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GENERALIZED COUNTABLE ITERATED FUNCTION SYSTEMS

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Abstract

One of the most common and most general way to generate fractals is by using iterated function systems which consists of a finite or infinitely many maps. Generalized countable iterated function systems (GCIFS) are a generalization of countable iterated function systems by considering contractions from $X \times X$ into X instead of contractions on the metric space X to itself, where (X, d) is a compact metric space. If all contractions of a GCIFS are Lipschitz with respect to a parameter and the supremum of the Lipschitz constants is finite, then the associated attractor depends continuously on the respective parameter.

1 Introduction

In the famous paper [2], J.E. Hutchinson proves that, given a set of contractions $(\omega_n)_{n=1}^N$ in a complete metric space X, there exists a unique nonempty compact set $A \subset X$, named the *attractor* of IFS. This attractor is, generally, a fractal set. These ideas has been extended to infinitely many contractions, a such generalization can be found, for example, in [3] and, for Countable Iterated Function Systems (CIFS) on a compact metric space, in [6]. There is a current effort to extend the classical Hutchinson's framework to more general spaces and infinite iterated function systems or, more generally, to multifunction systems. In [7] it is shown that any compact subset of a metric space ca be obtained as attractor of a CIFS. M. Barnsley (in [1]) and others show that, if the contractions of an IFS depend continuously on a parameter, then the corresponding attractor also depends continuously of the respective parameter with respect to the Hausdorff metric. The result has been extended to the countable case (see [8]).

We start with a short description of a Hausdorff metric on a metric space and on a product of two metric spaces and we prove some of its properties which will

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be used in the sequel. Next, the notions of iterated function system (IFS) and countable iterated function system on a complete and, respectively compact metric space (X, d) together with some its properties are presented.

In [4], A. Mihail introduces the Recurrent Iterated Functions Systems (RIFS) which is a finite family of contractions $\omega_n : X \times X \to X$, $n = 1, \ldots, N$, where (X, d) is a complete metric space and he proves some of its properties. That construction is extended in [5] by A. Mihail and R. Miculescu to a finite family of contractive mappings from X^m $(m \in \mathbb{N})$ into X, the space (X, d) being compact.

The main results of that paper are given in section 3 when it is introduced the Generalized Countable Iterated Functions Systems (GCIFS) of order two. A GCIFS consists of a sequence of contractions $\omega_n : X \times X \to X$, $n = 1, 2, \ldots$, where (X, d) is a compact metric space. Notice that the treatment of GCIFS of any order $m \in \mathbb{N}$, $m \geq 3$, (when the considered contractions are defined on X^m space) can be make in an analogous way as in the case when $m = 2$.

It is described some ways to characterize the attractor of a GCIFS as a limiting process and by means of the fixed points of a proper family of contractions. If the contractions which compose the GCIFS obey some continuity conditions with respect to a parameter, then the corresponding attractor depends continuously with respect to that parameter.

Some ways to write the attractor of a GCIFS as a limit of a sequence of sets are presented. They can be very beneficial in certain cases to use the computer to approximate the attractor. Finally, some examples in the compact subspace X of $\mathbb{R}$ and, respectively $\mathbb{R}^2$, is given.

2 Preliminary Facts

In this section we give some well known aspects on Fractal Theory used in the sequel (more complete and rigorous treatments may be found in [2], [1], [6], [8]).

Let us consider a function $f : X \to Y$, where (X, d), (Y, δ) are two metric spaces, and we define

$$\mathrm{Lip}(f) := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\delta\big(f(x), f(y)\big)}{\mathrm{d}(x_1, x_2)} \in \overline{\mathbb{R}}_+.$$

f is said to be a Lipschitz function if $\mathrm{Lip}(f) < \infty$ and a contraction if $\mathrm{Lip}(f) < 1$. If f is contraction, then any $r \in \big(0, \mathrm{Lip}(f)\big)$ is called contraction ratio.

2.1 Hausdorff metric

Let (X, d) be a metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X. The function $h : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$, $h(A, B) = \max\big\{\mathrm{D}(A, B), \mathrm{D}(B, A)\big\}$, where $\mathrm{D}(A, B) = \sup_{x \in A} \big(\inf_{y \in B} \mathrm{d}(x, y)\big)$, for all $A, B \in \mathcal{K}(X)$, is a metric, namely the *Hausdorff metric*. If (X, d) is complete, then $\mathcal{K}(X)$ is a complete metric space with respect to this metric h. Also, $\big(\mathcal{K}(X), h\big)$ is a compact metric space provided that (X, d) is compact.

Some simple standard facts in the space $\big(\mathcal{K}(X),h\big)$, which will be used in the sequel, are described in the following two lemmas:

Lemma 2.1. *If $(E_i)_{i\in\Im}$, $(F_i)_{i\in\Im}$ are two sequences of sets in $\mathcal{K}(X)$, then*

$$h\Big(\overline{\bigcup_{i\in\Im}E_i},\overline{\bigcup_{i\in\Im}F_i}\Big)=h\Big(\bigcup_{i\in\Im}E_i,\bigcup_{i\in\Im}F_i\Big)\le\sup_{i\in\Im}h(E_i,F_i).$$

Lemma 2.2. [6, Th.1.1] *Let $(A_n)_n$ be a sequence of nonempty compact subsets of X.*

(a) *If $A_n\subset A_{n+1}$, for all $n\ge 1$, and the set $A:=\bigcup_{n\ge1}A_n$ is relatively compact, then*

$$\overline{A}=\overline{\bigcup_{n\ge1}A_n}=\lim_n A_n,$$

the limiting process being taken with respect to the Hausdorff metric and the bar means the closure;

(b) *If $A_{n+1}\subset A_n$, for any $n\ge1$, then $\lim_n A_n=\bigcap_{n\ge1}A_n$.*

We now consider another metric space (Y,δ) and we use the same notation h for the Hausdorff metric on $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ and, analogously, for the function D. We equip the space $X\times Y$ with the "max" metric d_2, namely

$$\mathrm{d}_2\big((x_1,y_1),(x_2,y_2)\big):=\max\big\{\mathrm{d}(x_1,x_2),\delta(y_1,y_2)\big\}.$$

It is known that d_2 is a metric and the space $(X\times Y,\mathrm{d}_2)$ is complete, respectively compact, whenever (X,d) and (Y,δ) are completes, respectively compacts.

Let h_2 be the Hausdorff metric on $\mathcal{K}(X)\times\mathcal{K}(Y)$ induces by d_2. We denote D_2 the corresponding set function from the definition of Hausdorff metric,

$$\mathrm{D}_2\big((B_1,C_1),(B_2,C_2)\big):=\sup_{\substack{x_1\in B_1\\ y_1\in C_1}}\inf_{\substack{x_2\in B_2\\ y_2\in C_2}}\mathrm{d}_2\big((x_1,x_2),(y_1,y_2)\big).$$

Lemma 2.3. *Under the above conditions,*

$$h_2\big((B_1,C_1),(B_2,C_2)\big)=\max\big\{h(B_1,B_2),h(C_1,C_2)\big\},$$

for any $B_1,B_2\in\mathcal{K}(X)$ and any $C_1,C_2\in\mathcal{K}(Y)$.

Proof. We first prove that

$$\mathrm{D}_2\big\{(B_1,C_1),(B_2,C_2)\big\}=\max\big\{\mathrm{D}(B_1,B_2),\mathrm{D}(C_1,C_2)\big\},\tag{2.1}$$

that is

$$\sup_{\substack{x_1\in B_1\\ y_1\in C_1}}\inf_{\substack{x_2\in B_2\\ y_2\in C_2}}\max\big\{\mathrm{d}(x_1,x_2),\delta(y_1,y_2)\big\}$$
$$=\max\big\{\sup_{x_1\in B_1}\inf_{x_2\in B_2}\mathrm{d}(x_1,x_2),\ \sup_{y_1\in C_1}\inf_{y_2\in C_2}\delta(y_1,y_2)\big\}.$$

We suppose, by contradiction, that one has

$$\mathrm{D}_2\big\{(B_1,C_1),(B_2,C_2)\big\} > \max\big\{\mathrm{D}(B_1,B_2),\mathrm{D}(C_1,C_2)\big\}.$$

There exists $r\in\mathbb{R}$ such that

$$\sup_{\substack{x_1\in B_1\\ y_1\in C_1}}\ \inf_{\substack{x_2\in B_2\\ y_2\in C_2}} \max\big\{\mathrm{d}(x_1,x_2),\delta(y_1,y_2)\big\} > r,$$

$$r > \sup_{x_1\in B_1}\ \inf_{x_2\in B_2} \mathrm{d}(x_1,x_2) \text{ and } r > \sup_{y_1\in C_1}\ \inf_{y_2\in C_2} \delta(y_1,y_2).$$

Thence:

$$\exists\, x_1^0\in B_1,\ \exists\, y_1^0\in C_1 \text{ such that } \forall\, x_2\in B_2,\ \forall\, y_2\in C_2$$
$$\Rightarrow \mathrm{d}(x_1^0,x_2) > r \text{ or } \delta(y_1^0,y_2) > r. \tag{2.2}$$

At the same time,

$$\forall\, x_1\in B_1,\ \exists\, x_2\in B_2 \text{ such that } \mathrm{d}(x_1,x_2) < r, \quad \text{and}$$

$$\forall\, y_1\in C_1,\ \exists\, y_2\in C_2 \text{ such that } \delta(y_1,y_2) < r.$$

Next, for x_1^0 and y_1^0, there are $x_2^0\in B_2$ and $y_2^0\in C_2$ with $\mathrm{d}(x_1^0,x_2^0) < r$ and $\delta(y_1^0,y_2^0) < r$ contradicting (2.2). It follows that, in (2.1), one has the inequality "$\le$".

By using the similar arguments as before, we deduce the other inequality.

Finally, by symmetry, we find

$$\mathrm{D}_2\big\{(C_1,B_1),(C_2,B_2)\big\} = \max\big\{\mathrm{D}(B_2,B_1),\mathrm{D}(C_2,C_1)\big\}$$

and thence, with (2.1), the equality of statement comes.

□

Theorem 2.1. *Let $(X,\mathrm{d}),(Y,\delta),(Z,\rho)$ be three metric spaces and $\omega: X\times Y\to Z$ be a function. Then*
(i) if ω is a Lipschitz map, one has

$$h\big(\omega(B_1,C_1),\omega(B_2,C_2)\big) \le \mathrm{Lip}(\omega) h_2\big((B_1,C_1),(B_2,C_2)\big);$$

(ii) if ω is uniform continuous, then the set function $F_\omega:\mathcal{K}(X)\times\mathcal{K}(Y)\to\mathcal{K}(Z)$, $F_\omega(B,C):=\omega(B,C)$, is continuous (for simplicity, we use the same notation h for the Hausdorff metric on $\mathcal{K}(X),\mathcal{K}(Y)$ and $\mathcal{K}(Z)$).

Proof. (i) By using Lemma 2.3, we have

$$\sup_{\substack{x_1\in B_1\\ y_1\in C_1}}\ \inf_{\substack{x_2\in B_2\\ y_2\in C_2}} \rho\big(\omega(x_1,y_1),\omega(x_2,y_2)\big) \le \mathrm{Lip}(\omega) \sup_{\substack{x_1\in B_1\\ y_1\in C_1}}\ \inf_{\substack{x_2\in B_2\\ y_2\in C_2}} \max\big\{\mathrm{d}(x_1,x_2),\delta(y_1,y_2)\big\}$$

$$\le \mathrm{Lip}(\omega)\max\{\mathrm{D}(B_1,B_2),\mathrm{D}(C_1,C_2)\} \le \mathrm{Lip}(\omega)\max\{h(B_1,B_2),h(C_1,C_2)\}$$

$$= \mathrm{Lip}(\omega) h_2\big((B_1, C_1), (B_2, C_2)\big).$$

Therefrom, we deduce

$$\mathrm{D}\big(\omega(B_1, C_1), \omega(B_2, C_2)\big) \le \mathrm{Lip}(\omega) h_2\big((B_1, C_1), (B_2, C_2)\big),$$

consequently the assertion (i) follows.

(ii) We us consider the sequence of sets $(B_n, C_n)_n$ with $B_n \in \mathcal{K}(X)$, $C_n \in \mathcal{K}(Y)$ converging to $(B, C) \in \mathcal{K}(X) \times \mathcal{K}(Y)$ with respect to the Hausdorff metric h_2. Then $h(B_n, B) \to 0$ and $h(C_n, C) \to 0$. We suppose by reductio ad absurdum that $\big(\omega(B_n, C_n)\big)_n$ do not converging to $\omega(B, C)$. Then there exists $\varepsilon_0 > 0$ such that

$$\forall\, n \in \mathbb{N},\ \exists\, k_n \ge n \text{ such that } h\big(\omega(B_{k_n}, C_{k_n}), \omega(B, C)\big) \ge \varepsilon_0.$$

That is, for each $n = 1, 2, \ldots$, one has

$$\sup_{\substack{x_1 \in B_{k_n} \\ y_1 \in C_{k_n}}} \inf_{\substack{x_2 \in B \\ y_2 \in C}} \rho\big(\omega(x_1, y_1), \omega(x_2, y_2)\big) \ge \varepsilon_0$$

or

$$\sup_{\substack{x_2 \in B \\ y_2 \in C}} \inf_{\substack{x_1 \in B_{k_n} \\ y_1 \in C_{k_n}}} \rho\big(\omega(x_1, y_1), \omega(x_2, y_2)\big) \ge \varepsilon_0.$$

Case **I**: By considering, eventually, a subsequence, we can suppose that, for any $n \ge 1$,

$$\sup_{\substack{x_1 \in B_n \\ y_1 \in C_n}} \inf_{\substack{x_2 \in B \\ y_2 \in C}} \rho\big(\omega(x_1, y_1), \omega(x_2, y_2)\big) \ge \varepsilon_0.$$

So, for each $n \ge 1$, one can find $(x_n, y_n) \in (B_n, C_n)$ such that, for any $(x', y') \in (B, C)$, we have

$$\rho\big(\omega(x_n, y_n), \omega(x', y')\big) \ge \varepsilon_0. \tag{2.3}$$

Now, let be $\varepsilon > 0$, $\varepsilon < \varepsilon_0$. By the uniform continuity of ω, there is $\eta > 0$ so that

$$\forall\, (x, y), (x', y') \in X \times Y \text{ with } \max\{\mathrm{d}(x, x'), \delta(y, y')\} < \eta$$

$$\Rightarrow\ \rho\big(\omega(x, y), \omega(x', y')\big) < \varepsilon.$$

Next, by hypothesis, we have $B_n \to B$ and $C_n \to C$. Thence, there is $n_\eta \ge 1$ so that $h(B_n, B) < \eta$ and $h(C_n, C) < \eta$ for all $n \ge n_\eta$. It follows that

$$\sup_{x \in B_n} \big(\inf_{x' \in B} \mathrm{d}(x, x')\big) < \eta$$

and as well as $\sup_{y \in C_n} \big(\inf_{y' \in C} \delta(y, y')\big) < \eta$, for any $n \ge n_\eta$. Therefrom, for any $x \in B_n$ and any $y \in C_n$, there exist $x' \in B$ and $y' \in C$ with $\mathrm{d}(x, x') < \eta$ and more $\delta(y, y') < \eta$.

In particular, $\mathrm{d}(x_n, x') < \eta$, $\delta(y_n, y') < \eta$ and hence

$$\rho\big(\omega(x_n, y_n), \omega(x'_n, y'_n)\big) < \varepsilon < \varepsilon_0,$$

contradicting (2.3).

Case **II**: We proceed in a similar way as in the preceding case denoting, for simplicity, $(B_n, C_n)_n$ instead of $(B_{k_n}, C_{k_n})_n$.

Suppose that $\sup\limits_{\substack{x_2\in B\\ y_2\in C}} \inf\limits_{\substack{x_1\in B_n\\ y_1\in C_n}} \rho\big(\omega(x_1, y_1), \omega(x_2, y_2)\big) \geq \varepsilon_0$, for all $n \geq 1$. Then, for every $n = 1, 2, \dots$, there is $(x_0, y_0) \in (B, C)$ such that, for any $(x', y') \in (B_n, C_n)$, one has

$$\rho\big(\omega(x_0, y_0), \omega(x', y')\big) \geq \varepsilon_0. \tag{2.4}$$

At the same time, for an arbitrary $\varepsilon > 0$, $\varepsilon < \varepsilon_0$, there exists $\eta > 0$ such that, for all $(x, y), (x', y') \in X \times Y$ with $\max\big\{\mathrm{d}(x, x'), \delta(y, y')\big\} < \eta$, we have

$$\rho\big(\omega(x, y), \omega(x', y')\big) < \varepsilon. \tag{2.5}$$

Next,

$$B_n \to B,\ C_n \to C \ \Rightarrow\ \exists\, n_\eta \in \mathbb{N} \text{ such that}$$

$$\sup_{\substack{x\in B\\ y\in C}} \inf_{\substack{x'\in B_n\\ y'\in C_n}} \rho\big(\omega(x, y), \omega(x', y')\big) < \eta,\ \forall\, n \geq n_\eta,$$

namely, for any $(x, y) \in B \times C$, there is $(x', y') \in B_n \times C_n$ so that $\mathrm{d}(x, x') < \eta$ and $\delta(y, y') < \eta$. In particular, taking $x = x_0$ and $y = y_0$, we have, in view of (2.5), $\delta\big(\omega(x'_0), \omega(x)\big) < \varepsilon$ contradicting the relation (2.4).

Consequently, F_ω is continuous in the arbitrary point (B, C), so it is continuous.

The proof is complete. □

As a consequence of Lemma 2.1 and the above theorem, we have obviously:

Corollary 2.1. *We consider a sequence of Lipschitz functions $\omega_n : X \times Y \to Z$, the metric space (Z, ρ) being compact. We define a set function $\mathcal{S} : \mathcal{K}(X) \times \mathcal{K}(Y) \to \mathcal{K}(Z)$ by*

$$\mathcal{S}(B, C) := \overline{\bigcup_{n\geq 1} \omega_n(B, C)}. \tag{2.6}$$

Then $\mathrm{Lip}(\mathcal{S}) \leq \sup\limits_n \mathrm{Lip}(\omega_n)$. *In particular, if* $\sup\limits_n \mathrm{Lip}(\omega_n) < \infty$, *then $\mathcal{S}$ is a Lipschitz function.*

From Theorem 2.1 and Lemma 2.1 it follows easily:

Remark 2.1. *If we have a finite set of uniform continuous functions $(\omega_n)_{n=1}^N$, then the set function $\mathcal{S}_N : \mathcal{K}(X) \times \mathcal{K}(Y) \to \mathcal{K}(Z)$, $\mathcal{S}_N(B, C) = \bigcup\limits_{n=1}^{N} \omega_n(B, C)$, is continuous.*

2.2 Iterated Function Systems, Countable Iterated Function Systems

Let us consider a complete metric space (X, d). A finite set of contractions $\omega_n : X \to X$, $n = 1, 2, \ldots, N$, is called iterated function system, shortly IFS. Then the set function $\mathcal{S}_N : \mathcal{K}(X) \to \mathcal{K}(X)$, $\mathcal{S}_N(B) := \bigcup_{n=1}^{N} \omega_n(B)$, is a contraction in the space $(\mathcal{K}(X), h)$, whose unique set-fixed point A_N is named the attractor of the considered IFS.

Now, assume that (X, d) is a compact metric space and we consider a countable system of contractions $(\omega_n)_n$ on X into itself with contractivity factors, respectively r_n, $n = 1, 2, \cdots$. We say that $(\omega_n)_n$ is a countable iterated function system (abbreviated CIFS) if $\sup_n r_n < 1$. The associated set function $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$ given by

$$\mathcal{S}(B) = \overline{\bigcup_{n \geq 1} \omega_n(B)},$$

for any $B \in \mathcal{K}(X)$, is a contraction having the contractivity factor $r = \sup_n r_n$. According to the Banach contraction principle, there is a unique $A \in \mathcal{K}(X)$ such that $\mathcal{S}(A) = A$, namely the attractor of the considered CIFS.

The attractor of CIFS $(\omega_n)_n$ can be approximated in the Hausdorff metric by the attractors of partial IFSs, $N \geq 1$, $(\omega_n)_{n=1}^{N}$ ([6, Th.2.3]). Also, concerning the matter of the attractor A, one has the following result ([6, Cor.2.1]):

LEMMA 2.4. *The attractor of CIFS $(\omega_n)_n$ represents the adherence of the set of fixed points $e_{i_1 \ldots i_p}$ of all contractions $\omega_{i_1 \ldots i_p}$, $p \geq 1$ and $i_j \geq 1$, where $\omega_{i_1 \ldots i_p} := \omega_{i_1} \circ \cdots \circ \omega_{i_p}$. In symbols,*

$$A = \overline{\{e_{i_1 \ldots i_p};\ p, i_j = 1, 2, \ldots\}}.$$

We us consider further a metric space (T, d_T) and a sequences of mappings $\omega_n : T \times X \to X$ and $r_n : T \to [0, 1)$, $n = 1, 2, \ldots$, obeying the following three properties:

(i) for each $t \in T$, $\mathrm{d}(\omega_n(t, x), \omega_n(t, y)) \leq r_n(t)\mathrm{d}(x, y)$, for any $x, y \in X$, $n \geq 1$;

(ii) there is $C > 0$ such that $\mathrm{d}_T\big(\omega_n(t, x), \omega_n(s, x)\big) \leq C\mathrm{d}_T(t, s)$, for all $x \in X$, $t, s \in T$, $n \geq 1$;

(iii) $\sup_{n,t} r_n(t) < 1$.

We define $\mathcal{S} : T \times \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$, $\mathcal{S}(t, B) = \overline{\bigcup_{n \geq 1} \omega_n(t, B)}$, for any $t \in T$ and any $B \in \mathcal{K}(X)$. It follows that, for each $t \in T$, $\mathcal{S}(t, \cdot)$ is a contraction map on $\mathcal{K}(X)$ with the contraction ratio $r(t) = \sup_n r_n(t) < 1$.

The following theorem tell us that the attractor of a CIFS depends continuously on the parameter $t \in T$ ([8, Th.6]).

THEOREM 2.2. *Under the above conditions, the function* $t \mapsto A(t)$ *is continuous from* T *into* $\mathcal{K}(X)$, *where, for* $t \in T$, $A(t)$ *means the attractor of the CIFS* $(\omega_n(t,\cdot))_{n\geq 1}$.

3 Generalized Countable Iterated Function Systems

Throughout in this section (X, d) will be a compact metric space and we consider the metric

$$\mathrm{d}_2\big((x_1,y_1),(x_2,y_2)\big) = \max\{\mathrm{d}(x_1,x_2),\mathrm{d}(y_1,y_2)\big\}$$

on $X \times X$. Then $(X \times X, \mathrm{d}_2)$ is a compact metric space.

3.1 Definition. Continuity with respect to a parameter

DEFINITION 3.1. *A sequence of contractions* $\omega_n : X \times X \to X$ *with* $\sup_n \mathrm{Lip}(\omega_n) < 1$ *is said to be a* generalized countable iterated function system *of order two on* X, *abbreviated GCIFS.*

If $N \geq 1$ *is an integer, then the finite family of functions* $(\omega_n)_{n=1}^N$ *is called the* partial generalized iterated function system (*GIFS*) *of* $(\omega_n)_n$.

By corollary 2.1, it follows immediately that $\mathcal{S} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ given by (2.6) is a contraction having the contractivity factor $r = \sup r_n$, where r_n mains the contraction ratio of ω_n, $n = 1, 2, \cdots$. At the same time, the set function

$$\mathcal{S}_N : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X),\ \mathcal{S}_N(B,C) := \bigcup_{n=1}^{N} \omega_n(B,C)$$

is a contraction with the contractivity factor $r_N = \max_{1\leq n\leq N} r_n$.

THEOREM 3.1. [4, Th.2.1] (Banach Contraction Principle) *Let* (X, d) *be a complete metric space and* $f : X \times X \to X$ *be a contraction with contractivity factor* $c \in [0, 1)$. *Then there exists a unique* $e \in X$ *such that* $f(e, e) = e$. *Moreover, for any* $x_0, x_1 \in X$, *the sequence* $(x_k)_{k\geq 0}$ *defined by* $x_{k+1} = f(x_k, x_{k-1})$, $k \geq 1$, *is convergent to* e.

Furthermore,

$$\mathrm{d}(x_k, e) \leq \frac{2c^{[k/2]}}{1-c} \max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2)\}.$$

We say that $e \in X$ with $e = f(e, e)$ is a fixed point of f.

In view of the aforesaid, one obtain:

THEOREM 3.2. *Let (X, d) be a compact metric space and $(\omega_n)_n$ be a GCIFS on X. Then there is a unique $A \in \mathcal{K}(X)$ such that $\mathcal{S}(A, A) = A$.*

Moreover, if B_0 and B_1 be arbitrary sets in $\mathcal{K}(X)$, then the sequence $(B_k)_{k\geq 0}$ given by $B_{k+1} = \mathcal{S}(B_k, B_{k-1})$, $k \geq 1$, is converging to A.

Similarly, there uniquely exists a set $A_N \in \mathcal{K}(X)$ with $\mathcal{S}_N(A_N, A_N) = A_N$.

The sets $A, A_N \in \mathcal{K}(X)$ given in the above theorem are called the attractor of GCIFS $(\omega_n)_n$, respectively of GIFS $(\omega_n)_{n=1}^N$.

We give now a construction of a CIFS associated to the considered GCIFS. For each $n \geq 1$, we put

$$\widetilde{\omega}_n : X \to X,\ \widetilde{\omega}_n(x) := \omega_n(x, x).$$

Then $\widetilde{\omega}_n$ is a contraction map having the contraction ratio less than $\mathrm{Lip}(\omega_n)$ and the same fixed point as ω_n. The CIFS $(\widetilde{\omega}_n)_n$ is said to be associated to GCIFS. If $\widetilde{A} \in \mathcal{K}(X)$ is the attractor of the associated CIFS, then $\widetilde{A} \subset A$.

Let us define $\widetilde{\mathcal{S}} : \mathcal{K}(X) \to \mathcal{K}(X)$, $\widetilde{\mathcal{S}}(B) := \mathcal{S}(B, B)$. Then $\widetilde{\mathcal{S}}$ is also a contraction (see Corollary 2.1 and Lemma 2.3) and its unique set-fixed point is A. Further, according to the Banach contraction principle, for every $C \in \mathcal{K}(X)$, the sequence $\big(\widetilde{\mathcal{S}}^k(C)\big)_k$ converges to A. More precisely, one has

LEMMA 3.1. *Let us consider a set $C_0 \in \mathcal{K}(X)$. Then the sequence $(C_k)_k$ given by $C_k := \mathcal{S}(C_{k-1}, C_{k-1})$, $k \geq 1$, is converging in the Hausdorff metric to the attractor A of the considered GCIFS.*

Furthermore, we have

$$h(A, C_k) \leq \frac{r^{k+1}}{1-r} h\big(C_0, \mathcal{S}(C_0, C_0)\big).$$

In view of the aforesaid and Lemma 2.4, one can observe that the attractor of the GCIFS contain the fixed points of its contractions.

PROPOSITION 3.1. *The attractor A of a GCIFS $(\omega_n)_n$ contains the fixed points of all ω_n, $n = 1, 2, \cdots$. Furthermore, one has*

$$A \supset \overline{\{e_{i_1 \ldots i_p};\ p, i_j = 1, 2, \ldots\}},$$

where $e_{i_1 \ldots i_p}$ denotes the unique fixed point of the contraction $\widetilde{\omega}_{i_1} \circ \cdots \circ \widetilde{\omega}_{i_p}$.

REMARK 3.1. *Every CIFS can be seen as a GCIFS. Indeed, if $(\omega_n)_n$ constitutes a CIFS on X, then the sequence of mappings $\overline{\omega}_n : X \times X \to X$ defined by $\overline{\omega}_n(x, y) := \omega_n(x)$ is a GCIFS having the same attractor. Thence, the GCIFS represents an improvement of CIFS.*

Next, we will prove that, if the contractions of a GCIFS is Lipschitz maps with respect to a parameter and the supremum of the Lipschitz constants is finite, then the attractor depends continuously with respect to the respective parameter.

THEOREM 3.3. *Let us consider a metric space* (T, d_T) *and the sequences of maps* $\omega_n : T \times X \times X \to X$ *and* $r_n : T \to [0,1)$, $n = 1, 2, \dots$, *satisfying the following requirements:*

(i) *for each* $t \in T$, *we have*

$$\mathrm{d}(\omega_n(t, x_1, y_1), \omega_n(t, x_2, y_2)) \le r_n(t)\mathrm{d}_2\big((x_1, y_1), (x_2, y_2)\big),$$

for any $x_1, x_2, y_1, y_2 \in X$ *and any* $n \ge 1$;

(ii) *there is* $C > 0$ *such that*

$$\mathrm{d}\big(\omega_n(t, x, y), \omega_n(s, x, y)\big) \le C\mathrm{d}_T(t, s),$$

for all $x, y \in X$, $t, s \in T$, $n \ge 1$;

(iii) $r := \sup\limits_{n,t} r_n(t) < 1$.

Then, if $A(t)$ *denotes the attractor of the GCIFS* $(\omega_n(t, \cdot, \cdot))_n$, *then the mapping* $t \mapsto A(t)$ *has Lipschitz constant* $\dfrac{C}{1-r}$, *hence it is uniform continuous.*

Proof. Let us define $\mathcal{S} : T \times \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$,

$$\mathcal{S}(t, B, C) = \overline{\bigcup_{n \ge 1} \omega_n(t, B, C)},$$

for any $t \in T$, $B, C \in \mathcal{K}(X)$. It follows that, for each $t \in T$, $\mathcal{S}(t, \cdot, \cdot)$ is a contraction mapping on $\mathcal{K}(X) \times \mathcal{K}(X)$ with the contraction ratio $\sup\limits_n r_n(t) < 1$.

We will first show that

$$h\big(\omega_n(t, M, M), \omega_n(s, M, M)\big) \le C\mathrm{d}_T(t, s), \ \forall\, M \subset X, \ \forall\, t, s \in T. \tag{3.7}$$

By symmetry, it is enough to prove

$$D\big(\omega_n(t, M, M), \omega_n(s, M, M)\big) \le C\mathrm{d}_T(t, s). \tag{3.8}$$

Let be $t, s \in T$. Choose $w \in \omega_n(t, M, M)$. Then, there are $x, y \in M$ such that $w = \omega_n(t, x, y)$. Let be $z = \omega_n(s, x, y) \in \omega_n(s, M, M)$. By ($ii$) we deduce that $\mathrm{d}(w, z) \le C\mathrm{d}_T(t, s)$, hence $\sup\limits_{w \in \omega_n(t,M,M)} \inf\limits_{z \in \omega_n(s,M,M)} \mathrm{d}(w, z) \le C\mathrm{d}_T(t, s)$ which proves (3.8).

Next, by using Theorem 2.1(i) and Lemma 2.3, one has

$$h\big(\omega_n(s, M, M), \omega_n(s, N, N)\big) \le r_n(s)h(M, N), \ \forall\, M, N \subset X, s \in T, n \ge 1. \tag{3.9}$$

Now, for every $t, s \in T$, taking respectively $A(t)$, $A(s)$ in the place of M and N in (3.7) and (3.9), we obtain

$$h\big(A(t),A(s)\big)=h\Big(\overline{\bigcup_{n\geq 1}\omega_n\big(t,A(t),A(t)\big)},\overline{\bigcup_{n\geq 1}\omega_n\big(s,A(s),A(s)\big)}\Big)$$
$$\leq \sup_n h\big(\omega_n\big(t,A(t),A(t)\big),\omega_n\big(s,A(s),A(s)\big)$$
$$\leq \sup_n h\big(\omega_n\big(t,A(t),A(t)\big),\omega_n\big(s,A(t),A(t)\big)$$
$$+\sup_n h\big(\omega_n\big(s,A(t),A(t)\big),\omega_n\big(s,A(s),A(s)\big)$$
$$\leq C\mathrm{d}_T(t,s)+rh\big(A(t),A(s)\big).$$

It follows $h\big(A(t),A(s)\big)\leq \dfrac{C}{1-r}\mathrm{d}_T(t,s)$ which implies the Lipschitz property of $A(t)$ completing the proof. □

Under the preceding hypothesis, the associated CIFS $\widetilde{\omega}_n(t,x)=\omega_n(t,x,x)$, $n\geq 1$, obeys the conditions of Theorem 2.2. So, the attractor $\widetilde{A}(t)$ of the CIFS $(\widetilde{\omega}_n(t,\cdot))_n$, depends continuously on the parameter t.

3.2 Approximation of the attractor of a GCIFS

Lemma 3.2. *Under the conditions of Theorem* 3.2, *we have*

$$A_N \underset{N}{\longrightarrow} A,$$

with respect to the Hausdorff metric.

Proof. Let be $\varepsilon>0$. By applying Lemma 2.2 (a) to the increasing sequence $\Big(\bigcup\limits_{n=1}^{N}\omega_n(A,A)\Big)_N$, we can find $N_\varepsilon\geq 1$ such that, for any $N\geq N_\varepsilon$, we have

$$h\Big(\bigcup_{n=1}^{N}\omega_n(A,A),\overline{\bigcup_{n\geq 1}\omega_n(A,A)}\Big)<\varepsilon(1-\lambda),\tag{3.10}$$

where $\lambda=\sup\limits_n r_n$. Thereinafter, for every $N\geq N_\varepsilon$,

$$h(A_N,A)=h\big(\mathcal{S}_N(A_N,A_N),\mathcal{S}(A,A)\big)=h\Big(\bigcup_{n=1}^{N}\omega_n(A_N,A_N),\overline{\bigcup_{n\geq 1}\omega_n(A,A)}\Big)$$
$$\leq h\Big(\bigcup_{n=1}^{N}\omega_n(A_N,A_N),\bigcup_{n=1}^{N}\omega_n(A,A)\Big)+h\Big(\bigcup_{n=1}^{N}\omega_n(A,A),\overline{\bigcup_{n\geq 1}\omega_n(A,A)}\Big)$$
$$\leq \sup_{1\leq n\leq N} h\big(\omega_n(A_N,A_N),\omega_n(A,A)\big)+\varepsilon(1-\lambda)\leq \lambda h(A_N,A)+\varepsilon(1-\lambda).$$

Consequently, by using (3.10), Lemma 2.3 and Theorem 2.1, one obtain $h(A_N,A)<\varepsilon$, completing the proof. □

Lemma 3.3. *Let us consider two arbitrary sets $B_0, B_1 \in \mathcal{K}(X)$ and, for each $k \geq 1$, $B^N_{k+1} = \mathcal{S}_N(B^N_k, B^N_{k-1})$ and, respectively $B_{k+1} = \mathcal{S}(B_k, B_{k-1})$. Then $B^N_k \underset{N}{\longrightarrow} B_k$, for any $k = 0, 1, \cdots$.*

Proof. Firstly, by using the same argument as in the proof of (3.10), for some $k \geq 1$ and $\varepsilon > 0$, there is $N_\varepsilon \geq 1$ such that, whenever $N \geq N_\varepsilon$, one has

$$h\Big(\bigcup_{n=1}^{N} \omega_n(B_k, B_{k-1}), \overline{\bigcup_{n\geq 1} \omega_n(B_k, B_{k-1})}\Big) < \frac{\varepsilon}{2}.$$

Next, we proceed by mathematical induction with respect to k. We suppose that $h(B^N_m, B_m) \underset{N}{\longrightarrow} 0$ for all $m \leq k$. Hence there is $N^* \geq N_\varepsilon$ such that

$$h(B^N_m, B_m) < \frac{\varepsilon}{2 \sup r_n}$$

and withal $h(B^N_{m-1}, B_{m-1}) < \dfrac{\varepsilon}{2 \sup r_n}$, for any $N \geq N^*$.

In view of the aforesaid, we find

$$\begin{aligned}
h(B^N_{k+1}, B_{k+1}) &= h\big(\mathcal{S}(B^N_k, B^N_{k-1}), \mathcal{S}(B_k, B_{k-1})\big) \\
&= h\Big(\bigcup_{n=1}^{N} \omega_n(B^N_k, B^N_{k-1}), \overline{\bigcup_{n\geq 1} \omega_n(B_k, B_{k-1})}\Big) \\
&\leq h\Big(\bigcup_{n=1}^{N} \omega_n(B^N_k, B^N_{k-1}), \bigcup_{n=1}^{N} \omega_n(B_k, B_{k-1})\Big) \\
&\quad + h\Big(\bigcup_{n=1}^{N} \omega_n(B_k, B_{k-1}), \overline{\bigcup_{n\geq 1} \omega_n(B_k, B_{k-1})}\Big) \\
&\leq \sup_n r_n \cdot \max\big\{h(B^N_k, B_k), h(B^N_{k-1}, B_{k-1})\big\} + \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

□

According to Lemmas 3.2, 3.3 and Theorem 3.2 we deduce immediately the following result which is useful to approximate the attractor of a GCIFS.

Theorem 3.4. *Let A be the attractor of a GCIFS $(\omega_n)_{n\geq 1}$ and $B_0, B_1 \in \mathcal{K}(X)$ be some arbitrary sets. Then A is approximated with respect to the Hausdorff metric by the attractors A_N of the associated partial GIFS $(\omega_n)^N_{n=1}$ and, moreover, it is also approximated by the sequence $(B_k)_{k\geq 0}$, where $B_k = \mathcal{S}(B_{k-1}, B_{k-2})$ for $k \geq 2$.*

More precisely, we have the following diagram

$$\begin{array}{ccc}
B^N_k & \underset{k}{\longrightarrow} & A_N \\
\downarrow_N & & \downarrow_N \\
B_k & \underset{k}{\longrightarrow} & A
\end{array}$$

Another way to approximate the attractor of GCIFS are described below.

LEMMA 3.4. *Let us consider two sequences of sets $(B_k)_k$ and $(C_k)_k$ from $\mathcal{K}(X)$ converging with respect to the Hausdorff metric to B, respectively to C, where $B, C \in \mathcal{K}(X)$. Then $\mathcal{S}_k(B_k, C_k) \underset{k}{\longrightarrow} \mathcal{S}(B, C)$.*

Particulary, if $B_k = C_k \underset{k}{\longrightarrow} A$ (A being the attractor of the GCIFS), then

$$\mathcal{S}_k(B_k, B_k) \underset{k}{\longrightarrow} \mathcal{S}(A, A) = A.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Then, there exists $k_\varepsilon \geq 1$ such that

$$h_2\big((B_k, C_k), (B, C)\big) = \max\big\{h(B_k, B), h(C_k, C)\big\} < \frac{\varepsilon}{2},\ \forall\, k \geq k_\varepsilon.$$

According to Theorem 2.1, one has

$$h\Big(\bigcup_{n=1}^{k} \omega_n(B_k, C_k), \bigcup_{n=1}^{k} \omega_n(B, C)\Big) \leq \max_{1 \leq n \leq k} h\big(\omega_n(B_k, C_k), \omega_n(B, C)\big)$$

$$\leq \sup_n r_n h_2\big((B_k, C_k), (B, C)\big) < \frac{\varepsilon}{2},\ \forall\, k \geq k_\varepsilon. \tag{3.11}$$

By Lemma 2.2 (a) we can find $K_\varepsilon \geq k_\varepsilon$ such that, for any $k \geq K_\varepsilon$,

$$h\Big(\bigcup_{n=1}^{k} \omega_n(B, C), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\Big) < \frac{\varepsilon}{2}. \tag{3.12}$$

Finally, with (3.11) and (3.12), we have

$$h\Big(\mathcal{S}_k(B_k, C_k), \mathcal{S}(B, C)\Big) = h\Big(\bigcup_{n=1}^{k} \omega_n(B_k, C_k), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\Big)$$

$$\leq h\Big(\bigcup_{n=1}^{k} \omega_n(B_k, C_k), \bigcup_{n=1}^{k} \omega_n(B, C)\Big) + h\Big(\bigcup_{n=1}^{k} \omega_n(B, C), \overline{\bigcup_{n \geq 1} \omega_n(B, C)}\Big)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which implies the assertion of statement. □

LEMMA 3.5. *We suppose that $B_0, B_1 \in \mathcal{K}(X)$, $B_0 \subset B_1 \subset \mathcal{S}_N(B_0, B_1)$, for each $N \geq 1$. We consider further the sequence $(B_k^N)_{k,N}$ where $B_0^N = B_0$, $B_1^N = B_1$ and $B_{k+1}^N = \mathcal{S}_N(B_k^N, B_{k-1}^N)$, for all $k, N \geq 1$. Then $B_k^k \subset B_{k+1}^{k+1}$ and*

$$A = \lim_k B_k^k = \overline{\bigcup_{k \geq 1} B_k^k}.$$

Moreover, $(A_N)_N$ is increasing and $A = \overline{\bigcup_{N \geq 1} A_N}$

Proof. Let be $N \geq 1$. It is easy to establish by induction that $B_k^N \subset B_{k+1}^N$ for all k. Then, in view of Lemmas 2.2 and 3.2, one has

$$A_N = \lim_k B_k^N = \overline{\bigcup_{k \geq 1} B_k^N}.$$

Also it is obvious that $B_k^N \subset B_k^{N+1}$, for any k, N, thence $A_N \subset A_{N+1}$. Thus

$$A = \lim_{k,N} B_k^N = \overline{\bigcup_{k \geq 1} \bigcup_{N \geq 1} B_k^N}.$$

Consequently, the diagonal sequence $(B_k^k)_k$ is increasing and

$$A = \lim_k B_k^k = \overline{\bigcup_{k \geq 1} B_k^k}.$$

□

Remark 3.2. *According to the preceding lemma, if the sets B_0, B_1 are finite, then the attractor of a GCIFS can be approximated by the finite sets B_k^k, $k \geq 1$. This fact is very instrumental to represent, in certain cases, that attractor with the aim of computer.*

For every $N \geq 1$, we set $F_N := \{e_1, \dots, e_N\}$ and $B := \{e_1, e_2, \dots\}$, e_n being the fixed point of ω_n. For a fixed integer N let be $B_0^N = B_1^N = F_N$. It is evident that $F_N \subset \mathcal{S}_N(F_N, F_N)$. Then $B_{k+1}^N = \mathcal{S}_N(B_k^N, B_{k-1}^N) \nearrow_k A_N$. Thus

$$\overline{B} = \lim_N F_N \subset \lim_N A_N = A.$$

Thereafter, we deduce that such a finite sets $B_0 \subset B_1$ can be $B_0 = B_1 = F_1$ which obviously obey the requirement of Lemma 3.5.

Finally, we give two examples of GCIFS on a compact subset of $\mathbb{R}$, respectively on $\mathbb{R}^2$.

Example 3.1. *Let us consider the compact metric space $X := [0,1] \subset \mathbb{R}$ equipped with the Euclidean metric. Let $\alpha, p, q \in [0,1]$ be any fixed constants with $p+q \neq 0$ and $(\alpha_n)_n$ be an increasing sequence of real numbers from $[0,1]$ converging to $\alpha - \frac{p+q}{3}\alpha$. From each $n = 1, 2, \dots$, we define the mapping $\omega_n : [0,1] \times [0,1] \to [0,1]$, by*

$$\omega_n(x,y) = \frac{n(px+qy)}{3n+1} + \alpha_n.$$

Then $(\omega_n)_n$ is a GCIFS whose attractor is $[0, \alpha]$.

Proof. Firstly, we make evident that $\omega_n\big([0,1] \times [0,1]\big) = \Big[0, \frac{n(p+q)}{3n+1} + \alpha_n\Big] \subset [0,1]$, hence ω_n is well defined.

Next, it is simple to see that ω_n is a contraction having the contraction ratio $r_n = \dfrac{(p+q)n}{3n+1}$.

Moreover, since $\dfrac{(p+q)n\alpha}{3n+1} + \alpha_n \nearrow \alpha$, it follows

$$\mathcal{S}(A,A) = \overline{\bigcup_{n\geq 1} \omega_n(A,A)} = \overline{\bigcup_{n\geq 1} \Big[0, \frac{(p+q)n\alpha}{3n+1} + \alpha_n\Big]} = \overline{[0,\alpha)} = A,$$

where $A = [0,\alpha]$. □

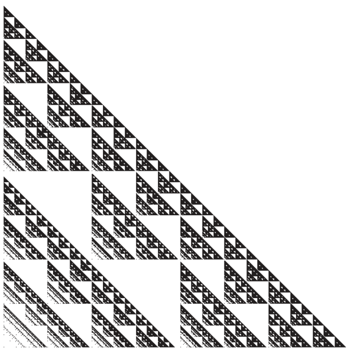
We present now as example a fractal of Sierpinski-infinite type as attractor of a proper GCIFS by generalizing a construction from [6].

Example 3.2. *Let* $X := \big\{(x,y) \in \mathbb{R}^2;\ 0 \leq x \leq 1,\ 0 \leq y \leq 1-x\big\}$ *be the plane surface of the closed triangle having its vertices in the points* $(0,0),\ (0,1),\ (1,0)$. *Next, we consider an integer* $p \geq 2$, $q \in [0,1]$ *and the contractions* $\omega_{ij} : X \times X \longrightarrow X$ *defined by*

$$\omega_{ij}\big((x_1,y_1),(x_2,y_2)\big)$$

$$= \Big(\frac{1}{p^i}\big(qx_1 + (1-q)x_2\big) + (j-1)\frac{1}{p^i}, \frac{1}{p^i}\big(qy_1 + (1-q)y_2\big) + \big(\frac{p^i-1}{p-1} - j\big)\frac{1}{p^i}\Big)$$

for all $i = 1,2,\dots,\ j = 1,2,\dots,\dfrac{p^i-1}{p-1}$. *Then* $(\omega_{i,j})_{i,j}$ *constitutes a GCIFS whose attractor is given in the following figure.*



The attractor associated to the considered GCIFS for $p = 2$

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