text{in}})\right) + 2(n-2)\chi\left(1 + \frac{d_{\text{in}} - d_{\text{out}}}{a}\right)\right].$$

Both cases lead to the same sufficient condition for global complete synchronization:

$$d_{\text{out}} > \frac{n-3}{2n-3} d_{\text{in}} + a. \quad (16)$$

Let us now check if the case $d_{\text{out}} = 0$ is compatible with our criterion for synchronization:

$$0 > \frac{n-3}{2n-3} d_{\text{in}} + a. \quad (17)$$

This condition can only be fulfilled for $n = 2$. In this case we are back to the previous example. On the other hand, for $n > 2$ it is obvious that synchronization is impossible. Indeed, secondary nodes of the network have no interaction at all and therefore they do not synchronize.

4.3. Irregular network

Consider the asymmetrical seven-node network of Fig. 4(a). For simplicity, we chose equal coupling coefficients d for all directed edges.

As before, we use the six-step process to derive the synchronization condition of Theorem 1.

Step 1. Calculate the difference between the sum of the coupling coefficients of all edges directed outward from node i and the sum of the coupling coefficients of all the edges directed to node i. Thus, determine the node balance for each node of the graph:

$$D_1^c = d - d = 0 \qquad D_2^c = d - 3d = -2d$$
$$D_3^c = d - d = 0$$
$$D_4^c = 3d - d = 2d \qquad D_5^c = 2d - d = d$$
$$D_6^c = d - 2d = -d \qquad D_7^c = d - d = 0.$$

Step 2. Symmetrize the graph by replacing each directed edge by an undirected edge with half the coupling strength: $d_k = \frac{d}{2}, k = 1, \dots, 10$ (see Fig. 4(b)).

Step 3. Choose a path P_{ij} between any pair of nodes i, j of the symmetrized graph. It turns out that it is often advantageous to choose paths that contain edges with negative mean node unbalance (this quantity will be calculated in Step 4.)

Our choice of paths is

P_{12} : edge 1 $\qquad P_{13}$: edges 1, 2
P_{14} : edge 8 $\qquad P_{15}$: edges 1, 10
P_{16} : edges 1, 7, 6 $\qquad P_{17}$: edges 1, 7
P_{23} : edge 2 $\qquad P_{24}$: edges 2, 3
P_{25} : edge 10 $\qquad P_{26}$: edges 7, 6
P_{27} : edge 7 $\qquad P_{34}$: edge 3 (18)
P_{35} : edges 2, 10 $\qquad P_{36}$: edges 2, 7, 6
P_{37} : edges 2, 7 $\qquad P_{45}$: edge 4
P_{46} : edge 9 $\qquad P_{47}$: edges 9, 6
P_{56} : edge 5 $\qquad P_{57}$: edges 5, 6
P_{67} : edge 6.

Step 4. For each path P_{ij} determine the mean node unbalance $\frac{D_i^c + D_j^c}{2}$ for endnodes i and j:

$$P_{12} : \frac{D_1^c + D_2^c}{2} = -d \qquad P_{13} : \frac{D_1^c + D_3^c}{2} = 0$$
$$P_{14} : \frac{D_1^c + D_4^c}{2} = d \qquad P_{15} : \frac{D_1^c + D_5^c}{2} = \frac{d}{2}$$
$$P_{16} : \frac{D_1^c + D_6^c}{2} = -\frac{d}{2} \qquad P_{17} : \frac{D_1^c + D_7^c}{2} = 0$$
$$P_{23} : \frac{D_2^c + D_3^c}{2} = -d \qquad P_{24} : \frac{D_2^c + D_4^c}{2} = 0$$
$$P_{25} : \frac{D_2^c + D_5^c}{2} = -\frac{d}{2} \qquad P_{26} : \frac{D_2^c + D_6^c}{2} = -\frac{3d}{2}$$
$$P_{27} : \frac{D_2^c + D_7^c}{2} = -d \qquad P_{34} : \frac{D_3^c + D_4^c}{2} = d$$
$$P_{35} : \frac{D_3^c + D_5^c}{2} = \frac{d}{2} \qquad P_{36} : \frac{D_3^c + D_6^c}{2} = -\frac{d}{2}$$
$$P_{37} : \frac{D_3^c + D_7^c}{2} = 0 \qquad P_{45} : \frac{D_4^c + D_5^c}{2} = \frac{3d}{2}$$
$$P_{46} : \frac{D_4^c + D_6^c}{2} = \frac{d}{2} \qquad P_{47} : \frac{D_4^c + D_7^c}{2} = \frac{d}{2}$$
$$P_{56} : \frac{D_5^c + D_6^c}{2} = 0 \qquad P_{57} : \frac{D_5^c + D_7^c}{2} = \frac{d}{2}$$
$$P_{67} : \frac{D_6^c + D_7^c}{2} = -\frac{d}{2}.$$

We now categorize the mean node unbalance terms as follows.

If $\frac{D_i^c + D_j^c}{2} < 0$ and there is an edge k of the symmetrized graph linking directly i and j, we set $D_k = \left|\frac{D_i^c + D_j^c}{2\cdot 7}\right|$ and add this additional coupling strength to d_k. This relates to edges 1, 2, 6, 7, 10 (see Fig. 4(b)):

$$D_1 = \left|\frac{D_1^c + D_2^c}{2\cdot 7}\right| = \frac{d}{7} \qquad D_2 = \left|\frac{D_2^c + D_3^c}{2\cdot 7}\right| = \frac{d}{7}$$
$$D_6 = \left|\frac{D_6^c + D_7^c}{2\cdot 7}\right| = \frac{d}{14} \qquad D_7 = \left|\frac{D_2^c + D_7^c}{2\cdot 7}\right| = \frac{d}{7}$$
$$D_{10} = \left|\frac{D_2^c + D_5^c}{2\cdot 7}\right| = \frac{d}{14}.$$

In all other cases, the terms $\frac{D_i^c + D_j^c}{2}$ are either nonnegative or negative but there is no direct link between i and j, so that all these terms become D_{ij}.

Step 5. For each edge of the graph determine the inequality (15).

Edge 1 (link between nodes 1 and 2):

$$d_1 + D_1 = \frac{d}{2} + \frac{d}{7} > \frac{a}{7} b_k, \quad \text{where } b_k = \sum_{j>i;k\in P_{ij}}^{n} L(P_{ij}).$$

The chosen paths that pass through the edge 1 are $P_{12}, P_{13}, P_{15}, P_{16}, P_{17}$ (cf. (15)). Their weighted lengths $L(P_{ij})$ are calculated in accordance with Eq. (13):

$L(P_{12}) = |P_{12}| = 1 \quad$ since $D_1^c + D_1^c < 0$;
and there is an edge between 1 and 2

$$L(P_{13}) = |P_{13}|\chi\left(1 + \frac{D_{13}}{a}\right) = |P_{13}|\left(1 + \frac{0}{a}\right) = 2$$
$$L(P_{15}) = |P_{15}|\chi\left(1 + \frac{D_{15}}{a}\right) = |P_{15}|\left(1 + \frac{d}{2a}\right) = 2\left(1 + \frac{d}{2a}\right)$$
$$L(P_{16}) = |P_{16}|\chi\left(1 + \frac{D_{16}}{a}\right) = |P_{16}|\psi\left(1 - \frac{d}{2a}\right) = |P_{16}| \cdot 0, \quad \text{by assumption: } d > 2a$$
$$L(P_{17}) = |P_{17}|\chi\left(1 + \frac{D_{17}}{a}\right) = |P_{17}| = 2.$$

Summing up all the lengths, we obtain

$$\frac{d}{2} + \frac{d}{7} > \frac{a}{7}\left[1 + 2 + 2\left(1 + \frac{d}{2a}\right) + 2\right] = \frac{7a + d}{7}.$$

Therefore, the synchronization condition for the edge 1 becomes $d > 2a$.

Exactly as for the edge 1, we can calculate the synchronization bounds for other edges. These bounds can be summarized as follows

edge 1: $d > 2a$ $\qquad$ edge 2: $d > \frac{18a}{7}$ $\qquad$ edge 3: $d > \frac{6a}{5}$

edge 4: $d > \frac{a}{2}$ edge 5: $d > \frac{3a}{5}$ edge 6: $d > 5a$

edge 7: $d > \frac{10a}{9}$ edge 8: $d > \frac{2a}{5}$ edge 9: $d > 3a$

edge 10: $d > \frac{5a}{2}$.

Step 6. Combining the synchronization criteria for all the edges, we take the maximum constraint to achieve global synchronization. This constraint corresponds to the weakest link. Here, the weakest link is the edge 6. This edge is a bottle neck for synchronization of the entire network and requires the maximum coupling strength to synchronize all oscillators of the network. Therefore we conclude that for

$$d > d^* = 5a \tag{19}$$

we can guarantee global synchronization of the network.

It is customary to discuss network synchronization in terms of eigenvalues of the connection matrix D. It allows one to give necessary and sufficient conditions for local synchronization depending on (usually numerically calculated) Lyapunov exponents of the individual systems. We have previously shown that the second largest eigenvalue also allows one to obtain a bound for global synchronization [40]. For asymmetrical coupling, the eigenvalues are typically complex and, therefore, difficult to derive. Usually, for irregular directed networks one can only calculate these eigenvalues numerically. Thus, by the eigenvalue approaches to global synchronization, the synchronization bound associated with the second largest eigenvalue of the connection matrix is

$$d > d^* = a/|\text{Re}\lambda_2|. \tag{20}$$

Note that this is true only for networks allowing global synchronization and for which the threshold a can be rigorously derived. Actually in the context of our quadratic Lyapunov function the criterion (20) is the optimal bound for the synchronization threshold.

The choice of paths P_{ij} we made for calculating the bound (19) is suboptimal so that the condition $d^* = 5a$ gives an overestimate: $5a$ versus $a/|\text{Re}\lambda_2|$, where $\lambda_2 = -1$ is the second largest eigenvalue of the connection matrix associated with the network of Fig. 4(a). Here, λ_2 is calculated numerically. A different choice of paths P_{ij} can lead to lower thresholds that are closer to the optimal bounds achievable by the eigenvalue method.

Typically, our graph method becomes more effective and gives more correct information on the qualitative dependence of the synchronization limits on parameters of the network, while the number of oscillators composing the network increases. Calculation of weighted path lengths can be quite a laborious task for networks with complicated coupling schemes. However, once the calculation scheme is constructed, a bound giving an explicit dependence of the synchronization threshold on the network size and topology can be obtained (for more complicated cases one can use MAPLE).

5. Conclusions

We have given a sufficient condition for global complete synchronization in an arbitrary network of diffusively coupled identical dynamical systems. The condition is composed of a set of inequalities which have to be satisfied, one inequality for each edge of the connection graph. Each inequality involves a term that depends only on the individual dynamical systems, namely the coupling strength that guarantees global synchronizing of two systems. The other terms of the inequality depend only on the graph structure and on the coupling coefficients.

The new component of the method for synchronization in asymmetrical networks is the use of the symmetrize-and-weight operation. This amounts to replacing each direct link between node i node j by an undirected edge with a coupling strength that depends on the node unbalance between the two nodes. Different weights are associated with each path between any two nodes of the network. These weights also depend on the node unbalance between the endnodes of the path. The synchronization criterion for this symmetrized network also guarantees global stability of synchronization in the original directed network.

In small and also in sufficiently regular networks, the condition can be written down explicitly. In other networks, a combinatorial algorithm of polynomial complexity can establish the inequalities on the coupling coefficients that guarantee global complete synchronization. The main computational task is to determine a path between any two nodes of the graph, typically the shortest path.

We impose no restriction on the interaction between the individual systems other than diffusive coupling, i.e. a coupling matrix with non-negative off-diagonal elements and zero row sums. In particular, we do not impose symmetry on the coupling matrix. This means that the coupling between any two systems may be either absent, unidirectional, or bidirectional with not necessarily equal coupling coefficients for both directions. Of course, when complete synchronization is never possible, such as in networks without a uniformly directed tree, the condition for synchronization is simply impossible to satisfy.

Since our approach is based on Lyapunov functions, the inequalities we obtain are conservative. However, comparing with numerical simulations, we have noticed that often they give correct information on the dependence of the synchronization limits on parameters of the network. Note that one can also use the eigenvalues of the connection matrix for the Lyapunov function approach. By the eigenvalue method, we may obtain, in the case of asymmetrical networks with fixed, time-independent connections, a better bound for global synchronization than with the connection graph method. However, the eigenvalues of connection matrices associated with irregular graphs are difficult to calculate analytically.

The generalized connection graph method has an advantage over the eigenvalue method in studying networks with time-dependent coupling coefficients. Specifically, within the framework of our method, the time-dependent coupling coefficients can be handled without problems, whereas inequalities in coupling coefficients do not necessarily result in corresponding inequalities in eigenvalues. This implies that, in general, the eigenvalue method cannot be applied to networks with a time-varying coupling structure.

We should remark that our generalized method is valid for networks of slightly nonidentical oscillators. In this case, perfect synchronization cannot exist anymore, but approximate synchronization is still possible. We have previously shown that in the case of symmetrically coupled networks, similar global stability conditions of approximate synchronization can be derived within the framework of the connection graph method [40]. This carries over to asymmetrical heterogenous networks.

Finally, let us remark that the results of this paper are generalizations of our previous papers on symmetric, or asymmetric node-balanced coupling to arbitrary asymmetric coupling.

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